ENTROPY OF SUBSHIFTS AND THE MACAEV NORM

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ABSTRACT. We obtain the exact value of Voiculescu's invariant $k_\infty^-(\tau)$, which is an obstruction of the existence of quasicentral approximate units relative to the Macaev ideal in perturbation theory, for a tuple τ of operators in the following two classes: (1) creation operators associated with a subshift, which are used to define Matsumoto algebras, (2) unitaries in the left regular representation of a finitely generated group.

1. INTRODUCTION

In the remarkable serial works [Voi1], [Voi2], [Voi3] and [DV] on perturbation of Hilbert space operators, Voiculescu investigated a numerical invariant $k_\Phi(\tau)$ for a family τ of bounded linear operators on a separable Hilbert space, where $k_\Phi(\tau)$ is the obstruction of the existence of quasicentral approximate units relative to the normed ideal $\mathfrak{S}_\Phi^{(0)}$ corresponding to a symmetric norming function Φ, (see definitions in Section 2). The invariant $k_\Phi(\tau)$ is considered to be a kind of dimension of τ with respect to the normed ideal $\mathfrak{S}_\Phi^{(0)}$ (see [Voi1] and [DV]).

In the present paper, we study the invariant $k_\Phi(\tau)$ for the Macaev ideal, which is denoted by $k_\infty^-(\tau)$. It is known that $k_\infty^-(\tau)$ possesses several remarkable properties: for instance, $k_\infty^-(\tau)$ is always finite and $k_\Phi(\tau) = 0$ if $\mathfrak{S}_\Phi^{(0)}$ is strictly larger than the Macaev ideal. In [Voi3], Voiculescu investigated the invariant $k_\infty^-(\tau)$ for several examples. He proved that $k_\infty^-(\tau) = \log N$ for an N-tuple τ of isometries in extensions of the Cuntz algebra $\mathcal{O}_N$. Here, $\log N$ can be interpreted as the value of the topological entropy of the N-full shift. Inspired by this result, we show that $k_\infty^-(\tau) = h_{\mathrm{top}}(X)$ for a general subshift X with a certain condition, where $h_{\mathrm{top}}(X)$ is the topological entropy of X and τ is the family of creation operators on the Fock space associated with the subshift X, which is used to define the Matsumoto algebra associated with X (e.g. see [Mat]). In particular, we show that $k_\infty^-(\tau) = h_{\mathrm{top}}(X)$ holds for every almost Sofic shift X (cf. [Pet]).

Let Γ be a countable finitely generated group and S its generating set. We also study $k_\infty^-((\lambda_a)_{a\in S})$, where λ is the left regular representation of Γ. For the related topic, see [Voi5], in which a relation between $k_\infty^-((\lambda_a)_{a\in S})$ and the entropy of random walks on groups is discussed. By using a method introduced in [Oka], we can compute the exact value of $k_\infty^-((\lambda_a)_{a\in S})$ for certain amalgamated free product groups. Voiculescu proved that $\log N \leq k_\infty^-((\lambda_a)_{a\in S}) \leq \log(2N-1)$ holds for the free group $\mathbb{F}_N$ with the canonical generating set S ([Voi3, Proposition 3.7. (a)]). As a particular case of our results, we show that $k_\infty^-((\lambda_a)_{a\in S}) = \log(2N-1)$ actually holds.
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2. Preliminary

Let H be a separable infinite dimensional Hilbert space. By $\mathbb{B}(H)$, $\mathbb{K}(H)$, $\mathbb{F}(H)$ a nd $\mathbb{F}(H)_1^+$, we denote the bounded linear operators, the compact operators, the finite rank operators and the finite rank positive contractions on H, respectively.

We begin by recalling some facts concerning normed ideal in [GK]. Let c_0 be the set of real valued sequences $\xi = (\xi_j)_{j\in\mathbb{N}}$ with $\lim_{j\to\infty}\xi_j = 0$, and $c_{0,0}$ the subspace of c_0 consisting of the sequences with finite support. A function Φ on $c_{0,0}$ is said to be a *symmetric norming function* if Φ satisfies:

(1) Φ is a norm on $c_{0,0}$;
(2) $\Phi((1,0,0,\dots)) = 1$;
(3) $\Phi((\xi_j)_{j\in\mathbb{N}}) = \Phi((|\xi_{\pi(j)}|)_{j\in\mathbb{N}})$ for any bijection $\pi : \mathbb{N} \to \mathbb{N}$.

For $\xi = (\xi_j)_{j\in\mathbb{N}} \in c_0$, we define

$$\Phi(\xi) = \lim_{n\to\infty} \Phi(\xi^*(n)) \in [0,\infty],$$

where $\xi^*(n) = (\xi_1^*,\dots,\xi_n^*,0,0,\dots) \in c_{0,0}$ and $\xi_1^* \geq \xi_2^* \geq \cdots$ is the decreasing rearrangement of the absolute value $(|\xi_j|)_{j\in\mathbb{N}}$. If $T \in \mathbb{K}(H)$ and Φ is a symmetric norming function, then let us denote

$$||T||_\Phi = \Phi((s_j(T))_{j\in\mathbb{N}}),$$

where $(s_j(T))_{j\in\mathbb{N}}$ is the singular numbers of T. We define two symmetrically normed ideals

$$\mathfrak{S}_\Phi = \{T \in \mathbb{K}(H) \mid ||T||_\Phi < \infty\},$$

and $\mathfrak{S}_\Phi^{(0)}$ by the closure of $\mathbb{F}(H)$ with respect to the norm $||\cdot||_\Phi$. Note that $\mathfrak{S}_\Phi^{(0)}$ does not coincide with $\mathfrak{S}_\Phi$ in general. If $\mathfrak{S}$ is a *symmetrically normed ideal*, i.e. $\mathfrak{S}$ is a ideal of $\mathbb{B}(H)$ and a Banach space with respect to the norm $||\cdot||_{\mathfrak{S}}$ satisfying:

(1) $||XTY||_{\mathfrak{S}} \leq ||X||\cdot||T||_{\mathfrak{S}}\cdot||Y||$ for $T\in\mathfrak{S}$ and $X,Y\in\mathbb{B}(H)$,
(2) $||T||_{\mathfrak{S}} = ||T||$ if T is of rank one,

where $||\cdot||$ is the operator norm in $\mathbb{B}(H)$, then there exists a unique symmetric norming function Φ such that $||T||_{\mathfrak{S}} = ||T||_\Phi$ for $T\in\mathbb{F}(H)$ and $\mathfrak{S}_\Phi^{(0)}\subseteq\mathfrak{S}\subseteq\mathfrak{S}_\Phi$.

We introduce some symmetrically normed ideals. For $1<p\leq\infty$, the symmetrically normed ideal $\mathcal{C}_p^-(H)$ is given by the symmetric norming function

$$\Phi_p^-(\xi) = \sum_{j=1}^{\infty}\frac{\xi_j^*}{j^{1-1/p}}.$$

We define $\mathcal{C}_p^-(H) = \mathfrak{S}^{(0)}_{\Phi_p^-}$. We remark that it coincides with $\mathfrak{S}_{\Phi_p^-}$. For $1\leq p<\infty$, the symmetrically normed ideal $\mathcal{C}_p^+(H)$ is given by the symmetric norming function

$$\Phi_p^+(\xi) = \sup_{n\in\mathbb{N}}\frac{\sum_{j=1}^n\xi_j^*}{\sum_{j=1}^n j^{1/p}}.$$

We define $\mathcal{C}_p^+(H) = \mathfrak{S}_{\Phi_p^+}$. However $\mathfrak{S}^{(0)}_{\Phi_p^+}$ is strictly smaller than $\mathcal{C}_p^+(H)$. For $1\leq p<q<r\leq\infty$, we have

$$\mathcal{C}_p(H)\subsetneqq\mathcal{C}_q^-(H)\subsetneqq\mathcal{C}_q(H)\subsetneqq\mathcal{C}_q^+(H)\subsetneqq\mathcal{C}_r(H),$$

where $\mathcal{C}_p(H)$ is the Schatten p class.

For a given symmetric norming function Φ, which is not equivalent to the l^1-norm, there is a symmetric norming function Φ^* such that $\mathfrak{S}_{\Phi^*}$ is the dual of $\mathfrak{S}_\Phi^{(0)}$, where the dual pairing is given by the bilinear form $(T,S)\mapsto\mathrm{Tr}(TS)$. If $1/p+1/q=1$, then $\mathcal{C}_p(H)^*\simeq\mathcal{C}_q(H)$ and $\mathcal{C}_p^-(H)^*\simeq\mathcal{C}_q^+(H)$. In particular, $\mathcal{C}_\infty^-(H)$ and $\mathcal{C}_1^+(H)$ are called the *Macaev ideal* and the *dual Macaev ideal*, respectively.

Let $\mathfrak{S}_\Phi^{(0)}$ be a symmetrically normed ideal with a symmetric norming function Φ. If $\tau=(T_1,\ldots,T_N)$ is an N-tuple of bounded linear operators, then the number $k_\Phi(\tau)$ is defined by

$$k_\Phi(\tau) = \liminf_{u\in\mathbb{F}(H)_1^+}\max_{1\leq a\leq N}||[u,T_a]||_\Phi,$$

where the inferior limit is taken with respect to the natural order on $\mathbb{F}(H)_1^+$ and $[A,B]=AB-BA$. We denote $k_{\Phi_p^-}$ by k_p^-. A relation between the invariant k_Φ and the existence of quasicentral approximate units relative to the symmetrically normed ideal $\mathfrak{S}_\Phi^{(0)}$ is discussed in [Voi1]. A quasicentral approximate unit for $\tau = (T_1, \ldots, T_N)$ relative to $\mathfrak{S}_\Phi^{(0)}$ is a sequence $\{u_n\}_{n=1}^\infty \subseteq \mathbb{F}(H)_1^+$ such that $u_n \nearrow I$ and $\lim_{n\to\infty} |||[u_n, T_a]|||_\Phi = 0$ for $1 \le a \le N$. Note that for an N-tuple $\tau = (T_1, \ldots, T_N)$, there exists a quasicentral approximate unit for τ relative to $\mathfrak{S}_\Phi^{(0)}$ if and only if $k_\Phi(\tau) = 0$ (e.g. see [Voi2, Lemma 1.1]).

We use the following propositions to prove our theorem.

Proposition 2.1 ([Voi1, Proposition 1.1]). *Let $\tau = (T_1, \ldots, T_N) \in \mathbb{B}(H)^N$ and $\mathfrak{S}_\Phi^{(0)}$ be a symmetrically normed ideal with a symmetric norming function Φ. If we take a sequence $\{u_n\}_{n=1}^\infty \subseteq \mathbb{F}(H)_1^+$ with* w-$\lim_{n\to\infty} u_n = I$*, then*

$$k_\Phi(\tau) \le \liminf_{n\to\infty} \max_{1\le a\le N} |||[u_n, T_a]|||_\Phi.$$

Proposition 2.2 ([Voi3, Proposition 2.1]). *Let $\tau = (T_1, \ldots, T_N) \in \mathbb{B}(H)^N$ and $X_a \in \mathcal{C}_1^+(H)$ for $a = 1, \ldots, N$. If*

$$\sum_{a=1}^N [X_a, T_a] \in \mathcal{C}_1(H) + \mathbb{B}(H)_+,$$

then we have

$$\left| \operatorname{Tr}\left(\sum_{a=1}^N [X_a, T_a] \right) \right| \le k_\infty^-(\tau) \sum_{a=1}^N ||X_a||_1^{\tilde{+}},$$

where $||X_a||_1^{\tilde{+}} = \inf_{Y\in\mathbb{F}(H)} ||X_a - Y||_1^+$.

The following proposition was shown in the proof of [GK, Theorem 14.1].

Proposition 2.3. *For $T \in \mathcal{C}_1^+(H)$, we have*

$$||T||_1^{\tilde{+}} = \limsup_{n\to\infty} \frac{\sum_{j=1}^n s_j(T)}{\sum_{j=1}^n 1/j}.$$

3. Subshifts and Macaev Norm

Let $\mathcal{A}$ be a finite set with the discrete topology, which we call the *alphabet*, and $\mathcal{A}^\mathbb{Z}$ the two-sided infinite product space $\prod_{i=-\infty}^\infty \mathcal{A}$ endowed with the product topology. The *shift map* σ on $\mathcal{A}^\mathbb{Z}$ is given by $(\sigma(x))_i = x_{i+1}$ for $i \in \mathbb{Z}$. The pair $(\mathcal{A}^\mathbb{Z}, \sigma)$ is called the *full shift*. In particular, if the cardinality of the alphabet $\mathcal{A}$ is N, then we call it the *N-full shift*.

Let X be a shift invariant closed subset of $\mathcal{A}^{\mathbb{Z}}$. The topological dynamical system (X, σ_X) is called a *subshift* of $\mathcal{A}^{\mathbb{Z}}$, where σ_X is the restriction of the shift map σ. We sometimes denote the subshift (X, σ_X) by X for short. A *word* over $\mathcal{A}$ is a finite sequence $w = (a_1, \dots, a_n)$ with $a_i \in \mathcal{A}$. For $x \in \mathcal{A}^{\mathbb{Z}}$ and a word $w = (a_1, \dots, a_n)$, we say that w *occurs in* x if there is an index i such that $x_i = a_1, \dots, x_{i+n-1} = a_n$. The empty word occurs in every $x \in \mathcal{A}^{\mathbb{Z}}$ by convention. Let $\mathcal{F}$ be a collection of words over $\mathcal{A}^{\mathbb{Z}}$. We define the subshift $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ in which *no* word in $\mathcal{F}$ occurs. It is well-known that any subshift X of $\mathcal{A}^{\mathbb{Z}}$ is given by $X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden words over $\mathcal{A}^{\mathbb{Z}}$. Note that for $\mathcal{F} = \emptyset$, the subshift $X_{\mathcal{F}}$ is the full shift $\mathcal{A}^{\mathbb{Z}}$.

Let X be a subshift of $\mathcal{A}^{\mathbb{Z}}$. We denote by $\mathcal{W}_n(X)$ the set of all words with length n that occur in X and we set

$$\mathcal{W}(X) = \bigcup_{n=0}^{\infty} \mathcal{W}_n(X).$$

Let $\varphi : \mathcal{W}_{m+n+1}(X) \to \mathcal{A}$ be a map, which we call a *block map*. The extension of φ from X to $\mathcal{A}^{\mathbb{Z}}$ is defined by $(x_i)_{i\in\mathbb{Z}} \mapsto (y_i)_{i\in\mathbb{Z}}$, where

$$y_i = \varphi((x_{i-m}, x_{i-m+1}, \dots, x_{i+n})).$$

We also denote this extension by φ and call it a *sliding block code*. Let X, Y be two subshifts and $\varphi : X \to Y$ a sliding block code. If φ is one-to-one, then φ is called an *embedding* of X into Y and we denote $X \subseteq Y$. If φ has an *inverse*, i.e. a sliding block code $\psi : Y \to X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$, then two subshifts X and Y are *topologically conjugate*.

The *topological entropy* of a subshift X is defined by

$$h_{\mathrm{top}}(X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{W}_n(X)|,$$

where $|\mathcal{W}_n(X)|$ is the cardinality of $\mathcal{W}_n(X)$. The reader is referred to [LM] for an introduction to symbolic dynamics.

For a given subshift X, we next construct the creation operators on the Fock space associated with X (cf. [Mat]). Let $\{\xi_a\}_{a\in\mathcal{A}}$ be an orthonormal basis of N-dimensional Hilbert space $\mathbb{C}^N$, where N is the cardinality of $\mathcal{A}$. For $w = (a_1, \dots, a_n) \in \mathcal{W}_n(X)$, we denote $\xi_w = \xi_{a_1} \otimes \cdots \otimes \xi_{a_n}$. We define the Fock space $\mathcal{F}_X$ for a subshift X by

$$\mathcal{F}_X = \mathbb{C}\xi_0 \oplus \bigoplus_{n\in\mathbb{N}} \mathrm{span}\{\xi_w \mid w \in \mathcal{W}_n(X)\},$$

where ξ_0 is the vacuum vector. The creation operator T_a on $\mathcal{F}_X$ for $a \in \mathcal{A}$ is given by

$$\begin{aligned} T_a\xi_0 &= \xi_a, \\ T_a\xi_w &= \begin{cases} \xi_a \otimes \xi_w & \text{if} \quad aw \in \mathcal{W}(X), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that T_a is a partial isometry such that

$$P_0 + \sum_{a\in\mathcal{A}} T_a T_a^* = 1,$$

where P_0 is the rank one projection onto $\mathbb{C}\xi_0$. We denote by P_n the projection onto the subspace spanned by ξ_w for all $w \in \mathcal{W}_n(X)$. For $w = (a_1, \ldots, a_n) \in \mathcal{W}_n(X)$, we set $T_w = T_{a_1} \cdots T_{a_n}$. The following proposition is essentially proved in [Voi3].

Proposition 3.1. *If $\tau = (T_a)_{a\in\mathcal{A}}$, then we have*

$$k_\infty^-(\tau) \leq h_{\mathrm{top}}(X).$$

Proof. We first assume that the topological entropy of X is non-zero. Let us denote $h = h_{\mathrm{top}}(X)$. By definition, for a given $\epsilon > 1$, there exists $K \in \mathbb{N}$ such that for any $n \geq K$, we have

$$\frac{1}{n}\log|\mathcal{W}_n(X)| < \varepsilon h.$$

Thus

$$|\mathcal{W}_n(X)| < e^{n\varepsilon h},$$

for all $n \geq K$. We set

$$X_n = \sum_{j=0}^{n-1}\left(1 - \frac{j}{n}\right) P_j.$$

One can show that

$$||[X_n, T_a]|| \leq \frac{1}{n}.$$

Since

$$r_n = \mathrm{rank}([X_n, T_a]) \leq \sum_{j=1}^{n} |\mathcal{W}_j(X)| \leq \sum_{j=1}^{K-1} |\mathcal{W}_j(X)| + \sum_{j=K}^{n} e^{j\varepsilon h}$$

for $n \geq K$, we obtain

$$k_\infty^-(\tau) \leq \limsup_{n\to\infty} \max_{a\in\mathcal{A}} ||[X_n, T_a]||_\infty^- \leq \limsup_{n\to\infty} \frac{\sum_{j=1}^{r_n} 1/j}{n} \leq h.$$

In the case of $h = 0$, for any $\varepsilon > 0$, we have

$$|\mathcal{W}_n(X)| < e^{n\varepsilon}$$

for sufficiently large n. By the same argument, we can get

$$k_\infty^-(\tau) \leq \limsup_{n\to\infty} \max_{a\in\mathcal{A}} |||[X_n, T_a]|||_\infty^- \leq \varepsilon,$$

for arbitrary $\epsilon > 0$. □

Next we obtain the lower bound of $k_\infty^-(\tau)$ by using Proposition 2.2. Before it, we prepare some notations. For any $m \in \mathbb{Z}$ and $w = (a_1, \ldots, a_n) \in \mathcal{W}_n(X)$, let us denote

$${}_m[w] = \{(x_i)_{i\in\mathbb{Z}} \in X \mid x_m = a_1, \ldots, x_{m+n-1} = a_n\}.$$

We sometimes denote the cylinder set ${}_0[w]$ by $[w]$ for short. Let μ be a shift invariant probability measure on X. The following holds:

(1) $\sum_{a\in\mathcal{A}} \mu([a]) = 1$;
(2) $\mu([a_1, \ldots, a_n]) = \sum_{a_0\in\mathcal{A}} \mu([a_0, a_1, \ldots, a_n])$;
(3) $\mu([a_1, \ldots, a_n]) = \sum_{a_{n+1}\in\mathcal{A}} \mu([a_1, \ldots, a_n, a_{n+1}])$.

For any partition $\beta = (B_1, \ldots, B_n)$ of X, we define a function on X by

$$I(\beta) = -\sum_{B\in\beta} \log \mu(B)\chi_B,$$

where χ_B is the characteristic function of B. Let $\beta_1, \ldots, \beta_k$ be partitions of X. The partition $\bigvee_{i=1}^k \beta_i$ is defined by

$$\left\{ \bigcap_{i=1}^k B_i \mid (B_i)_{1\leq i\leq k} \in \prod_{i=1}^k \beta_i \right\}.$$

The value

$$H_\mu(\beta) = -\sum_{B\in\beta} \mu(B) \log \mu(B)$$

is called the *entropy of the partition* β. We define

$$h_\mu(\beta, \sigma_X) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} \sigma_X^{-i}(\beta)).$$

The *entropy* of (X, σ_X, μ) is defined by

$$h_\mu(\sigma_X) = \sup\{h_\mu(\beta, \sigma_X) \mid H_\mu(\beta) < \infty\}.$$

Note that $h_\mu(\sigma_X) \leq h_{\mathrm{top}}(X)$ in general. A shift invariant probability measure μ is said to be a *maximal measure* if $h_{\mathrm{top}}(X) = h_\mu(\sigma_X)$. The reader is referred to [DGS] for details.

Theorem 3.2. *Let $\tau = (T_a)_{a\in\mathcal{A}}$ be the creation operators for a subshift X. If there exists a shift invariant probability measure μ on X such that for any $\varepsilon > 0$ we have*

$$\sum_{n=0}^{\infty} \mu\left(\left\{x \in X : \left|\frac{1}{n+1} I\left(\bigvee_{i=0}^{n} \sigma_X^{-i}\beta\right)(x) - h_\mu(\sigma_X)\right| > \varepsilon\right\}\right) < \infty,$$

where β is the generating partition $\{[a]\}_{a\in\mathcal{A}}$ of X, then

$$h_\mu(\sigma_X) \leq k_\infty^-(\tau).$$

In particular, if we can take a maximal measure μ with the above condition, then we have

$$k_\infty^-(\tau) = h_{\mathrm{top}}(X).$$

Proof. Let μ be a shift invariant probability measure on X. For $a \in \mathcal{A}$, we set

$$X_a = \sum_{n\geq 0} \sum_{w\in\mathcal{W}_n(X)} \mu([aw]) T_w P_0 T_{aw}^*.$$

Then

$$\begin{aligned} \sum_{a\in\mathcal{A}} T_a X_a &= \sum_{n\geq 0}\sum_{a\in\mathcal{A}}\sum_{w\in\mathcal{W}_n(X)} \mu([aw]) T_{aw} P_0 T_{aw}^* \\ &= \sum_{n\geq 1}\sum_{w\in\mathcal{W}_n(X)} \mu([w]) T_w P_0 T_w^*, \end{aligned}$$

and

$$\begin{aligned} \sum_{a\in\mathcal{A}} X_a T_a &= \sum_{n\geq 0}\sum_{w\in\mathcal{W}_n(X)} \left(\sum_{a\in\mathcal{A}} \mu([aw])\right) T_w P_0 T_w^* \\ &= \sum_{n\geq 0}\sum_{w\in\mathcal{W}_n(X)} \mu([w]) T_w P_0 T_w^*. \end{aligned}$$

Hence we have

$$\sum_{a\in\mathcal{A}} [X_a, T_a] = P_0.$$

We assume that $h_\mu(\sigma_X) \neq 0$ and denote it by h for short. To apply Proposition 2.2, we need an estimate of $||X_a||_1^{\widetilde{+}}$. Fix $\varepsilon > 0$ and $a \in \mathcal{A}$. We set

$$D_n = \left\{w \in \mathcal{W}_n(X) \mid e^{-(n+1)(h+\varepsilon)} \leq \mu([aw]) \leq e^{-(n+1)(h-\varepsilon)}\right\},$$

and

$$\varepsilon_n = \sum_{w\in\mathcal{W}_n(X)\setminus D_n} \mu([aw]).$$

If μ satisfies the assumption, then we have

$$\sum_{n\geq 0}\varepsilon_n < \infty. \tag{$\star$}$$

Note that $s_j(X_a) = s_j(X_aT_a)$ for all $j \in \mathbb{N}$. Thus we have $||X_a||_1^{\tilde{+}} = ||X_aT_a||_1^{\tilde{+}}$. We put

$$\widetilde{X_a} = \sum_{n\geq 0}\sum_{w\in D_n}\mu([aw])T_wP_0T_w^*.$$

We remark that for each $j \in \mathbb{N}$, there are $n \in \mathbb{N}$, $w \in \mathcal{W}_n(X)$ such that $s_j(X_aT_a) = \mu([aw])$. By ($\star$), we obtain

$$\begin{aligned} ||X_a||_1^{\tilde{+}} = ||X_aT_a||_1^{\tilde{+}} &= \limsup_{n\to\infty}\frac{\sum_{j=1}^n s_j(X_aT_a)}{\sum_{j=1}^n 1/j} \\ &\leq ||\widetilde{X_a}||_1^{\tilde{+}} + \limsup_{n\to\infty}\frac{\sum_{k=0}^\infty \varepsilon_k}{\sum_{j=1}^n 1/j} = ||\widetilde{X_a}||_1^{\tilde{+}}. \end{aligned}$$

Hence it suffices to give an estimate of $||\widetilde{X_a}||_1^{\tilde{+}}$. Let $d_n = \sum_{j=0}^n |D_j|$, where $|D_j|$ is the cardinality of D_j. One can easily check that

$$||\widetilde{X_a}||_1^{\tilde{+}} \leq \limsup_{n\to\infty}\frac{\sum_{j=1}^{d_n} s_j(\widetilde{X_a})}{\sum_{j=1}^{d_n} 1/j}.$$

Note that if $s_j(\widetilde{X_a}) = \mu([aw])$ for some $w \in D_n$, then we have

$$e^{-(n+1)(h+\varepsilon)} \leq s_j(\widetilde{X_a}) = \mu([aw]) \leq e^{-(n+1)(h-\varepsilon)}.$$

Assume that there are $m > n$ such that $s_j(\widetilde{X_a}) = \mu([aw])$ for some $w \in D_m$ and $j \leq d_n$. Then it holds that

$$e^{-(m+1)(h-\varepsilon)} \geq e^{-(n+1)(h+\varepsilon)}. \tag{$\star\star$}$$

Indeed, if $e^{-(m+1)(h-\varepsilon)} < e^{-(n+1)(h+\varepsilon)}$, then

$$s_j(\widetilde{X_a}) = \mu([aw]) \leq e^{-(m+1)(h-\varepsilon)} < e^{-(n+1)(h+\varepsilon)} \leq \mu([au]),$$

for all $u \in D_k$ $(1 \leq k \leq n)$. However, by our assumption, we have $\mu([av]) \leq s_j(\widetilde{X_a}) = \mu([aw])$ for some $v \in D_l$ and $1 \leq l \leq n$. This is a contradiction.

Hence, by ($\star\star$), we have

$$m + 1 \leq (n+1)\frac{h+\varepsilon}{h-\varepsilon}.$$

Let $k \in \mathbb{N}$ with

$$(n+1)\frac{h+\varepsilon}{h-\varepsilon} - 1 < k + 1 \leq (n+1)\frac{h+\varepsilon}{h-\varepsilon}.$$

Since

$$\begin{aligned}\frac{\sum_{j=1}^{d_n} s_j(\widetilde{X_a})}{\sum_{j=1}^{d_n} 1/j} &\leq \frac{\sum_{i=0}^{k}\sum_{w\in D_i}\mu([aw])}{\log d_n}\\ &\leq \frac{\sum_{i=0}^{k}\mu([a])}{\log d_n}\\ &\leq \frac{n+1}{\log d_n}\cdot\frac{h+\varepsilon}{h-\varepsilon}\mu([a]),\end{aligned}$$

we obtain

$$||\widetilde{X_a}||_1^{\widetilde{+}} \leq \limsup_{n\to\infty}\frac{n+1}{\log d_n}\cdot\frac{h+\varepsilon}{h-\varepsilon}\mu([a]).$$

Moreover, because

$$\mu([a]) = \sum_{w\in D_n}\mu([aw]) + \sum_{w\in\mathcal{W}_n(X)\setminus D_n}\mu([aw]) \leq |D_n|e^{-(n+1)(h-\varepsilon)} + \varepsilon_n,$$

we have

$$(\mu([a]) - \varepsilon_n)\, e^{(n+1)(h-\varepsilon)} \leq |D_n|.$$

Note that $\varepsilon_n \to 0$ $(n\to\infty)$ by $(\star)$. Therefore

$$\begin{aligned}||\widetilde{X_a}||_1^{\widetilde{+}} &\leq \limsup_{n\to\infty}\frac{n+1}{\log|D_n|}\cdot\frac{h+\varepsilon}{h-\varepsilon}\mu([a])\\ &\leq \limsup_{n\to\infty}\frac{n+1}{\log(\mu([a])-\varepsilon_n)+(n+1)(h-\varepsilon)}\cdot\frac{h+\varepsilon}{h-\varepsilon}\mu([a])\\ &= \frac{h+\varepsilon}{(h-\varepsilon)^2}\mu([a]).\end{aligned}$$

Since ε is arbitrary, we have

$$||X_a||_1^{\widetilde{+}} \leq \frac{1}{h}\mu([a]).$$

By Proposition 2.2, the proof is complete. □

We now give some examples of subshifts with a maximal measure satisfying the condition in Theorem 3.2.

Corollary 3.3. *Let A be a 0-1 $N\times N$ matrix. We denote by Σ_A the Markov shift associated with A, i.e.*

$$\Sigma_A = \{(a_i)_{i\in\mathbb{Z}} \in S^{\mathbb{Z}} \mid A(a_i, a_{i+1}) = 1\},$$

where $S = \{1,\ldots,N\}$ is an alphabet. If $\tau = (T_a)_{a\in S}$ is the creation operators for the Markov shift Σ_A, then we have

$$k_\infty^-(\tau) = h_{\mathrm{top}}(\Sigma_A).$$

Proof. It suffices to show that the unique maximal measure of Σ_A satisfies the condition in Theorem 3.2. For simplicity, we may assume that A is irreducible with the Perron value α. Note that the topological entropy $h_{\mathrm{top}}(\Sigma_A)$ is equal to $\log\alpha$. If l and r are the left and right Perron vectors with $\sum_{a=1}^N l_a r_a = 1$, then the unique maximal measure μ is given by

$$\mu([a_0, a_1, \ldots, a_n]) = \frac{l_{a_0} r_{a_n}}{\alpha^n},$$

where $(a_0, a_1, \ldots, a_n) \in \mathcal{W}_{n+1}(\Sigma_A)$ (e.g. see [Kit]). For any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for any $n \geq K$, we have

$$\left|\frac{\log l_a r_b \alpha}{n+1}\right| < \varepsilon,$$

for all $1 \leq a, b \leq N$. Therefore for any $w \in \mathcal{W}_{n+1}(\Sigma_A)$, we have

$$\left|-\frac{1}{n+1}\log\mu([w]) - \log\alpha\right| < \varepsilon,$$

for all $n \geq K$, i.e. the maximal measure μ satisfies the condition in Theorem 3.2. □

More generally, there is a class of subshifts, which is called almost Sofic (see [Pet]). A subshift X is said to be *almost Sofic* if for any $\varepsilon > 0$, there is a SFT $\Sigma \subseteq X$ such that $h_{\mathrm{top}}(X) - \varepsilon < h_{\mathrm{top}}(\Sigma)$, where a *shift of finite type* or SFT is a subshift that can be described by a finite set of forbidden words, i.e. a subshift having the form $X_{\mathcal{F}}$ for some finite set $\mathcal{F}$ of words. Every SFT is topologically conjugate to a Markov shift. However, it is not trivial that $h_{\mathrm{top}}(\Sigma) = k_\infty^-(\tau)$ holds for a SFT Σ.

Corollary 3.4. *If $\tau = (T_a)_{a\in\mathcal{A}}$ is the creation operators for a SFT Σ, then we have*

$$k_\infty^-(\tau) = h_{\mathrm{top}}(\Sigma).$$

Proof. We recall that every SFT Σ is topologically conjugate to a Markov shift Σ_A associated with a 0-1 matrix A. Now we give a short proof of this result. Let Σ be a SFT that can be described by a finite set $\mathcal{F}$ of forbidden words. We may assume that all words in $\mathcal{F}$ have length $N+1$. We set $\mathcal{A}_\Sigma^{[N]} = \mathcal{W}_N(\Sigma)$ and the block map $\varphi : \mathcal{W}_N(\Sigma) \to \mathcal{A}_\Sigma^{[N]}$, $w \mapsto w$. We define the *N-th higher block code* $\beta_N : \Sigma \to (\mathcal{A}_\Sigma^{[N]})^{\mathbb{Z}}$ by

$$(\beta_N(x))_i = (x_i, \ldots, x_{i+N-1}) \in \mathcal{A}_\Sigma^{[N]},$$

for $x = (x_i)_{i\in\mathbb{N}} \in \Sigma$. Note that β_N is the sliding block code with respect to φ. The subshift $\beta_N(\Sigma)$ is given by a Markov shift, i.e. there is a 0-1 matrix A with $\beta_N(\Sigma) = \Sigma_A$.

Let μ be the maximal measure of Σ_A. The maximal measure of Σ is given by $\nu = \mu \circ \beta_N$. We recall that μ is the Markov measure given by the left and right eigenvectors l, r and the eigenvalue α. For $w \in \mathcal{W}_n(\Sigma)$ with $n \geq N$, we have

$$\begin{aligned} \nu([w]) &= \mu([\varphi(w_{[1,N]}), \ldots, \varphi(w_{[n-N+1,n]})]) \\ &= \frac{l_a r_b}{\alpha^{n-N}}, \end{aligned}$$

where $a = \varphi(w_{[1,N]})$, $b = \varphi(w_{[n-N+1,n]})$ and $w_{[k,l]} = (w_k, \ldots, w_l)$ for $k \leq l$. Hence one can show that the maximal measure ν of Σ satisfies the condition in Theorem 3.2 by the same argument as in the proof of Corollary 3.3. □

Corollary 3.5. *Let X be a almost Sofic shift. If $\tau = (T_a)_{a \in \mathcal{A}}$ is the creation operators for X, then we have*

$$k_\infty^-(\tau) = h_{\text{top}}(X).$$

Proof. Let $\varepsilon > 0$. Since X is almost Sofic, there is a SFT $\Sigma \subseteq X$ such that $h_{\text{top}}(X) - \varepsilon < h_{\text{top}}(\Sigma)$. Let $\varphi : \Sigma \to X$ be an embedding. Note that the subshift $\varphi(\Sigma)$ is also a SFT. Thus we may identify $\varphi(\Sigma)$ with Σ. Let μ be the unique maximal measure of Σ. For $a \in \mathcal{A}$, we set

$$X_a = \sum_{n \geq 0} \sum_{w} \mu([aw]) T_w P_0 T_{aw}^*,$$

where w runs over all elements in $\mathcal{W}_n(\Sigma)$ with $aw \in \mathcal{W}(\Sigma)$. We have shown that the maximal measure μ of Σ satisfies the condition of Theorem 3.2 in the proof of Corollary 3.4. Hence by the same argument as in the proof of Theorem 3.2, we have

$$h_{\text{top}}(\Sigma) \leq k_\infty^-(\tau).$$

Thus for arbitrary $\varepsilon > 0$, the following holds:

$$h_{\text{top}}(X) - \varepsilon < h_{\text{top}}(\Sigma) \leq k_\infty^-(\tau).$$

It therefore follows from Proposition 3.1 that $h_{\text{top}}(X) = k_\infty^-(\tau)$ if X is an almost Sofic shift. □

For $\beta > 1$, the β-transformation T_β on the interval $[0, 1]$ is defined by the multiplication with β (mod 1), i.e. $T_\beta(x) = \beta x - [\beta x]$, where $[t]$ is the integer part of t. Let $N \in \mathbb{N}$ with $N - 1 < \beta \leq N$ and $\mathcal{A} = \{0, 1, \ldots, N-1\}$. The *$\beta$-expansion* of $x \in [0, 1]$ is a sequence $d(x, \beta) = \{d_i(x, \beta)\}_{i \in \mathbb{N}}$ of $\mathcal{A}$ determined by

$$d_i(x, \beta) = [\beta T_\beta^{i-1}(x)].$$

We set

$$\zeta_\beta = \left\{ \sup_{x\in[0,1)} d_i(x,\beta) \right\}_{i\in\mathbb{N}},$$

and we define the shift invariant closed subset Σ_β^+ of the full one-sided shift $\mathcal{A}^{\mathbb{N}}$ by

$$\Sigma_\beta^+ = \left\{ x \in \mathcal{A}^{\mathbb{N}} \mid \sigma^i(x) \le \zeta_\beta,\ i = 0, 1, \dots \right\},$$

where $\le$ is the lexicographical order on $\mathcal{A}^{\mathbb{N}} = \{0, 1, \dots, N-1\}^{\mathbb{N}}$. The *$\beta$-shift* Σ_β is the natural extension given by

$$\Sigma_\beta = \left\{ (x_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} \mid (x_i)_{i\ge k} \in \Sigma_\beta^+,\ k \in \mathbb{Z} \right\}.$$

It is known that $h_{\text{top}}(\Sigma_\beta) = \log\beta$, (see [Hof]).

The following result might be known among specialists. However, we give a proof here as we can not find it in the literature.

Proposition 3.6. *For $\beta > 1$, the β-shift Σ_β is an almost Sofic shift.*

Proof. In [Par], it is shown that Σ_β is a SFT if and only if $d(1,\beta)$ is finite, i.e. there is $K \in \mathbb{N}$ such that $d_k(1,\beta) = 0$ for all $k \ge K$. Thus we may assume that $d(1,\beta)$ is not finite. Let $\zeta_\beta = (\xi_i)_{i\in\mathbb{N}}$. For $n \in \mathbb{N}$, there is $\beta(n) < \beta$ such that

$$1 = \frac{\xi_1}{\beta(n)} + \frac{\xi_2}{\beta(n)^2} + \cdots + \frac{\xi_n}{\beta(n)^n}.$$

In [Par, Theorem 5], it is proved that

$$\lim_{n\to\infty} \beta(n) = \beta.$$

Hence we may assume that $N - 1 \le \beta(n) < \beta$ for sufficiently large n. Since the maximal element $\zeta_{\beta(n)}$ has the form

$$(\xi_1, \xi_2, \dots, (\xi_n - 1), \xi_1, \xi_2, \dots, (\xi_n - 1), \xi_1, \dots),$$

we have $\zeta_{\beta(n)} < \zeta$, where $<$ is the lexicographical order. Therefore we obtain

$$\Sigma_{\beta(n)}^+ \subseteq \Sigma_\beta^+ \subseteq \{0, 1, \dots, N-1\}^{\mathbb{N}}.$$

It follows that $\Sigma_{\beta(n)}$ is the shift invariant closed subset of Σ_β with topological entropy $\log\beta(n)$. Since $d(1,\beta(n))$ is finite, the subshift $\Sigma_{\beta(n)}$ is a SFT. It therefore follows form [Par, Theorem 5] that Σ_β is an almost Sofic. □

Hence it holds that $k_\infty^-(\tau) = h_{\text{top}}(\Sigma_\beta)$ for every β-shift by Corollary 3.5.

Corollary 3.7. *Let Σ_β be the β-shift for $\beta > 1$. If $\tau = (T_a)_{a\in\mathcal{A}}$ is the creation operators for Σ_β, then we have*

$$k_\infty^-(\tau) = h_{\text{top}}(\Sigma_\beta) = \log\beta.$$

4. Groups and Macaev Norm

We discuss a relation between groups and the Macaev norm. Let Γ be a countable finitely generated group, S a symmetric set of generators of Γ. We denote by $|\cdot|_S$ the word length and by $\mathcal{W}_n(\Gamma, S)$ the set of elements in Γ with length n, with respect to the system of generators S. The *logarithmic volume* of a group Γ in a given system of generators S is the number

$$v_S = \lim_{n\to\infty} \frac{\log|\mathcal{W}_n(\Gamma, S)|}{n},$$

(cf. [Ver]). The following proposition can be proved in the same way as in the free group case [Voi3, Proposition 3.7. (a)].

Proposition 4.1. *Let Γ be a finitely generated group with a finite generating set S and λ the left regular representation of Γ. If we set $\lambda_S = (\lambda_a)_{a\in S}$, then*

$$k_\infty^-(\lambda_S) \le v_S.$$

Proof. Let us denote by P_n the projection onto the subspace $\text{span}\{\delta_g \in l^2(\Gamma) \mid |g|_S = n\}$. If we set

$$X_n = \sum_{j=0}^{n-1}\left(1 - \frac{j}{n}\right)P_j,$$

then we have

$$||X_n\lambda_a - \lambda_a X_n|| = ||\lambda_a^* X_n \lambda_a - X_n|| \le \frac{1}{n}.$$

for $a \in S$. Hence

$$k_\infty^-(\lambda_S) \le \limsup_{n\to\infty}\max_{a\in S}|||[X_n,\lambda_a]|||_\infty^- \le \lim_{n\to\infty}\frac{\log\sum_{j=0}^n|\mathcal{W}_n(\Gamma,S)|}{n} = v_S.$$

$\square$

Now we compute the exact value of $k_\infty^-(\lambda_S)$ for certain amalgamated free product groups.

Proposition 4.2. *Let A be a finite group, $G_1, \cdots, G_M$ nontrivial finite groups containing A as a subgroup and $H_1, \ldots, H_N$ the product group of the infinite cyclic group $\mathbb{Z}$ and the finite group A, $(N + M > 1)$. Let Γ be the amalgamated free product group of $G_1, \cdots, G_M, H_1, \cdots, H_N$ with amalgamation over A. Set $S = G_1 \cup \cdots \cup G_M \cup (S_1 \times A) \cup \cdots \cup (S_N \times A) \setminus \{e\}$, where S_j is the canonical generating set $\{x_j, x_j^{-1}\}$ of* *the infinite cyclic group* $\mathbb{Z}$ *and* e *is the group unit. Let* λ *be the left regular representation of* Γ *and* $\lambda_S = (\lambda_a)_{a\in S}$*. Then we have*

$$k_\infty^-(\lambda_S) = v_S.$$

In particular, for the free group $\mathbb{F}_N$ $(N \geq 2)$*, we have*

$$k_\infty^-(\lambda_S) = \log(2N-1).$$

Proof. By Proposition 4.1, it suffices to show that $v_S \leq k_\infty^-(\lambda_S)$. Let Ω_i be the set of the representatives of G_i/A with $e \in \Omega_i$ for $i = 1, \ldots, M$. We identify x_j with $(x_j, e) \in H_j$ for $j = 1, \ldots, N$, and set $\Omega_{M+j} = \{x_j, x_j^{-1}, e\}$. Let

$$\widetilde{S} = \bigcup_{i=1}^{M+N} \Omega_i \setminus \{e\}.$$

We define the 0-1 matrix A with index $\widetilde{S}$ by

$$A(a,b) = \begin{cases} 1 & \text{if } |ab|_S = 2; \\ 0 & \text{otherwise.} \end{cases}$$

One can easily check that the above matrix A is irreducible and the topological entropy $h_{\text{top}}(\Sigma_A)$ of the Markov shift Σ_A coincides with the logarithmic volume v_S of Γ with respect to the generating set S.

We denote by Γ_0 the subset of Γ consisting of the group unit e and elements $a_1 \cdots a_n \in \Gamma, (n \in \mathbb{N})$ of the form

$$\begin{cases} a_k \in \Omega_{i_k} \setminus \{e\} & \text{for } \; k = 1, \ldots, n, \\ i_k \neq i_{k+1} & \text{if } \; 1 \leq i_k \leq M, \\ a_k = a_{k+1} & \text{if } \; M+1 \leq i_k \leq M+N, i_k = i_{k+1}. \end{cases}$$

Note that the subspace $l^2(\Gamma_0)$ can be identified with the Fock space $\mathcal{F}_A$ of the Markov shift Σ_A by the following correspondence:

$$\begin{aligned} \delta_e &\longleftrightarrow \xi_0, \\ \delta_{a_1\cdots a_n} &\longleftrightarrow \xi_{a_1} \otimes \cdots \otimes \xi_{a_n}. \end{aligned}$$

Let us denote by P_n the projection onto the subspace

$$\overline{\text{span}}\{\delta_g \in l^2(\Gamma) \mid |g|_S = n\}.$$

For $a \in S$, we define the partial isometry $T_a \in \mathbb{B}(l^2(\Gamma))$ by

$$T_a = \sum_{n\geq 0} P_{n+1}\lambda_a P_n.$$

Under the identification with $\mathcal{F}_A$, the partial isometry $T_a|_{l^2(\Gamma_0)}$ for $a \in \widetilde{S}$ is the creation operator on $\mathcal{F}_A$, (cf. [Oka]). We also identify Γ_0 and

$\mathcal{W}(\Sigma_A)$. For $w = a_1 \cdots a_n \in \Gamma_0$, we set $T_w = T_{a_1} \cdots T_{a_n}$. Let μ be the maximal measure of Σ_A. For $a \in \widetilde{S}$, we put

$$X_a = \sum_{n \geq 0} \sum_{w} \mu([aw]) T_w P_0 T_{aw}^*,$$

where w runs over all $w \in \Gamma_0$ with $|w| = n$ and $|aw| = |w| + 1$. For $a \in S \setminus \widetilde{S}$, we set $X_a = 0$. It can be easily checked that $[\lambda_a, X_a] = [T_a, X_a]$ for $a \in \widetilde{S}$. Therefore by the same proof as in the sufshift case, we obtain

$$v_S = h_{\text{top}}(\Sigma_A) = k_\infty^-(\lambda_S).$$

□

Remark 4.3. Let Γ be a finitely generated group with a finite generating set S. In [Voi5], Voiculescu proved that if the entropy $h(\Gamma, \mu)$ of a random walk μ on Γ with support S is non-zero, then $k_\infty^-((\lambda_a)_{a \in S})$ is non-zero. However the above proposition suggests that the volume v_S of Γ is more related to the invariant $k_\infty^-((\lambda_a)_{a \in S})$ rather than the entropy $h(\Gamma, \mu)$. It is an interesting problem to ask whether v_S being non-zero implies $k_\infty^-((\lambda_a)_{a \in S})$ being non-zero. We also remark here that there is a relation between v_S and $h(\Gamma, \mu)$: If $h(\Gamma, \mu) \neq 0$, then $v_S \neq 0$, (see [Ver, Theorem 1]). If the above mentioned problem was solved affirmatively, then it would follow from Proposition 4.1 that $k_\infty^-((\lambda_a)_{a \in S}) \neq 0$ if and only if $v_S \neq 0$, i.e. Γ has exponential growth.

References

[DGS] Denker, M.; Grillenberger, C.; Sigmund, K.: *Ergodic theory on compact spaces.* Lecture Notes in Mathematics, Vol. **527**. Springer-Verlag, Berlin-New York, 1976.

[DV] David, G.; Voiculescu, D.:*s-numbers of singular integrals for the invariance of absolutely continuous spectra in fractional dimensions.* J. Funct. Anal. **94** (1990), no. 1, 14–26.

[GK] Gohberg, I. C.; Kreĭn, M. G.: *Introduction to the theory of linear nonselfadjoint operators.* Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. **18** American Mathematical Society, Providence, R.I. 1969.

[Hof] Hofbauer, F.: *β-shifts have unique maximal measure.* Monatsh. Math. **85** (1978), no. 3, 189–198.

[Kit] Kitchens, B.P.: *Symbolic dynamics. One-sided, two-sided and countable state Markov shifts*, Universitext. Springer-Verlag, Berlin, 1998.

[LM] Lind, D.; Marcus, B.: *An introduction to symbolic dynamics and coding.* Cambridge University Press, Cambridge, 1995.

[Mat] Matsumoto, K.: *On C^*-algebras associated with subshifts.* Internat. J. Math. **8** (1997), no. 3, 357–374.

[Oka] Okayasu, R.: *Cuntz-Krieger-Pimsner algebras associated with amalgamated free product groups.* Publ. Res. Inst. Math. Sci. **38** (2002), no. 1, 147–190.

[Par] Parry, W.: *On the β-expansions of real numbers.* Acta Math. Acad. Sci. Hungar. **11** (1960) 401–416.

[Pet] Petersen, K.: *Chains, entropy, coding.* Ergodic Theory Dynam. Systems **6** (1986), no. 3, 415–448

[Ver] Vershik, A.M.: *Dynamic theory of growth in groups: entropy, boundaries, examples.* Russian Math. Surveys **55** (2000), no. 4, 667–733.

[Voi1] Voiculescu, D.: *Some results on norm-ideal perturbations of Hilbert space operators.* J. Operator Theory **2** (1979), no. 1, 3–37.

[Voi2] Voiculescu, D.: *Some results on norm-ideal perturbations of Hilbert space operators. II.* J. Operator Theory **5** (1981), no. 1, 77–100.

[Voi3] Voiculescu, D.: *On the existence of quasicentral approximate units relative to normed ideals. Part I.* J. Funct. Anal. **91** (1990), no. 1, 1–36.

[Voi4] Voiculescu, D.: *Entropy of dynamical systems and perturbations of operators.* Ergodic Theory Dynam. Systems **11** (1991), no. 4, 779–786.

[Voi5] Voiculescu, D.: *Entropy of random walks on groups and the Macaev norm.* Proc. Amer. Math. Soc. **119** (1993), no. 3, 971–977.

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