KO-theory of flag manifolds

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1 Introduction

The purpose of this paper is to determine the KO^*-groups of flag manifolds which are the homogeneous spaces $G(n)/T$ for $G = U, Sp, SO$ and T is the maximal torus of $G(n)$. We compute it by making use of the Atiyah-Hirzebruch spectral sequence and obtain the following.

Theorem . *The KO^i-groups of $G(n)/T$ for $G = U, Sp, SO$ are as follows, where $s = n!/2, 2^{n-1}n!$ for $G = U, Sp$ and $s = 2^{m-2}m!, 2^{m-1}m!$ for $G = SO$ and $n = 2m, 2m+1$ respectively.*

i	
0	$s\mathbf{Z} \oplus t_3\mathbf{Z}_2$
-1	$t_0\mathbf{Z}_2$
-2	$s\mathbf{Z} \oplus t_0\mathbf{Z}_2$
-3	$t_1\mathbf{Z}_2$
-4	$s\mathbf{Z} \oplus t_1\mathbf{Z}_2$
-5	$t_2\mathbf{Z}_2$
-6	$s\mathbf{Z} \oplus t_2\mathbf{Z}_2$
-7	$t_3\mathbf{Z}_2$

$U(n)/T$

n	t_0	t_1	t_2	t_3
$4k$	$2^{4k-2}+(-1)^k2^{2k-1}$	2^{4k-2}	$2^{4k-2}-(-1)^k2^{2k-1}$	2^{4k-2}
$4k+1$	$2^{4k-1}+(-1)^k2^{2k-1}$	$2^{4k-1}-(-1)^k2^{2k-1}$	$2^{4k-1}-(-1)^k2^{2k-1}$	$2^{4k-1}+(-1)^k2^{2k-1}$
$4k+2$	$2^{4k}+(-1)^k2^{2k}$	2^{4k}	$2^{4k}-(-1)^k2^{2k}$	2^{4k}
$4k+3$	$2^{4k+1}-(-1)^k2^{2k}$	$2^{4k+1}+(-1)^k2^{2k}$	$2^{4k+1}+(-1)^k2^{2k}$	$2^{4k+1}-(-1)^k2^{2k}$

$Sp(n)/T$

n	t_0	t_1	t_2	t_3
$2k$	$2^{2k-2}+2^{k-1}$	2^{2k-1}	$2^{2k-2}-2^{k-1}$	2^{2k-1}
$2k+1$	$2^{2k-1}+2^{k-1}$	$2^{2k-1}+2^{k-1}$	$2^{2k-1}-2^{k-1}$	$2^{2k-1}-2^{k-1}$

$SO(n)/T$

n	t_0	t_1	t_2	t_3
$8k$	2^{4k-1}	$2^{4k-1}-(-1)^k2^{2k-1}$	2^{4k-1}	$2^{4k-1}+(-1)^k2^{2k-1}$
$8k+1$	2^{4k-1}	$2^{4k-1}-(-1)^k2^{2k}$	2^{4k-1}	$2^{4k-1}+(-1)^k2^{2k}$
$8k+2$	$2^{4k-1}+(-1)^k2^{2k-1}$	2^{4k-1}	$2^{4k-1}-(-1)^k2^{2k-1}$	2^{4k-1}
$8k+3$	$2^{4k}+(-1)^k2^{2k-1}$	$2^{4k}-(-1)^k2^{2k-1}$	$2^{4k}-(-1)^k2^{2k-1}$	$2^{4k}+(-1)^k2^{2k-1}$
$8k+4$	$2^{4k+1}+(-1)^k2^{2k}$	2^{4k+1}	$2^{4k+1}-(-1)^k2^{2k}$	2^{4k+1}
$8k+5$	2^{4k+1}	$2^{4k+1}-(-1)^k2^{2k+1}$	2^{4k+1}	$2^{4k+1}+(-1)^k2^{2k+1}$
$8k+6$	2^{4k+1}	$2^{4k+1}-(-1)^k2^{2k}$	2^{4k+1}	$2^{4k+1}+(-1)^k2^{2k}$
$8k+7$	$2^{4k+2}+(-1)^k2^{2k}$	$2^{4k+2}-(-1)^k2^{2k}$	$2^{4k+2}-(-1)^k2^{2k}$	$2^{4k+2}+(-1)^k2^{2k}$

2 The Atiyah-Hirzebruch spectral sequence

First we recall that the coefficient ring of KO-theory is that

$$KO^* = \mathbf{Z}[\alpha, x, \beta, \beta^{-1}]/(2\alpha, \alpha^3, \alpha x, x^2-4\beta),$$

where $|\alpha| = 1$, $|x| = 4$ and $|\beta| = 8$.

Let X be a finite CW-complex. The Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ is the spectral sequence with $E_2^{p,q} \cong H^p(X; KO^q)$ converging to $KO^*(X)$. It is well known that the differential d_2 of the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ is given by the following. (See [F])

$$d_2^{*,q} = \begin{cases} Sq^2\,\pi_2 & q \equiv 0\ (8) \\ Sq^2 & q \equiv -1\ (8) \\ 0 & \text{otherwise,} \end{cases}$$

where π_2 is the modulo 2 reduction.

It is well known that G/T is a CW-complex with only even cells, where G is a compact connected Lie group and T is the maximal torus of G. ([B]) The next proposition, given in [HK1,2], is concerned with the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ for the special X which can be G/T.

Proposition 2.1. *Let X be a CW-complex whose cohomology is torsion free and concentrated in even dimension, and $E_r(X)$ be the r-th term of the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$. Then we have the following.*

1. $\iota : E_3^{p,q}(X) \cong H^p(H^*(X;\mathbf{Z}_2); Sq^2)$ *for* $q \equiv -1\ (8)$
2. *Let d_r be the first non-trivial differential.* $(r \geq 3)$
   - (a) $r \equiv 2\ (8)$.
   - (b) *There exists $x \in E_r^{p,0}(X)$ such that $\alpha x \neq 0$ and $\alpha d_r x \neq 0$.*
   - (c) *If X admits a map $\mu : X \times X \to X$ which makes $H^*(H^*(X;\mathbf{Z}_2); Sq^2)$ to be a Hopf algebra, then $\iota(\alpha x)$ is indecomposable and $\iota(d_r x)$ is primitive for the least p and $x \in E_r^{p,0}(X)$ in (b).*

3 The Sq^2-cohomology of flag manifolds

Recall that the cohomology of the flag manifold $U(n)/T$ is

$$H^*(U(n)/T;\mathbf{Z}) \cong \mathbf{Z}[x_1,\dots x_n]/(c_1,\dots,c_n),$$

where $|x_i| = 2$ and c_j is the j-th elementary symmetric function in $x_1,\cdots,x_n$.

We determine the Sq^2-cohomology of $U(n)/T$ by the similar way of [HK1, Proposition 2].

Proposition 3.1.

$$H^*(H^*(U(n)/T;\mathbf{Z}_2);Sq^2) \cong \begin{cases} \bigwedge(y_6,y_{14},\dots,y_{8m-2}) & n=2m+1 \\ \bigwedge(y_6,y_{14},\dots,y_{8m-10},z) & n=2m, \end{cases}$$

where y_{8k-2} and z are represented by $\sum_{i_1<\dots<i_{2k}} x_{i_1}x_{i_2}^2x_{i_3}^2\cdots x_{i_{2k}}^2$ and x_1^{n-1} respectively.

Proof. Let R be a differential graded algebra $(\,\mathbf{Z}_2[x_1,\dots,x_n],d\,)$ with $|x_i|=2$ and $dx_i = x_i^2$, and c_j be the j-th elementary symmetric function in $x_1,\dots,x_n$. Then we have

$$dc_{2i} = c_{2i+1} + c_1c_{2i},\ dc_{2i+1} = c_1c_{2i+1},$$

where $c_j = 0$ for $j > n$.

Let R_1 be the graded differential algebra $R_1 = R/(c_1)$ with the differential induced from R. We construct the differential graded algebra R_k $(k \le n)$ inductively by the following short exact sequences.

$$\begin{array}{llll} 0 \to R_{2k-1} \xrightarrow{\cdot c_{2k+1}} R_{2k-1} \to R_{2k} & \to 0 & (2k<n) \\ 0 \to R_{2k} \xrightarrow{\cdot c_{2k}} R_{2k} \to R_{2k+1} & \to 0 & (2k+1 \le n) \\ 0 \to R_{n-1} \xrightarrow{\cdot c_n} R_{n-1} \to R_n & \to 0 & (n \text{ is even}) \end{array}$$

It is obvious that $R_n \cong (H^*(U(n)/T;\mathbf{Z}_2), Sq^2)$ as a differential graded algebra.

We have the following long exact sequences.

$$\cdots \to H^i(R_{2k-1}) \xrightarrow{H(\cdot c_{2k+1})} H^{i+4k+2}(R_{2k-1}) \to H^{i+4k+2}(R_{2k}) \xrightarrow{\delta} H^{i+2}(R_{2k-1}) \to \cdots \quad (2k<n)$$

$$\cdots \to H^i(R_{2k}) \xrightarrow{H(\cdot c_{2k})} H^{i+4k}(R_{2k}) \to H^{i+4k}(R_{2k+1}) \xrightarrow{\delta} H^{i+2}(R_{2k}) \to \cdots \quad (2k+1 \le n)$$

Inductively we obtain

$$\begin{aligned} H^*(R_{2k}) &\cong \bigwedge(y_6,y_{14},\dots,y_{8k-10},c_{2k}) \\ H^*(R_{2k+1}) &\cong \bigwedge(y_6,y_{14},\dots,y_{8k-2}), \qquad \delta y_{8k-2} = c_{2k}.\ (2k+1 \le n). \end{aligned}$$

Then y_{8k-2} is represented by

$$\sum_{i_1<\ldots<i_{2k}} x_{i_1}x_{i_2}^2x_{i_3}^2\cdots x_{i_{2k}}^2$$

and this completes the case that n is odd.

When n is even we have the following exact sequence.

$$\cdots \to H^i(R_{n-1}) \xrightarrow{H(\cdot c_n)} H^{i+2n}(R_{n-1}) \to H^{i+2n}(R_n) \xrightarrow{\delta} H^{i+2}(R_{n-1}) \to \cdots$$

Then we have

$$H^*(R_n) \cong \bigwedge(y_6, y_{14}, \ldots, y_{8m-10}, z),\ \delta z = 1.\ (n = 2m)$$

Therefore z is represented by $x_2x_3\cdots x_n = x_1^{n-1} \in R_n$ and this completes the proof. □

It is well known that

$$H^*(Sp(n)/T; \mathbf{Z}) \cong \mathbf{Z}[x_1, \ldots, x_n]/(c_1^2, \ldots, c_n^2),$$

where $|x_i| = 2$ and c_j is the j-th elementary symmetric function in $x_1, \ldots, x_n$.

Proposition 3.2.

$$H^*(H^*(Sp(n)/T; \mathbf{Z}_2); Sq^2) \cong \bigwedge(y_2, y_6, \ldots, y_{4n-2}),$$

where y_{4k-2} is represented by $\sum_{i_1<\ldots<i_k} x_{i_1}x_{i_2}^2x_{i_3}^2\cdots x_{i_k}^2$.

Proof. Let R_0 be the differential graded algebra $\mathbf{Z}_2[x_1, \ldots, x_n]$ with $dx_i = x_i^2$. We construct the differential graded algebra R_k for $k \leq n$ inductively by the following exact sequence.

$$0 \to R_k \xrightarrow{\cdot c_{k+1}^2} R_k \to R_{k+1} \to 0$$

It is obvious that R_n is isomorphic to $(H^*(Sp(n)/T; \mathbf{Z}_2), Sq^2)$ as differential graded algebras. We have the following exact sequence.

$$\cdots \to H^i(R_{k-1}) \xrightarrow{H(\cdot c_k^2)} H^{i+4k}(R_{k-1}) \to H^{i+4k}(R_k) \xrightarrow{\delta} H^{i+2}(R_{k-1}) \to \cdots$$

Then we obtain inductively

$$H^*(R_k) \cong \bigwedge(y_2, y_6, \ldots, y_{4k-2}),\ \delta y_{4k-2} = 1.$$

Therefore y_{4k-2} is represented by $\sum_{i_1<\ldots<i_k} x_{i_1}x_{i_2}^2x_{i_3}^2\cdots x_{i_k}^2$ and this completes the proof. □

It is known that

$$H^*(SO(2n+\epsilon)/U(n);\mathbf{Z}_2) \cong \Delta(e_2, e_4, \ldots, e_{2(n+\epsilon+1)}), \;\; e_{2i}^2 = e_{4i},$$

where $\epsilon = 0, 1$, $|e_i| = i$, $e_i = 0$ for $i > 2(n+\epsilon-1)$ and $\Delta(e_2, \ldots)$ is the $\mathbf{Z}_2$-algebra whose $\mathbf{Z}_2$-module basis are $e_{i_1} \cdots e_{i_k}$ $(i_1 < \ldots < x_{i_k})$. ([KI],[T]) We see the following by making use of the fibration $U(n)/T \xrightarrow{j} SO(2n+\epsilon)/T \xrightarrow{p} SO(2n+\epsilon)/U(n)$.

$$H^*(SO(2n+\epsilon)/T;\mathbf{Z}_2) \cong \mathbf{Z}_2[x_1, \ldots, x_n]/(c_1, \ldots, c_n) \otimes \Delta(e_2, e_4, \ldots, e_{2(n+\epsilon+1)}),$$

where $Sq^2 e_{4i-2} = e_{4i}$, $j^*(x_i) = x_i \in H^2(U(n)/T;\mathbf{Z}_2)$ and $p^*(e_i) = e_i \in H^i(SO(2n+\epsilon)/T;\mathbf{Z}_2)$. ([KI],[T])

Proposition 3.3.

$$H^*(H^*(SO(2n+\epsilon)/T;\mathbf{Z}_2);Sq^2)$$

$$\cong \begin{cases} \bigwedge(y_6, y_{14} \ldots y_{8m-10}, z) \otimes \bigwedge(e'_6, e'_{14}, \ldots e'_{8m-10}, [e_{4m-2}]) & \epsilon = 0, n = 2m \\ \bigwedge(y_6, y_{14} \ldots y_{8m-10}, z) \otimes \bigwedge(e'_6, e'_{14}, \ldots e'_{8m-2}) & \epsilon = 1, n = 2m \\ \bigwedge(y_6, y_{14} \ldots y_{8m-2}) \otimes \bigwedge(e'_6, e'_{14}, \ldots e'_{8m-2}) & \epsilon = 0, n = 2m+1 \\ \bigwedge(y_6, y_{14} \ldots y_{8m-2}) \otimes \bigwedge(e'_6, e'_{14}, \ldots e'_{8m-2}, [e_{4m+2}]) & \epsilon = 1, n = 2m+1, \end{cases}$$

where y_{8k-2}, z, e'_{8k-2} *are represented by* $\sum_{i_1<\ldots<i_{2k}} x_{i_1} x_{i_2}^2 \cdots x_{i_{2k}}^2, x_1^{n-1}, e_{4k-2}e_{4k} + e_{8k-2}$ *respectively.*

Proof. We have the following isomorphism as differential graded algebras with the differential Sq^2.

$$H^*(SO(2n+\epsilon)/T;\mathbf{Z}_2) \cong H^*(U(n)/T;\mathbf{Z}_2) \otimes H^*(SO(2n+\epsilon)/U(n);\mathbf{Z}_2)$$

By Proposition 3.1, we obtain $H^*(H^*(U(n)/T;\mathbf{Z}_2);Sq^2)$. Then we compute $H^*(H^*(SO(2n+\epsilon)/U(n);\mathbf{Z}_2);Sq^2)$.

Let M_i be the following module, where $e'_{8i-2} = e_{4i-2}e_{4i} + e_{8i-2}$.

$$M_i = \mathbf{Z}_2\langle 1, e_{4i-2}, e_{4i}, e'_{8i-2}\rangle$$

Then we see that M_i is the differential graded submodule of $H^*(SO(2n+\epsilon)/U(n);\mathbf{Z}_2)$ with the differential Sq^2. We have the following isomorphisms as differential graded modules with the differential Sq^2.

$$H^*(SO(2n+\epsilon)/U(n);\mathbf{Z}_2) \cong \begin{cases} M_1 \otimes \ldots \otimes M_{m-1} \otimes \bigwedge(e_{4m-2}) & \epsilon = 0, n = 2m \\ M_1 \otimes \ldots \otimes M_m & \epsilon = 1, n = 2m \\ M_1 \otimes \ldots \otimes M_m & \epsilon = 0, n = 2m+1 \\ M_1 \otimes \ldots \otimes M_m \otimes \bigwedge(e_{4m+2}) & \epsilon = 1, n = 2m+1 \end{cases}$$

Since $H^*(M_i;Sq^2) \cong \mathbf{Z}_2\langle 1, [e'_{8i-2}]\rangle$ and ${e'_{8i-2}}^2 = Sq^2(e_{8i-6}e_{8i} + e_{16i-6})$, the proof is completed. □

4 Proof of Theorem

Let BT^n be the classifying space of an n-torus and $\mu_n : BT^n \times BT^n \to BT^{2n}$ be the identity. We can set $H^*(BT^{2n};\mathbf{Z}) \cong \mathbf{Z}[x_1,\dots,x_{2n}]$, $H^*(BT^n \times BT^n;\mathbf{Z}) \cong \mathbf{Z}[x_1,\dots,x_n] \otimes \mathbf{Z}[x_{n+1},\dots,x_{2n}]$ and

$$\mu_n^*(x_i) = \begin{cases} x_i \otimes 1 & i \le n \\ 1 \otimes x_i & i > n. \end{cases}$$

Then we have the following.

$$\begin{aligned}
&\mu_n^*(\sum_{i_1<\ldots<i_k\le 2n} x_{i_1}x_{i_2}^2x_{i_3}^2\cdots x_{i_k}^2) \\
&\quad= \sum_{i_1<\ldots<i_k\le n} x_{i_1}x_{i_2}^2\ldots x_{i_k}^2 \otimes 1 + \sum_{i_1<\ldots<i_{k-1}\le n<i_k} x_{i_1}x_{i_2}^2\ldots x_{i_{k-1}}^2 \otimes x_{i_k}^2 \\
&\qquad+ \sum_{i_1<\ldots<i_{k-2}\le n<i_{k-1}<i_k} x_{i_1}x_{i_2}^2\ldots x_{i_{k-2}}^2 \otimes x_{i_{k-1}}^2x_{i_k}^2 + \cdots \\
&\qquad\quad+ \sum_{i_1\le n<i_2<\ldots<i_k} x_{i_1} \otimes x_{i_2}^2\ldots x_{i_k}^2 + \sum_{n<i_1<\ldots<i_k} 1 \otimes x_{i_1}x_{i_2}^2\ldots x_{i_k}^2 \\
&\quad= \sum_{i_1<\ldots<i_k\le n} x_{i_1}x_{i_2}^2x_{i_3}^2\cdots x_{i_k}^2 \otimes 1 + \sum_{i_1<\ldots<i_{k-1}\le n<i_k} x_{i_1}x_{i_2}^2\ldots x_{i_{k-1}}^2 \otimes c_1^2 \\
&\qquad+ \sum_{i_1<\ldots<i_{k-2}\le n<i_{k-1}<i_k} x_{i_1}x_{i_2}^2\ldots x_{i_{k-2}}^2 \otimes c_2^2 + \cdots \\
&\qquad\quad+ \sum_{i_1\le n<i_2<\ldots<i_k} x_{i_1} \otimes c_{k-1}^2 + 1 \otimes \sum_{n\le i_1<\ldots<i_k} x_{i_1}x_{i_2}^2x_{i_3}^2\cdots x_{i_k}^2,
\end{aligned}$$

where c_i is the i-th elementary symmetric function in $x_{n+1},\dots,x_{2n}$. Then we have the following for $y_k = \sum_{i_1<\ldots<i_k} x_{i_1}x_{i_2}^2x_{i_3}^2\cdots x_{i_k}^2 \in H^*(BT^\infty;\mathbf{Z})$.

$$\mu_\infty^*(y_k) = y_k \otimes 1 + 1 \otimes y_k + \sum_{i=1}^{k-1} y_{k-i} \otimes c_i^2 \qquad (*)$$

Let $\mu_{G/T} : G/T \times G/T \to G/T$ be the natural inclusion for $G = U, Sp, SO$, then we have the following commutative diagram.

$$\begin{array}{ccc}
G/T \times G/T & \longrightarrow & BT \times BT \\
\downarrow{\scriptstyle \mu_{G/T}} & & \downarrow{\scriptstyle \mu_\infty} \\
G/T & \longrightarrow & BT
\end{array}$$

Note Proposition 3.1, 3.2 and $(*)$, then we see that $H^*(H^*(G/T;\mathbf{Z}_2);Sq^2)$ is a Hopf algebra by $\mu_{G/T}$ for $G = U, Sp$. Consider the following commutative diagram, where $\bar{\mu}$ is the natural inclusion.

$$\begin{array}{ccccc} U/T \times U/T & \longrightarrow & SO/T \times SO/T & \longrightarrow & SO/U \times SO/U \\ {\scriptstyle \mu_{U/T}} \downarrow & & \downarrow {\scriptstyle \mu_{SO/T}} & & {\scriptstyle \bar{\mu}} \downarrow \\ U/T & \longrightarrow & SO/T & \longrightarrow & SO/U \end{array}$$

Since SO/U is a Hopf space with the multiplication $\bar{\mu}$ and Proposition 3.3 holds, we see that $H^*(H^*(SO/T; \mathbf{Z}_2); Sq^2)$ is a Hopf algebra by $\mu_{SO/T}$.

Proposition 4.1. *$H^*(H^*(G/T; \mathbf{Z}_2); Sq^2)$ is a Hopf algebra by $\mu_{G/T}$ for $G = U, Sp, SO$.*

Lemma 4.1. *$E_r(G/T)$ collapses at $r = 3$ for $G = U, Sp, SO$.*

Proof. Let $d_r : E_r(U/T) \to E_r(U/T)$ be the first non-trivial differential for $r \geq 3$, then we have $r \equiv 2$ (8) by Proposition 2.1,*2,(a)*. There exists $x \in E_r^{p,0}(U/T)$ such that $\iota(\alpha x)$ is indecomposable, $\iota(d_r x)$ is primitive and $\alpha x \neq 0$, $\alpha d_r x \neq 0$ by Proposition 2.1,*2,(c)* and 4.1, where ι is as in Proposition 2.1,*1*. By [MM, Proposition 4.23] and Proposition 3.1, $\iota(\alpha x)$ and $\iota(d_r x)$ have degree $\equiv -2$ (8). Then we have $r \equiv |\iota(d_r x)| - |\iota(\alpha x)| \equiv 0$ (8) and this contradicts to $r \equiv 2$ (8). By the same way we see that $E_r(Sp/T)$ and $E_r(SO/T)$ collapse at $r = 3$. □

Consider the homomorphism $E_r(G/T) \to E_r(G(n)/T)$ induced from the natural inclusion

$$G(n)/T \to G/T,$$

for $G = U, Sp, SO$, then we obtain the following for $r \geq 3$ by Proposition 3.1, 3.2, 3.3 and Lemma 4.1, where we identify $H^*(H^*(G(n)/T; \mathbf{Z}_2); Sq^2)$ with $E_3^{*,-1}(G(n)/T)$ by Proposition 2.1,*1*.

Proposition 4.2. *We have the following for $r \geq 3$.*

$$\begin{array}{ll} d_r y_{8k-2} = 0 & y_{8k-2} \in E_r^{*,-1}(U(n)/T) \\ d_r y_{4k-2} = 0 & y_{4k-2} \in E_r^{*,-1}(Sp(n)/T) \\ d_r y_{8k-2} = d_r e'_{8k-2} = 0 & y_{8k-2}, e'_{8k-2} \in E_r^{*,-1}(SO(n)/T) \end{array}$$

Proposition 4.3. *We have the following for $r \geq 3$.*

$$\begin{array}{ll} d_r e_{4n+2} = 0 & e_{4n+2} \in E_r^{*,-1}(SO(4n+3)/T) \\ d_r e_{4n-2} = 0 & e_{4n-2} \in E_r^{*,-1}(SO(4n)/T) \end{array}$$

Proof. Consider the following projection.

$$p : SO(4n+3)/T \to SO(4n+3)/SO(4n+2) = S^{4n+2}$$

Then we have $p^*(s) = e_{4n+2} \in H^*(SO(4n+3)/T; \mathbf{Z}_2)$, where s is a generator of $H^{4n+2}(S^{4n+2}; \mathbf{Z}_2) \cong \mathbf{Z}_2$. It is easily seen that

$$E_3^{*,-1}(S^{4n+2}) \cong H^*(H^*(S^{4n+2}; \mathbf{Z}_2); Sq^2) \cong \bigwedge([s]).$$

Since $d_r([s]) = 0$ $(r \geq 3)$, we have $d_r e_{4n+2} = 0$ $(r \geq 3)$ for $e_{4n+2} \in E_r^{*,-1}(SO(4n+3)/T)$.

Since it is shown in [HK2, Lemma 2.2] that $d_r e_{4n-2} = 0$ $(r \geq 3)$ for $e_{4n-2} \in E_r^{*,-1}(SO(4n)\ /U(2n))$, we have $d_r e_{4n-2} = 0$ $(r \geq 3)$ for $e_{4n-2} \in E_r^{*,-1}(SO(4n)/T)$ by considering the homomorphism $E_r(SO(4n)/U(2n)) \to E_r(SO(4n)/T)$ induced from the projection $SO(4n)/T \to SO(4n)/U(2n)$. $\square$

Proposition 4.4. *We have the following for $r \geq 3$.*

$$\begin{array}{ll} d_r z = 0 & z \in E_r(SO(4n+\epsilon)/T) \ \ (\epsilon = 0, 1) \\ d_r z = 0 & z \in E_r(U(2n)/T) \end{array}$$

Proof. It is shown in [HK2, (2-6) and Theorem 2.5] that $E_r(SO(4n+\epsilon)/SO(2) \times SO(4n+\epsilon-2))$ collapses at $r = 3$ and

$$E_3^{*,-1}(SO(4n)/SO(2) \times SO(4n+\epsilon-2)) \cong \begin{cases} \bigwedge([t^{2n-1}], s_{4n-2}) & \epsilon = 0 \\ \bigwedge([t^{2n-1}]) & \epsilon = 1, \end{cases}$$

where $t = i^*(s \otimes 1) \in H^2(SO(4n+\epsilon)/SO(2) \times SO(4n+\epsilon-2); \mathbf{Z}_2)$, s is a generator of $H^2(BSO(2); \mathbf{Z}_2) \cong \mathbf{Z}_2$ and the map i is as in the following commutative diagram.

$$\begin{array}{ccc} SO(4n+\epsilon)/T & \longrightarrow & BT \\ {\scriptstyle p}\downarrow & & \downarrow \\ SO(4n+\epsilon)/SO(2) \times SO(4n+\epsilon-2) & \underset{i}{\longrightarrow} & BSO(2) \times BSO(4n+\epsilon-2) \end{array}$$

Then we have $p^*(t) = x_1 \in H^*(SO(4n+\epsilon)/T; \mathbf{Z}_2)$ and $p^*([t^{2n-1}]) = z \in E_3^{*,-1}(SO(4n+\epsilon)/T)$ by Proposition 3.3. Since $d_r([t^{2n-1}]) = 0$ $(r \geq 3)$, we have $d_r z = 0$ $(r \geq 3)$.

Consider the homomorphism $j^* : E_r(SO(4n+\epsilon)/T) \to E_r(U(2n)/T)$ induced from the following inclusion.

$$j : U(2n)/T \to SO(4n+\epsilon)/T$$

Then we have $j^*(z) = z \in E_3^{*,-1}(U(2n)/T)$ for $z \in E_3^{*,-1}(SO(4n+\epsilon)/T)$. Since $d_r z = 0$ $(r \geq 3)$ for $z \in E_r(SO(4n+\epsilon)/T)$, $d_r z = 0$ $(r \geq 3)$ for $z \in E_r(U(2n)/T)$. $\square$

By Proposition 4.2, 4.3 and 4.4, we have the following.

Lemma 4.2. *$E_r(G(n)/T)$ collapses at $r = 3$ for $G = U, Sp, SO$.*

Proof of Theorem. Let X be a finite CW-complex such that $H^*(X;\mathbf{Z})$ is torsion free and concentrated in even dimension. Consider the Bott sequence

$$\cdots \to K^n(X) \to KO^{n+2}(X) \to KO^{n+1}(X) \xrightarrow{\mathbf{c}} K^{n+1}(X) \to \cdots ,$$

where $\mathbf{c} : KO^i(X) \to K^i(X)$ is the complexification map. Since $\mathbf{rc} = 2$ for the realization map $\mathbf{r} : K^i(X) \to KO^i(X)$ and $K^i(X)$ is torsion free and concentrated in even dimension, we have the following. ([H])

$$\begin{aligned} KO^{2i+1}(X) &\cong s\mathbf{Z}_2 \\ KO^{2i}(X) \quad &\cong r\mathbf{Z} \oplus s\mathbf{Z}_2, \end{aligned}$$

$$\begin{aligned} \operatorname{rank} KO^0(X) \quad &= \operatorname{rank} KO^{-4}(X) = \sum_i \operatorname{rank} H^{4i}(X;\mathbf{Z}) \\ \operatorname{rank} KO^{-2}(X) &= \operatorname{rank} KO^{-6}(X) = \sum_i \operatorname{rank} H^{4i+2}(X;\mathbf{Z}) \end{aligned}$$

Hence the extension of $\bigoplus_{p+q=2i-1} E_\infty^{p,q}(X) \cong \bigoplus_k E_\infty^{8k+2i,-1}$ to $KO^{2i-1}(X)$ is trivial.

It is well known that the Poincaré series of $G(n)/T$ is as follows. ([KI])

$$P_t(G(n)/T) = \begin{cases} \dfrac{(1-t^2)\cdots(1-t^{2n})}{(1-t^2)\cdots(1-t^2)} & G = U \\ \dfrac{(1-t^4)\cdots(1-t^{4n})}{(1-t^2)\cdots(1-t^2)} & G = Sp \\ \dfrac{1}{1+t^{2m}} \cdot \dfrac{(1-t^4)\cdots(1-t^{4m})}{(1-t^2)\cdots(1-t^2)} & G = SO, n = 2m \\ \dfrac{(1-t^4)\cdots(1-t^{4m})}{(1-t^2)\cdots(1-t^2)} & G = SO, n = 2m+1 \end{cases}$$

By substituting $t = 1, \sqrt{-1}$ with $P_t(G(n)/T)$ we have the following.

$$\sum_i \operatorname{rank} H^{4i}(X;\mathbf{Z}) = \sum_i \operatorname{rank} H^{4i+2}(X;\mathbf{Z}) = \begin{cases} n!/2 & G = U \\ 2^{n-1}n! & G = Sp \\ 2^{m-2}m! & G = SO, n = 2m \\ 2^{m-1}m! & G = SO, n = 2m+1 \end{cases}$$

By Proposition 2.1,*1* and Lemma 4.2, we see that $E_\infty^{*,-1}(G(n)/T) \cong E_3^{*,-1}(G(n)/T)$ $\cong H^*(H^*(G(n)/T;\mathbf{Z}_2); Sq^2)$. Then the Poincaré series of $E_\infty^{*,-1}(G(n)/T)$ are as follows by Proposition 3.1, 3.2 and 3.3, where degrees are taken by $*$.

$P_t(E_\infty^{*,-1}(G(n)/T))$

$$= \begin{cases} (1+t^6)\cdots(1+t^{8m-10})(1+t^{4m-2}) & G=U, n=2m \\ (1+t^6)\cdots(1+t^{8m-2}) & G=U, n=2m+1 \\ (1+t^2)\cdots(1+t^{4n-2}) & G=Sp \\ ((1+t^6)\cdots(1+t^{8m-10})(1+t^{4m-2}))^2 & G=SO, n=4m \\ ((1+t^6)\cdots(1+t^{8m-10}))^2(1+t^{4m-2})(1+t^{8m-2}) & G=SO, n=4m+1 \\ ((1+t^6)\cdots(1+t^{8m-2}))^2 & G=SO, n=4m+2 \\ ((1+t^6)\cdots(1+t^{8m-2}))^2(1+t^{4m-2}) & G=SO, n=4m+3 \end{cases}$$

By substituting $t = 1, \sqrt{-1}, e^{\sqrt{-1}\pi/4}$ with the Poincaré series above we completes the proof. □

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