KO\text{-}theory of flag manifolds

Daisuke KISHIMOTO, Akira KONO and Akihiro OHSHITA

1 Introduction

The purpose of this paper is to determine the $KO^*$-groups of flag manifolds which are the homogeneous spaces $G(n)/T$ for $G = U, Sp, SO$ and $T$ is the maximal torus of $G(n)$. We compute it by making use of the Atiyah-Hirzebruch spectral sequence and obtain the following.

Theorem. The $KO^i$-groups of $G(n)/T$ for $G = U, Sp, SO$ are as follows, where $s = n!/2, 2^{n-1}n!$ for $G = U, Sp$ and $s = 2^{m-2}m!, 2^{m-1}m!$ for $G = SO$ and $n = 2m, 2m + 1$ respectively.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$U(n)/T$</th>
<th>$Sp(n)/T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$sZ \oplus t_1Z_2$</td>
<td>$2^{2k-2} + 2^{k-1}$</td>
</tr>
<tr>
<td>-1</td>
<td>$t_0Z_2$</td>
<td>$2^{2k-1} - 2^{k-1}$</td>
</tr>
<tr>
<td>-2</td>
<td>$sZ \oplus t_1Z_2$</td>
<td>$2^{4k-1} - (-1)^k2^{2k-1}$</td>
</tr>
<tr>
<td>-3</td>
<td>$t_2Z_2$</td>
<td>$2^{4k-1} - (-1)^k2^{2k-1}$</td>
</tr>
<tr>
<td>-4</td>
<td>$sZ \oplus t_1Z_2$</td>
<td>$2^{4k-1} - (-1)^k2^{2k-1}$</td>
</tr>
<tr>
<td>-5</td>
<td>$t_2Z_2$</td>
<td>$2^{4k-1} - (-1)^k2^{2k-1}$</td>
</tr>
<tr>
<td>-6</td>
<td>$sZ \oplus t_1Z_2$</td>
<td>$2^{4k-1} - (-1)^k2^{2k-1}$</td>
</tr>
<tr>
<td>-7</td>
<td>$t_2Z_2$</td>
<td>$2^{4k-1} - (-1)^k2^{2k-1}$</td>
</tr>
</tbody>
</table>
$SO(n)/T$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$l_0$</th>
<th>$l_1$</th>
<th>$l_2$</th>
<th>$l_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$8k$</td>
<td>$2^{4k-1}$</td>
<td>$2^{4k-1} - (-1)^k 2^{2k-1}$</td>
<td>$2^{4k-1}$</td>
<td>$2^{4k-1} + (-1)^k 2^{2k-1}$</td>
</tr>
<tr>
<td>$8k + 1$</td>
<td>$2^{4k-1}$</td>
<td>$2^{4k-1} - (-1)^k 2^{2k}$</td>
<td>$2^{4k-1}$</td>
<td>$2^{4k-1} + (-1)^k 2^{2k}$</td>
</tr>
<tr>
<td>$8k + 2$</td>
<td>$2^{4k-1} + (-1)^k 2^{2k-1}$</td>
<td>$2^{4k-1}$</td>
<td>$2^{4k-1} - (-1)^k 2^{2k-1}$</td>
<td>$2^{4k-1}$</td>
</tr>
<tr>
<td>$8k + 3$</td>
<td>$2^{4k} + (-1)^k 2^{2k-1}$</td>
<td>$2^{4k} - (-1)^k 2^{2k-1}$</td>
<td>$2^{4k} - (-1)^k 2^{2k-1}$</td>
<td>$2^{4k} + (-1)^k 2^{2k-1}$</td>
</tr>
<tr>
<td>$8k + 4$</td>
<td>$2^{4k+1} + (-1)^k 2^{2k}$</td>
<td>$2^{4k+1}$</td>
<td>$2^{4k+1} - (-1)^k 2^{2k}$</td>
<td>$2^{4k+1}$</td>
</tr>
<tr>
<td>$8k + 5$</td>
<td>$2^{4k+1}$</td>
<td>$2^{4k+1} - (-1)^k 2^{2k+1}$</td>
<td>$2^{4k+1}$</td>
<td>$2^{4k+1} + (-1)^k 2^{2k+1}$</td>
</tr>
<tr>
<td>$8k + 6$</td>
<td>$2^{4k+1}$</td>
<td>$2^{4k+1} - (-1)^k 2^{2k}$</td>
<td>$2^{4k+1}$</td>
<td>$2^{4k+1} + (-1)^k 2^{2k}$</td>
</tr>
<tr>
<td>$8k + 7$</td>
<td>$2^{4k+2} + (-1)^k 2^{2k}$</td>
<td>$2^{4k+2} - (-1)^k 2^{2k}$</td>
<td>$2^{4k+2} - (-1)^k 2^{2k}$</td>
<td>$2^{4k+2} + (-1)^k 2^{2k}$</td>
</tr>
</tbody>
</table>

2. The Atiyah-Hirzebruch spectral sequence

First we recall that the coefficient ring of $KO$-theory is that

$$KO^* = \mathbb{Z}[\alpha, x, \beta, \beta^{-1}]/(2\alpha, \alpha^3, ax^2 - 4\beta),$$

where $|\alpha| = 1$, $|x| = 4$ and $|\beta| = 8$.

Let $X$ be a finite CW-complex. The Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ is the spectral sequence with $E_2^{p, q} \cong H^p(X; KO^q)$ converging to $KO^*(X)$. It is well known that the differential $d_2$ of the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ is given by the following. (See [F])

$$d_2^{p,q} = \begin{cases} Sq^2 \pi_2 & q \equiv 0 \ (8) \\ Sq^2 & q \equiv -1 \ (8) \\ 0 & \text{otherwise,} \end{cases}$$

where $\pi_2$ is the modulo 2 reduction.

It is well known that $G/T$ is a CW-complex with only even cells, where $G$ is a compact connected Lie group and $T$ is the maximal torus of $G$. ([B]) The next proposition, given in [HK1,2], is concerned with the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ for the special $X$ which can be $G/T$.

**Proposition 2.1.** Let $X$ be a CW-complex whose cohomology is torsion free and concentrated in even dimension, and $E_r^*(X)$ be the $r$-th term of the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$. Then we have the following.

1. $r : E_3^{p,q}(X) \cong H^p(H^*(X; \mathbb{Z}_2); Sq^2)$ for $q \equiv -1 \ (8)$

2. Let $d_r$ be the first non-trivial differential. ($r \geq 3$)

   (a) $r \equiv 2 \ (8)$.

   (b) There exists $x \in E_3^{p,0}(X)$ such that $ax \neq 0$ and $axd_r x \neq 0$.

   (c) If $X$ admits a map $\mu : X \times X \to X$ which makes $H^*(H^*(X; \mathbb{Z}_2); Sq^2)$ to be a Hopf algebra, then $r(ax)$ is indecomposable and $r(d_r x)$ is primitive for the least $p$ and $x \in E_r^{p,0}(X)$ in (b).
3 The $Sq^2$-cohomology of flag manifolds

Recall that the cohomology of the flag manifold $U(n)/T$ is

$$H^*(U(n)/T; \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n]/(c_1, \ldots, c_n),$$

where $|x_i| = 2$ and $c_j$ is the $j$-th elementary symmetric function in $x_1, \ldots, x_n$.

We determine the $Sq^2$-cohomology of $U(n)/T$ by the similar way of [HK1, Proposition 2].

Proposition 3.1.

$$H^*(H^*(U(n)/T; \mathbb{Z}_2); Sq^2) \cong \begin{cases} \bigwedge(y_6, y_{14}, \ldots, y_{8m-2}) & n = 2m + 1 \\
\bigwedge(y_6, y_{14}, \ldots, y_{8m-10}, z) & n = 2m, \end{cases}$$

where $y_{8k-2}$ and $z$ are represented by $\sum_{i_1 < \cdots < i_{2k}} x_{i_1} x_{i_2}^2 \cdots x_{i_{2k}}^2$ and $x_1^{n-1}$ respectively.

Proof. Let $R$ be a differential graded algebra $(\mathbb{Z}_2[x_1, \ldots, x_n], d)$ with $|x_i| = 2$ and $dx_i = x_i^2$, and $c_j$ be the $j$-th elementary symmetric function in $x_1, \ldots, x_n$.

Then we have

$$dc_i = c_{2i+1} + c_1 c_{2i}, \quad dc_{2i+1} = c_{2i} c_{2i+1},$$

where $c_j = 0$ for $j > n$.

Let $R_1$ be the graded differential algebra $R_1 = R/(c_1)$ with the differential induced from $R$. We construct the differential graded algebra $R_k$ ($k \leq n$) inductively by the following short exact sequences.

$$0 \to R_{2k-1} \xrightarrow{c_{2k+1}} R_{2k-1} \to R_{2k} \to 0 \quad (2k < n)$$
$$0 \to R_{2k} \xrightarrow{c_{2k}} R_{2k} \to R_{2k+1} \to 0 \quad (2k + 1 \leq n)$$
$$0 \to R_{n-1} \xrightarrow{c_n} R_{n-1} \to R_n \to 0 \quad (n \text{ is even})$$

It is obvious that $R_n \cong (H^*(U(n)/T; \mathbb{Z}_2), Sq^2)$ as a differential graded algebra.

We have the following long exact sequences.

$$\cdots \to H^i(R_{2k-1}) \xrightarrow{H(c_{2k+1})} H^{i+4k+2}(R_{2k-1}) \to H^{i+4k}(R_{2k}) \xrightarrow{\delta} H^{i+2}(R_{2k-1}) \to \cdots \quad (2k < n)$$
$$\cdots \to H^i(R_{2k}) \xrightarrow{H(c_{2k})} H^{i+4k}(R_{2k}) \to H^{i+4k}(R_{2k+1}) \xrightarrow{\delta} H^{i+2}(R_{2k}) \to \cdots \quad (2k + 1 \leq n)$$

Inductively we obtain

$$H^*(R_{2k}) \cong \bigwedge(y_6, y_{14}, \ldots, y_{8k-10}, c_{2k})$$
$$H^*(R_{2k+1}) \cong \bigwedge(y_6, y_{14}, \ldots, y_{8k-2}), \quad \delta y_{8k-2} = c_{2k}, \quad (2k + 1 \leq n).$$

3
Then $y_{8k-2}$ is represented by
$$
\sum_{i_1 < \ldots < i_k} x_{i_1}^2 x_{i_2}^2 \cdots x_{i_k}^2
$$
and this completes the case that $n$ is odd.

When $n$ is even we have the following exact sequence.
$$
\cdots \rightarrow H^i(R_{n-1}) \xrightarrow{H(c_n)} H^{i+2n}(R_{n-1}) \rightarrow H^{i+2n}(R_n) \xrightarrow{\delta} H^{i+2}(R_{n-1}) \rightarrow \cdots
$$
Then we have
$$
H^*(R_n) \cong \bigwedge(y_6, y_{14}, \ldots, y_{8n-10}, z), \quad \delta z = 1, \quad (n = 2m)
$$
Therefore $z$ is represented by $x_2 x_3 \cdots x_n = x_1^{n-1} \in R_n$ and this completes the proof.

It is well known that
$$
H^*(Sp(n)/T; \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n]/(c_1^2, \ldots, c_n^2),
$$
where $|x_i| = 2$ and $c_j$ is the $j$-th elementary symmetric function in $x_1, \ldots, x_n$.

**Proposition 3.2.**

$$
H^*(H^*(Sp(n)/T; \mathbb{Z}); Sq^2) \cong \bigwedge(y_2, y_6, \ldots, y_{4n-2}),
$$
where $y_{4k-2}$ is represented by $\sum_{i_1 < \ldots < i_k} x_{i_1}^2 x_{i_2}^2 \cdots x_{i_k}^2$.

**Proof.** Let $R_0$ be the differential graded algebra $\mathbb{Z}_2[x_1, \ldots, x_n]$ with $dx_i = x_1^2$. We construct the differential graded algebra $R_k$ for $k \leq n$ inductively by the following exact sequence.
$$
0 \rightarrow R_k \xrightarrow{c_{k+1}^2} R_k \rightarrow R_{k+1} \rightarrow 0
$$
It is obvious that $R_n$ is isomorphic to $(H^*(Sp(n)/T; \mathbb{Z}_2), Sq^2)$ as differential graded algebras. We have the following exact sequence.
$$
\cdots \rightarrow H^i(R_{k-1}) \xrightarrow{H(c_{k+1}^2)} H^{i+4k}(R_{k-1}) \rightarrow H^{i+4k}(R_k) \xrightarrow{\delta} H^{i+2}(R_{k-1}) \rightarrow \cdots
$$
Then we obtain inductively
$$
H^*(R_k) \cong \bigwedge(y_2, y_6, \ldots, y_{4k-2}), \quad \delta y_{4k-2} = 1.
$$
Therefore $y_{4k-2}$ is represented by $\sum_{i_1 < \ldots < i_k} x_{i_1}^2 x_{i_2}^2 \cdots x_{i_k}^2$ and this completes the proof.
It is known that

\[ H^*(SO(2n + \epsilon)/U(n); \mathbb{Z}_2) \cong \Delta(e_2, e_4, \ldots, e_{2(\alpha + \epsilon + 1)}), \quad e_2 = e_{4i}. \]

where \( \epsilon = 0, 1, |e_i| = i, e_i = 0 \) for \( i > 2(n + \epsilon - 1) \) and \( \Delta(e_2, \ldots) \) is the \( \mathbb{Z}_2 \)-algebra whose \( \mathbb{Z}_2 \)-module basis are \( e_i \cdots e_k \) \( (i_1 < \ldots < x_{\epsilon}) \). ([KI],[T]) We see the following by making use of the fibration \( U(n)/T \xrightarrow{\rho} SO(2n + \epsilon)/T \xrightarrow{\rho} SO(2n + \epsilon)/U(n). \)

\[ H^*(SO(2n + \epsilon)/T; \mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, \ldots, x_n]/(e_2, e_4, \ldots, e_{2(\alpha + \epsilon + 1)}), \]

where \( S q^2 e_{8i-2} = e_{4i}, \quad j^*(x_i) = x_i \in H^2(U(n)/T; \mathbb{Z}_2) \) and \( p^*(e_1) = e_i \in H^1(SO(2n + \epsilon)/T; \mathbb{Z}_2). \) ([KI],[T])

**Proposition 3.3.**

\[ H^*(H^*(SO(2n + \epsilon)/T; \mathbb{Z}_2); S q^2) \]

\[ \cong \begin{cases} \bigwedge (y_6, y_{14} \cdots y_{8m-10}, z) \otimes \bigwedge (e'_{0}, e'_{14}, \ldots, e'_{8m-10}, [e_{8m-2}]) & \epsilon = 0, n = 2m \\ \bigwedge (y_6, y_{14} \cdots y_{8m-10}, z) \otimes \bigwedge (e'_{0}, e'_{14}, \ldots, e'_{8m-2}) & \epsilon = 1, n = 2m \\ \bigwedge (y_6, y_{14} \cdots y_{8m-2}) \otimes \bigwedge (e'_{0}, e'_{14}, \ldots, e'_{8m-2}, [e_{8m+2}]) & \epsilon = 1, n = 2m + 1 \end{cases} \]

where \( y_{8k-2}, z, e'_{8k-2} \) are represented by \( \sum_{i_1 \cdots i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, x_{i_2}^{2i-1}, e_{4k-2}^1 e_{4k} + e_{8k-2} \) respectively.

**Proof.** We have the following isomorphism as differential graded algebras with the differential \( S q^2. \)

\[ H^*(SO(2n + \epsilon)/U(n); \mathbb{Z}_2) \cong H^*(U(n)/T; \mathbb{Z}_2) \otimes H^*(SO(2n + \epsilon)/U(n); \mathbb{Z}_2) \]

By Proposition 3.1, we obtain \( H^*(H^*(U(n)/T; \mathbb{Z}_2); S q^2). \) Then we compute \( H^*(H^*(SO(2n + \epsilon)/U(n); \mathbb{Z}_2); S q^2). \)

Let \( M_i \) be the following module, where \( e'_{8i-2} = e_{4i-2} e_{4i} + e_{8i-2}. \)

\[ M_i = \mathbb{Z}_2(1, e_{4i-2}, e_{4i}, e'_{8i-2}) \]

Then we see that \( M_i \) is the differential graded submodule of \( H^*(SO(2n + \epsilon)/U(n); \mathbb{Z}_2) \) with the differential \( S q^2. \) We have the following isomorphisms as differential graded modules with the differential \( S q^2. \)

\[ H^*(SO(2n+\epsilon)/U(n); \mathbb{Z}_2) \cong \begin{cases} M_1 \otimes \cdots \otimes M_{m-1} \otimes \bigwedge (e_{4m-2}) & \epsilon = 0, n = 2m \\ M_1 \otimes \cdots \otimes M_m & \epsilon = 1, n = 2m \\ M_1 \otimes \cdots \otimes M_m \otimes \bigwedge (e_{4m+2}) & \epsilon = 1, n = 2m + 1 \end{cases} \]

Since \( H^*(M_i; S q^2) \cong \mathbb{Z}_2(1, |e'_{8i-2}|) \) and \( e'_{8i-2}^2 = S q^2(e_{8i-2} e_{8i} + e_{16i-6}), \) the proof is completed. \[ \Box \]
4 Proof of Theorem

Let $BT^n$ be the classifying space of an $n$-torus and $\mu_n : BT^n \times BT^n \rightarrow BT^{2n}$ be the identity. We can set $H^*(BT^n; \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_{2n}]$, $H^*(BT^n \times BT^n; \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n] \otimes \mathbb{Z}[x_{n+1}, \ldots, x_{2n}]$ and

$$\mu_n(x_i) = \begin{cases} x_i \otimes 1 & i \leq n \\ 1 \otimes x_i & i > n. \end{cases}$$

Then we have the following.

$$\mu_n^* \left( \sum_{i_1 < \ldots < i_k \leq 2n} x_{i_1} x_{i_2}^2 \cdots x_{i_k}^2 \right) = \sum_{i_1 < \ldots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} \otimes 1 + \sum_{i_1 < \ldots < i_{k-1} \leq n < i_k} x_{i_1} x_{i_2} \cdots x_{i_{k-1}} \otimes x_{i_k}^2 + \ldots + \sum_{i_1 < \ldots < i_{k-2} < n < i_{k-1} < i_k} x_{i_1} x_{i_2} \cdots x_{i_{k-1}} \otimes x_{i_k}^2 + \ldots + \sum_{i_1 < \ldots < i_{k-2} < n < i_{k-1} < i_k} x_{i_1} x_{i_2} \cdots x_{i_{k-1}} \otimes x_{i_k}^2 + \ldots + \sum_{i_1 < \ldots < i_{k-2} < n < i_{k-1} < i_k} x_{i_1} x_{i_2} \cdots x_{i_{k-1}} \otimes c_i^2 + \ldots + \sum_{i_1 < \ldots < i_{k-1} < n \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_{k-1}} \otimes c_i^2 + \ldots + \sum_{i_1 < \ldots < i_{k-1} < n \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_{k-1}} \otimes c_i^2,$$

where $c_i$ is the $i$-th elementary symmetric function in $x_{n+1}, \ldots, x_{2n}$. Then we have the following for $y_k = \sum_{i_1 < \ldots < i_k} x_{i_1} x_{i_2}^2 \cdots x_{i_k}^2 \in H^*(BT^n; \mathbb{Z})$.

$$\mu_n^*(y_k) = y_k \otimes 1 + 1 \otimes y_k + \sum_{i=1}^{k-1} y_{k-i} \otimes c_i^2 \quad (\ast)$$

Let $\mu_{G/T} : G/T \times G/T \rightarrow G/T$ be the natural inclusion for $G = U, Sp, SO$, then we have the following commutative diagram.

$$\begin{array}{ccc}
G/T \times G/T & \longrightarrow & BT \times BT \\
\mu_{G/T} \downarrow & & \downarrow \mu_n \\
G/T & \longrightarrow & BT
\end{array}$$

Note Proposition 3.1, 3.2 and (\ast), then we see that $H^*(H^*(G/T; \mathbb{Z}); Sq^2)$ is a Hopf algebra by $\mu_{G/T}$ for $G = U, Sp$. Consider the following commutative
diagram, where $\bar{\mu}$ is the natural inclusion.
\[
\begin{array}{ccc}
    U/T \times U/T & \longrightarrow & SO/T \times SO/T \\
    \downarrow & & \downarrow \mu_{SO/T} \\
    U/T & \longrightarrow & SO/T
\end{array}
\]
Since $SO/U$ is a Hopf space with the multiplication $\bar{\mu}$ and Proposition 3.3 holds, we see that $H^*(H^*(SO/T; \mathbb{Z}_2); Sq^2)$ is a Hopf algebra by $\mu_{SO/T}$.

**Proposition 4.1.** $H^*(H^*(G/T; \mathbb{Z}_2); Sq^2)$ is a Hopf algebra by $\mu_{G/T}$ for $G = U, Sp, SO$.

**Lemma 4.1.** $E_r(G/T)$ collapses at $r = 3$ for $G = U, Sp, SO$.

**Proof.** Let $d_r : E_r(U/T) \to E_r(U/T)$ be the first non-trivial differential for $r \geq 3$, then we have $r \equiv 2 \pmod{8}$ by Proposition 2.1, 2.2, and 2.3. There exists $x \in E_r^{p,0}(U/T)$ such that $\iota(ax)$ is indecomposable, $\iota(dx)$ is primitive and $ax \neq 0$, $ax = 0$ by Proposition 2.1, 2.2, and 4.1, where $\iota$ is as in Proposition 2.1. By [MM, Proposition 4.23] and Proposition 3.1, $\iota(ax)$ and $\iota(dx)$ have degree $\equiv -2 \pmod{8}$. Then we have $r \equiv \iota(x) - \iota(ax) = 0 \pmod{8}$ and this contradicts to $r \equiv 2 \pmod{8}$. By the same way we see that $E_r(Sp/T)$ and $E_r(SO/T)$ collapse at $r = 3$. \qed

Consider the homomorphism $E_r(G/T) \to E_r(G(n)/T)$ induced from the natural inclusion

$G(n)/T \to G/T,$

for $G = U, Sp, SO$, then we obtain the following for $r \geq 3$ by Proposition 3.1, 3.2, 3.3 and Lemma 4.1, where we identify $H^*(H^*(G(n)/T; \mathbb{Z}_2); Sq^2)$ with $E_r^{*,-1}(G(n)/T)$ by Proposition 2.1.

**Proposition 4.2.** We have the following for $r \geq 3$.

\[
\begin{align*}
d_r y_{sk-2} &= 0 & y_{sk-2} &\in E_r^{*,-1}(U(n)/T) \\
d_r y_{sk-2} &= 0 & y_{sk-2} &\in E_r^{*,-1}(Sp(n)/T) \\
d_r y_{sk-2} = d_r' y_{sk-2} &= 0 & y_{sk-2}, y_{sk-2}' &\in E_r^{*,-1}(SO(n)/T)
\end{align*}
\]

**Proposition 4.3.** We have the following for $r \geq 3$.

\[
\begin{align*}
d_r e_{4n+2} &= 0 & e_{4n+2} &\in E_r^{*,-1}(SO(4n+3)/T) \\
d_r e_{4n-2} &= 0 & e_{4n-2} &\in E_r^{*,-1}(SO(4n)/T)
\end{align*}
\]

**Proof.** Consider the following projection.

$p : SO(4n+3)/T \to SO(4n+3)/SO(4n+2) = S^{4n+2}$

Then we have $p^*(s) = e_{4n+2} \in H^*(SO(4n+3)/T; \mathbb{Z}_2)$, where $s$ is a generator of $H^*(S^{4n+2}; \mathbb{Z}_2) \cong \mathbb{Z}_2$. It is easily seen that

$E_r^{*-1}(S^{4n+2}) \cong H^*(H^*(S^{4n+2}; \mathbb{Z}_2); Sq^2) \cong \bigwedge ([s]).$
Lemma 4.2. \( e_{4n+2} = 0 \) for \( n \geq 3 \) for \( e_{4n+2} \in E_{r}^{n-1}(SO(4n+3)/T) \).

Since it is shown in [HK2, Lemma 2.2] that \( d_r e_{4n-2} = 0 \) for \( r \geq 3 \) for \( e_{4n-2} \in E_{r}^{n-1}(SO(4n)/U(2n)) \), we have \( d_r e_{4n-2} = 0 \) for \( r \geq 3 \) for \( e_{4n-2} \in E_{r}^{n-1}(SO(4n)/T) \) by considering the homomorphism \( E_r(SO(4n)/U(2n)) \rightarrow E_r(SO(4n)/T) \) induced from the projection \( SO(4n)/T \rightarrow SO(4n)/U(2n) \).

Proposition 4.4. We have the following for \( r \geq 3 \).

\[
\begin{align*}
d_r z &= 0 & z &= E_r(SO(4n+\epsilon)/T) \\
d_r z &= 0 & z &= E_r(U(2n)/T)
\end{align*}
\]

Proof. It is shown in [HK2, (2-6) and Theorem 2.5] that \( E_r(SO(4n+\epsilon)/SO(2) \times SO(4n+\epsilon - 2)) \) collapses at \( r = 3 \) and

\[
E_r^{3-1}(SO(4n)/SO(2) \times SO(4n+\epsilon - 2)) \cong \begin{cases} 
\Lambda([t^{2n-1}], s_{4n-2}) & \epsilon = 0 \\
\Lambda([t^{2n-1}]) & \epsilon = 1,
\end{cases}
\]

where \( t = i^*(s \otimes 1) \in H^2(SO(4n+\epsilon)/SO(2) \times SO(4n+\epsilon - 2); \mathbb{Z}_2) \), \( s \) is a generator of \( H^2(BSO(2); \mathbb{Z}_2) \cong \mathbb{Z}_2 \) and the map \( i \) is as in the following commutative diagram.

\[
\begin{array}{ccc}
SO(4n+\epsilon)/T & \longrightarrow & BT \\
\downarrow & & \downarrow \\
SO(4n+\epsilon)/SO(2) \times SO(4n+\epsilon - 2) & \longrightarrow & BSO(2) \times BSO(4n+\epsilon - 2)
\end{array}
\]

Then we have \( p^*(t) = x_1 \in H^*(SO(4n+\epsilon)/T; \mathbb{Z}_2) \) and \( p^*([t^{2n-1}]) = z \in E_r^{3-1}(SO(4n+\epsilon)/T) \) by Proposition 3.3. Since \( d_r([t^{2n-1}]) = 0 \) for \( r \geq 3 \), we have \( d_r z = 0 \) for \( r \geq 3 \).

Consider the homomorphism \( j^*: E_r(SO(4n+\epsilon)/T) \rightarrow E_r(U(2n)/T) \) induced from the following inclusion.

\[
j: U(2n)/T \rightarrow SO(4n+\epsilon)/T
\]

Then we have \( j^*(z) = z \in E_r^{3-1}(U(2n)/T) \) for \( z \in E_r^{3-1}(SO(4n+\epsilon)/T) \).

Since \( d_r z = 0 \) for \( r \geq 3 \) for \( z \in E_r(SO(4n+\epsilon)/T) \), \( d_r z = 0 \) for \( r \geq 3 \) for \( z \in E_r(U(2n)/T) \).

By Proposition 4.2, 4.3 and 4.4, we have the following.

Lemma 4.2. \( E_r(G^n)/T \) collapses at \( r = 3 \) for \( G = U, Sp, SO \).
Proof of Theorem. Let $X$ be a finite CW-complex such that $H^*(X;\mathbb{Z})$ is torsion free and concentrated in even dimension. Consider the Bott sequence

$$\cdots \to K^n(X) \to KO^{n+2}(X) \to KO^{n+1}(X) \xrightarrow{c} K^{n+1}(X) \to \cdots,$$

where $c: KO^i(X) \to K^i(X)$ is the complexification map. Since $rc = 2$ for the realization map $r: K^i(X) \to KO^i(X)$ and $K^i(X)$ is torsion free and concentrated in even dimension, we have the following. ([H])

$$KO^{2i+1}(X) \cong s\mathbb{Z}_2$$

$$KO^{2i}(X) \cong r\mathbb{Z} \oplus s\mathbb{Z}_2,$$

$$\text{rank } KO^0(X) = \text{rank } KO^{-4}(X) = \sum_i \text{rank } H^{4i}(X;\mathbb{Z})$$

$$\text{rank } KO^{-2}(X) = \text{rank } KO^{-6}(X) = \sum_i \text{rank } H^{4i+2}(X;\mathbb{Z})$$

Hence the extension of $\bigoplus_{p+q=2i-1} E_{\infty}^{pq}(X) \cong \bigoplus_k E_{\infty}^{2k+2i-1}$ to $KO^{2i-1}(X)$ is trivial.

It is well known that the Poincaré series of $G(n)/T$ is as follows. ([KI])

$$P_t(G(n)/T) = \begin{cases} (1 - t^2) \cdots (1 - t^{2n}) & G = U \\ (1 - t^2) \cdots (1 - t^2) & G = Sp \\ (1 - t^2) \cdots (1 - t^4) & G = SO, n = 2m \\ (1 - t^2) \cdots (1 - t^4) & G = SO, n = 2m + 1 \\ 1 + \frac{(1 - t^4) \cdots (1 - t^{4m})}{(1 - t^2) \cdots (1 - t^2)} & G = SO, n = 2m + 1 \end{cases}$$

By substituting $t = 1, \sqrt{-1}$ with $P_t(G(n)/T)$ we have the following.

$$\sum_i \text{rank } H^{4i}(X;\mathbb{Z}) = \sum_i \text{rank } H^{4i+2}(X;\mathbb{Z}) = \begin{cases} n!/2 & G = U \\ 2^{n-1}n! & G = Sp \\ 2^{m-2}m! & G = SO, n = 2m \\ 2^{m-1}m! & G = SO, n = 2m + 1 \end{cases}$$

By Proposition 2.1, 4.2, we see that $E_{\infty}^{-1}(G(n)/T) \cong E_{\infty}^{-1}(G(n)/T) \cong H^*(H^*(G(n)/T;\mathbb{Z}_2); Sq^2)$. Then the Poincaré series of $E_{\infty}^{-1}(G(n)/T)$ are as follows by Proposition 3.1, 3.2 and 3.3, where degrees are taken by $\ast$. 

9
$P_1(E_{\infty}^{-1}(G(n)/T))$

\[
\begin{aligned}
(1 + t^6) \cdots (1 + t^{2m}) & \quad G = U, n = 2m \\
(1 + t^6) \cdots (1 + t^{2m-2}) & \quad G = U, n = 2m + 1 \\
(1 + t^4) \cdots (1 + t^{4m-2}) & \quad G = Sp \\
(1 + t^6) \cdots (1 + t^{8m-10})(1 + t^{4m-2}) & \quad G = SO, n = 4m \\
(1 + t^6) \cdots (1 + t^{8m-10})(1 + t^{4m-2})(1 + t^{8m-2}) & \quad G = SO, n = 4m + 1 \\
(1 + t^6) \cdots (1 + t^{8m-2})^2 & \quad G = SO, n = 4m + 2 \\
(1 + t^6) \cdots (1 + t^{8m-2})(1 + t^{4m-2}) & \quad G = SO, n = 4m + 3
\end{aligned}
\]

By substituting $t = 1, \sqrt{-1}, e^{\sqrt{-1} \pi/4}$ with the Poincaré series above we completes the proof.

References


