# $KO$-theory of complex Stiefel manifolds

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## 1 Introduction

The purpose of this paper is to determine the $KO^*$-groups of complex Stiefel manifolds $V_{n,q}$ which is $q$-frames in $\mathbf{C}^n$. We compute it by using the Atiyah-Hirzebruch spectral sequence of $KO^*(V_{n,q})$ and obtain the following.

**Theorem .** *Let $P(t)$ be the polynomial $\prod_{i=n-q+1}^{n}(1+t^{2i-1})$, $Q(t)$ be the following and $a_k, b_k$ be the sum of coefficients of $t^{4i+k}$ in $P(t)$, $t^{8i+k+1}$ in $Q(t)$ respectively.*

$$Q(t) = \begin{cases} (1+t^{2(n-q)+1})(1+t^{2n-1})\prod_{i=n-q+2}^{n-2}(1+t^{4i}) & (n,q)=(2k,2l) \\ (1+t^{2(n-q)+1})\prod_{i=n-q+2}^{n-1}(1+t^{4i}) & (n,q)=(2k+1,2l) \\ (1+t^{2n-1})\prod_{i=n-q+1}^{n-2}(1+t^{4i}) & (n,q)=(2k,2l+1) \\ \prod_{i=n-q+1}^{n-1}(1+t^{4i}) & (n,q)=(2k+1,2l+1) \end{cases}$$

*Then the $KO^i$-groups of $V_{n,q}$ is $r\mathbf{Z}\oplus s\mathbf{Z}_2$ for $(r,s)$ below.*

| $i$ | $0$ | $-1$ | $-2$ | $-3$ |
|---|---|---|---|---|
| $(r,s)$ | $(a_0, b_{-7}+b_0)$ | $(a_{-1}, b_{-1}+b_0)$ | $(a_{-2}, b_{-1}+b_{-2})$ | $(a_{-3}, b_{-3}+b_{-2})$ |
| $i$ | $-4$ | $-5$ | $-6$ | $-7$ |
| $(r,s)$ | $(a_0, b_{-3}+b_{-4})$ | $(a_{-1}, b_{-5}+b_{-4})$ | $(a_{-2}, b_{-5}+b_{-6})$ | $(a_{-3}, b_{-7}+b_{-6})$ |

## 2 The Atiyah-Hirzebruch spectral sequence

First we recall that the coefficient ring of $KO$-theory is that

$$KO^* = \mathbf{Z}[\alpha, x, \beta, \beta^{-1}]/(2\alpha, \alpha^3, \alpha x, x^2 - 4\beta),$$

where $|\alpha| = 1$, $|x| = 4$ and $|\beta| = 8$.

Let $X$ be a finite CW-complex. The Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ is the spectral sequence with $E_2^{p,q} \cong H^p(X; KO^q)$ converging to $KO^*(X)$. It is well known that the differential $d_2$ of the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ is given by the following. (See [F])

$$d_2^{*,q} = \begin{cases} Sq^2\,\pi_2 & q \equiv 0\ (8) \\ Sq^2 & q \equiv -1\ (8) \\ 0 & \text{otherwise,} \end{cases}$$

where $\pi_2$ is the modulo 2 reduction.

In this paper we compute the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ with $X$ in two special classes of CW-complexes.

Let $\mathcal{E}$ be the class of CW-complexes with only even cells and $\mathcal{O}$ be the one with only odd cells and 0-cells. The Atiyah-Hirzebruch spectral sequence of $KO^*(X)$ for $X$ in $\mathcal{E}$ is considered in [H-K]. It is easily seen that [HK, Proposition 1] is valid for a CW-complex in $\mathcal{O}$ and we have the following.

**Proposition 1.** *Let $X$ be a finite CW-complex in either $\mathcal{E}$ or $\mathcal{O}$ and $E_r(X)$ be the $r$-th term of the Atiyah-Hirzebruch spectral sequence of $KO^*(X)$. Then we have the following.*

1. *$E_3^{p,-1}(X) \cong H^p(H^*(X; \mathbf{Z}_2); Sq^2)$.*
2. *Let $d_r$ be the first non-trivial differential for $r \geq 3$.*
   - *(a) $r \equiv 2\ (8)$.*
   - *(b) There exists $x \in E_r^{*,0}(X)$ such that $\alpha x \neq 0$ and $\alpha d_r x \neq 0$.*

## 3 The $Sq^2$-cohomology of $V_{n,q}$

It is well known that $V_{n,q} \simeq U(n)/U(n-q)$ and

$$H^*(V_{n,q}; \mathbf{Z}) \cong \bigwedge(e_{2(n-q)+1}, e_{2(n-q)+3}, \ldots, e_{2n-1}),$$

where $U(k)$ is the $k$-dimensional unitary group and $|e_i| = i$. Since $Sq^2 e_{4i-1} = e_{4i+1}$ $(4i+1 \leq 2n-1)$, we have the following.

**Proposition 2.** $H^*(H^*(V_{n,q}; \mathbf{Z}_2); Sq^2)$ *is the exterior algebra generated by the elements in the table below.*

| | |
|---|---|
| $(n,q)=(2k,2l)$ | $e_{2(n-q)+1}, e_{2(n-q)+3}e_{2(n-q)+5}, \dots, e_{2n-5}e_{2n-3}, e_{2n-1}$ |
| $(n,q)=(2k+1,2l)$ | $e_{2(n-q)+1}e_{2(n-q)+3}, \dots, e_{2n-3}e_{2n-1}$ |
| $(n,q)=(2k,2l+1)$ | $e_{2(n-q)+1}e_{2(n-q)+3}, \dots, e_{2n-5}e_{2n-3}, e_{2n-1}$ |
| $(n,q)=(2k+1,2l+1)$ | $e_{2(n-q)+1}, e_{2(n-q)+3}e_{2(n-q)+5}, \dots, e_{2n-3}e_{2n-1}$ |

# 4 Collapse problem of $E_r(V_{n,q})$

Let $G_{q,k}$ be the complex Grassmannian of $k$-planes in $\mathbf{C}^q$ which is the homogeneous space $U(q)/U(k) \times U(q-k)$. Let $\mathrm{ad}_k : U(k) \to GL(k^2, \mathbf{R})$ and $\mathrm{can}_k : U(k) \to GL(2k, \mathbf{R})$ be the adjoint and the canonical representation. By abuse of notation, $\mathrm{ad}_k \oplus m\mathrm{can}_k$ denotes the real vector bundle associated to the representation $\mathrm{ad}_k \oplus m\mathrm{can}_k$ and the $U(k)$-principal bundle $U(k) \to V_{q,k} \to G_{q,k}$.

In [M] it is shown that there exists a stable homotopy equivalence as follows.

$$V_{n,q} \underset{s}{\simeq} \bigvee_{k=1}^{q} G_{q,k}^{\mathrm{ad}_k \oplus (n-q)\mathrm{can}_k}, \tag{$*$}$$

where $G_{q,k}^{\mathrm{ad}_k \oplus (n-q)\mathrm{can}_k}$ is the Thom space of the real vector bundle $\mathrm{ad}_k \oplus (n-q)\mathrm{can}_k$ on $G_{q,k}$. Then $E_r(V_{n,q})$ splits into $E_r(G_{q,k}^{\mathrm{ad}_k \oplus (n-q)\mathrm{can}_k})$. Note that $G_{q,k}^{\mathrm{ad}_k \oplus (n-q)\mathrm{can}_k}$ is either in $\mathcal{E}$ or in $\mathcal{O}$.

**Proposition 3.** *Let $E \to G_{q,k}$ be a real vector bundle with $w_2(E) = 0$ and either $k$ be even or $q$ be odd. Then $E_r(G_{q,k}^E)$ collapses at the third term.*

*Proof.* By Thom isomorphism, we have

$$E_2(G_{q,k}^E) \cong KO^* \oplus \phi_E H^*(G_{q,k}; KO^*) \cong KO^* \oplus \phi_E E_2(G_{q,k}),$$

where $\phi_E$ is the Thom class of $E$. Since $d_2\phi_E = Sq^2\pi_2\phi_E = w_2(E)\pi_2\phi_E = 0$, we have

$$E_3(G_{q,k}^E) \cong KO^* \oplus \phi_E E_3(G_{q,k}).$$

It is shown in [HK] that $E_r(G_{q,k})$ collapses at the third term for any $k, q$ and $H^*(H^*(G_{q,k}; \mathbf{Z}_2); Sq^2)$ has only elements of $8i$ degree if $k$ is even or $q$ is odd. Then we see $d_r\phi_E = 0$ for $r \geq 3$ by degree argument and Proposition 1,*2,(a)*. Therefore we obtain that $d_r = 0$ for $r \geq 3$ by Proposition 1,*2,(b)*. □

By the naturality of the Thom class, we have the following.

**Corollary 1.** *Let $E \to G_{q,k}$ be a real vector bundle with $w_2(E) = 0$, either $k$ be even or $q$ be odd and $\imath : G_{q-1,k} \to G_{q,k}$ be the natural inclusion. Then $E_r(G_{q-1,k}^{\imath^* E})$ collapses at the third term.*

**Lemma 1.** *$E_r(V_{n,q})$ collapses at the third term.*

*Proof.* We show that the elements of $E_3^{*,-1}(V_{n,q}) \cong H^*(H^*(V_{n,q};\mathbf{Z}_2);Sq^2)$ in the table of Proposition 2 are permanent cycles.

It is easily seen that $w_2(\mathrm{can}_k) \neq 0$ and

$$w_2(\mathrm{ad}_k) \begin{cases} \neq 0 & k \text{ is even} \\ = 0 & k \text{ is odd.} \end{cases}$$

Since $w_2(\mathrm{ad}_k \oplus (n-q)\mathrm{can}_k) = w_2(\mathrm{ad}_k) + (n-q)w_2(\mathrm{can}_k)$ and $H^2(G_{q,k};\mathbf{Z}_2) \cong \mathbf{Z}_2$, $E_r(G_{q,k}^{\mathrm{ad}_k \oplus (n-q)\mathrm{can}_k})$ collapses at $r=3$ when $n-q$ is even and $k$ is odd, or, $n-q$ is odd and $k$ is even by Proposition 3 and Corollary 1. Note that $G_{q,k}^{\mathrm{ad}_k \oplus (n-q)\mathrm{can}_k}$ is in $\mathcal{E}$ (resp. $\mathcal{O}$) if $k$ is even (resp. odd) and that $E_r^{2l+1,*}(X) = 0$ ($E_r^{2l,*}(X) = 0$) if $X$ is in $\mathcal{E}$ (resp. $\mathcal{O}$), then we see that $x \in E_3^{*,-1}(V_{n,q})$ is a permanent cycle if $n-q$ is even and $|x|$ is odd, or, $n-q$ is odd and $|x|$ is even. Therefore $e_{2(n-q)+1}$ is a permanent cycle. We also see that $e_{4i-1}e_{4i+1}$ is permanent cycle for any $n, q$ by considering the homomorphisms $E_3(V_{n,q}) \to E_3(V_{n,q+1})$ and $E_3(V_{n+1,q+1}) \to E_3(V_{n,q})$ induced by the natural projection $V_{n,q+1} \to V_{n,q}$ and the natural inclusion $V_{n,q} \to V_{n+1,q+1}$. Note that we have the homomorphism $E_3(S^{2n-1}) \to E_3(V_{n,q})$ induced from the projection $V_{n,q} \to V_{n,1} = S^{2n-1}$, then we see that $e_{2n-1}$ is the permanent cycle. □

# 5 Proof of Theorem

It is easily seen that $K^n(X)$ is torsion free and concentrated in even (odd) dimension, if $X$ is in $\mathcal{E}$ (resp.$\mathcal{O}$). Consider the Bott sequence

$$\cdots \to K^n(X) \to KO^{n+2}(X) \to KO^{n+1}(X) \xrightarrow{\mathbf{c}} K^{n+1}(X) \to \cdots,$$

where $\mathbf{c} : KO^i(X) \to K^i(X)$ is the complexification map. Since $\mathbf{rc} = 2$ we have the following, where $\mathbf{r} : K^i(X) \to KO^i(X)$ is the realization map. (See [H].)

**Proposition 4.** *If $X$ is in $\mathcal{E}$, we have*

$$\begin{aligned} KO^{2i+1}(X) &\cong s\mathbf{Z}_2 \\ KO^{2i}(X) &\cong r\mathbf{Z} \oplus s\mathbf{Z}_2. \end{aligned}$$

*If $X$ is in $\mathcal{O}$, we have*

$$\begin{aligned} KO^{2i}(X) &\cong s\mathbf{Z}_2 \\ KO^{2i-1}(X) &\cong r\mathbf{Z} \oplus s\mathbf{Z}_2. \end{aligned}$$

*Proof of Theorem.* By Proposition 4 we have

$$\bigoplus_{p+q=2n-1} E_\infty^{p,q}(X) \cong \bigoplus_i E_\infty^{2n+8i,-1} \cong KO^{2n-1}(X), \ \text{ for } X \text{ in } \mathcal{E}$$

$$\bigoplus_{p+q=2n} E_\infty^{p,q}(X) \cong \bigoplus_i E_\infty^{2n+8i+1,-1} \cong KO^{2n}(X), \quad \text{for } X \text{ in } \mathcal{O}.$$

Note that the Thom space of a vector bundle on $G_{q,k}$ as in the stable splitting $(*)$ is either in $\mathcal{E}$ or $\mathcal{O}$ and that $E_r^{2i-1,*}(X) = 0$ ($E_r^{2i,*}(X) = 0$ for $i > 0$) if $X$ is in $\mathcal{E}$ (resp. $\mathcal{O}$). Then we obtain that $KO^i(V_{n,q}) \cong r\mathbf{Z} \oplus s\mathbf{Z}_2$ for (r,s) below, where $s_k = \sum_i \mathrm{rank} H^{4i+k}(V_{n,q};\mathbf{Z}), t_k = \sum_i \dim_{\mathbf{Z}_2} E_\infty^{8i+k+1,-1}(V_{n,q})$.

| $i$ | $0$ | $-1$ | $-2$ | $-3$ |
|---|---|---|---|---|
| $(r,s)$ | $(s_0, t_{-7}+t_0)$ | $(s_{-1}, t_{-1}+t_0)$ | $(s_{-2}, t_{-1}+t_{-2})$ | $(s_{-3}, t_{-3}+t_{-2})$ |
| $i$ | $-4$ | $-5$ | $-6$ | $-7$ |
| $(r,s)$ | $(s_0, t_{-3}+t_{-4})$ | $(s_{-1}, t_{-5}+t_{-4})$ | $(s_{-2}, t_{-5}+t_{-6})$ | $(s_{-3}, t_{-7}+t_{-6})$ |

It is easily seen that the Poincaré series $P_t(H^*(V_{n,q};\mathbf{Z})) = \sum_i \mathrm{rank} H^*(V_{n,q};\mathbf{Z})t^i$, $P_t(E_\infty^{*,-1}(V_{n,q})) = \sum_i \dim_{\mathbf{Z}_2} E_\infty^{i,-1}(V_{n,q})t^i = \sum_i \dim_{\mathbf{Z}_2} H^i(H^*(V_{n,q};\mathbf{Z}_2);Sq^2)t^i$ are $P(t), Q(t)$ respectively by Proposition 2. Then we have $a_i = s_i, b_i = t_i$ and complete the proof. $\square$

# References


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