FINITE GROUP ACTIONS ON C^*-ALGEBRAS WITH THE ROHLIN PROPERTY, I

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Abstract. Basic properties of finite group actions with the Rohlin property on unital C^*-algebras are investigated. A characterization of finite group actions with the Rohlin property on the Cuntz algebra $\mathcal{O}_2$ is given in terms of central sequences, which may be considered as an equivariant version of E. Kirchberg and N. C. Phillips' characterization of $\mathcal{O}_2$. A large class of symmetries on $\mathcal{O}_2$ are classified in terms of the fixed point algebras for conjugacy and the crossed products for cocycle conjugacy. Model actions of symmetries of $\mathcal{O}_2$ are constructed for given K-theoretical invariants.

1. Introduction

Group actions on C^*-algebras and von Neumann algebras are one of the most fundamental subjects in the theory of operator algebras. Even if one focuses on classification of finite or compact group actions on C^*-algebras, a number of results are found in the literature [27], [28], [29], [30], [42], [6], [21], [24], [60], [25], [26]. However, "classifiable classes" known so far are very restricted and they are divided into the following two classes: inductive limit actions on AF or A$\mathbb{T}$ algebras and those coming from the linear transformations of generators of the Cuntz algebras [14] or the Cuntz-Krieger algebras [17]. In particular, most of the above papers deal with so-called locally representable actions, a special class of inductive limit actions, and their classification invariants are given by the K-theory of the crossed products and the dual actions.

On the other hand, due to the recent progress of G. E. Elliott's classification program [19] of amenable C^*-algebras, "classifiable classes" of C^*-algebras are no longer specified by the way they are constructed. There are two such classes: separable purely infinite simple nuclear C^*-algebras satisfying the universal coefficient theorem (abbreviated as UCT in the sequel) [52], and separable simple nuclear C^*-algebras of tracial topological rank zero satisfying the UCT [46]. (It is practically impossible to exhaust all the contributions to this field here, and interested readers are recommended to consult recent monographs [47], [56] and references in them.) In the study of single automorphisms of C^*-algebras, a similar progress has been made in the last decade. Here, the "classifiable classes" consist of automorphisms with the Rohlin property [43], [44], [51] (see also [35] and references in it).

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One of the purposes of this series of papers is to provide a new "classifiable class" of finite group actions and to establish fundamental results about them. More specifically, we investigate finite group actions with the Rohlin property (Definition 3.1), a notion which has already appeared in [40], [31], [32] with a different term. Indeed, R. Herman and V. F. R. Jones investigated in [31], [32] some class of finite group actions with what we call here the Rohlin property. Since they followed the model action splitting argument developed in the von Neumann algebra setting [11], [12], [37], they were forced to treat a rather small class of actions from the view point of the "post classification era". In order to obtain an outer conjugacy result of automorphisms with the Rohlin property, D. E. Evans and A. Kishimoto introduced in [23] an intertwining argument for automorphisms, an equivariant version of Elliott's intertwining argument for classification of C^*-algebras. A variant of the Evans-Kishimoto intertwining argument enables us to compare two finite group actions with the Rohlin property directly without using the model action splitting argument, and to obtain a conjugacy result (Theorem 3.5). It turns out that when the group is finite abelian, the dual action of every locally representable action has the Rohlin property, which also explains why the locally representable actions are classifiable in terms of the K-theoretical invariants of the crossed products and the dual actions.

One of the difficulties in the study of finite group actions on C^*-algebras is that one can hardly control the K-groups of the crossed products, not as in the case of the integer group actions [54] or the real number group actions [13]. Therefore, it is reasonable for us to begin our analysis with the Cuntz algebra $\mathcal{O}_2$, as a test case, whose K-theory is known to be trivial [15]. Thanks to E. Kirchberg and N. C. Phillips [39], $\mathcal{O}_2$ is characterized in terms of central sequences, which has several important consequences. For example, it follows that the tensor product of $\mathcal{O}_2$ and any separable simple unital nuclear C^*-algebra is isomorphic to $\mathcal{O}_2$ [39]. We give a similar characterization of finite group actions on $\mathcal{O}_2$ with the Rohlin property in terms of relative central sequences, a notion first introduced by D. Bisch in the subfactor context [1]. We show the Rohlin property for several concrete examples of finite group actions on $\mathcal{O}_2$, which does not seem to be possible without using our characterization. We classify a large class of symmetries on $\mathcal{O}_2$ up to conjugacy (respectively cocycle conjugacy) in terms of the fixed point algebras (respectively the crossed products) using the fact that their dual actions have the Rohlin property. For a given pair of uniquely 2-divisible countable abelian groups, we also construct a model action with the K-groups of the crossed product isomorphic to the prescribed ones. Unique 2-divisibility here is a necessary constraint pointed out by B. Blackadar [3], and he also constructed model actions for uniquely 2-divisible torsion groups.

This paper is organized as follows. In Section 2, we collect basic facts on group actions, central sequences, and purely infinite C^*-algebras. We give a characterization of the first cohomology of some class of compact group actions in terms of K-theory.

In Section 3, we introduce the Rohlin property, and deduce basic properties of actions possessing it. We also introduce two classes of finite abelian group actions, strongly approximately inner actions and approximately representable actions. The latter includes all locally representable actions, and their dual actions have the Rohlin property. We give a strong K-theoretical constraint for the Rohlin property using our K-theoretical characterization of the first cohomology.

Section 4 is devoted to the analysis of finite group actions on $\mathcal{O}_2$ described above. In Section 5, we treat examples, mainly coming from quasi-free actions of finite group on the Cuntz algebra $\mathcal{O}_n$.

We try to collect all fundamental results related to the Rohlin property for finite group actions, and some results are included even though they have no contribution to the main results of the present paper. The second part of these serial works will be devoted to the classification of finite group actions with the Rohlin property in terms of K-theoretical invariants, and part of the purposes of the present paper is to establish fundamental results used in the second part.

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2. Preliminaries

In this section, we collect notations and basic facts frequently used in the paper.

2.1. **Notations.** C^*-algebras are always assumed to be unital unless particularly mentioned. The symbol $\otimes$ stands for the minimal tensor product of C^*-algebras. For a C^*-algebra A, we denote by $U(A)$ the unitary group of A, and by $\mathrm{Aut}(A)$ the automorphism group of A. $\mathrm{Aut}(A)$ is equipped with the topology of pointwise norm convergence. We denote by $\mathrm{Inn}(A)$ and $\overline{\mathrm{Inn}(A)}$ the inner automorphism group and the approximately inner automorphism group of A respectively. Two automorphisms α and β are said to be approximately unitary equivalent if $\alpha\cdot\beta^{-1}\in\overline{\mathrm{Inn}(A)}$.

For a topological group G, a G-action α on A is a continuous homomorphism from G into $\mathrm{Aut}(A)$. For a G-action (or a single automorphism) α on A, A^α denotes the fixed-point subalgebra of A under α. Two G-actions α on A and β on B are said to be *conjugate* if there exists an isomorphism θ from A onto B such that $\theta\cdot\alpha_g=\beta_g\cdot\theta$ holds for every $g\in G$.

A continuous map u from G to the unitary group $U(A)$ of A is said to be an α-cocycle if the following cocycle relation holds:

$$u(g)\alpha_g(u(h))=u(gh),\quad g,h\in G.$$

When u is an α-cocycle, we define the perturbed action α^u by

$$\alpha^u_g=\mathrm{Ad}(u(g))\cdot\alpha_g.$$

There exists a natural isomorphism from $A\rtimes_{\alpha^u}G$ onto $A\rtimes_\alpha G$ that intertwines the dual actions (or the dual coactions if the group is not commutative) [50]. α and β are said to be *cocycle conjugate* if there exists an α-cocycle u such that α^u and β are conjugate. Two α-cocycles u and w are said to be *equivalent* (or *cohomologous*) if there exists a unitary v satisfying

$$w(g)=vu(g)\alpha_g(v^*),\quad g\in G.$$

An α-cocycle equivalent to 1 is said to be a coboundary. We denote by $Z^1_\alpha(G,U(A))$ the set of α-cocycles, and by $H^1_\alpha(G,U(A))$ the quotient of $Z^1_\alpha(G,U(A))$ by the above equivalence relation, which is called the first cohomology of α. When $H^1_\alpha(G,U(A))$ is trivial, α is said to be *stable*.

A cocycle G-action α on A is a continuous map α from G into $\mathrm{Aut}(A)$ with a 2-cocycle u, a continuous map from $G\times G$ into $U(A)$, satisfying

$$\alpha_g\cdot\alpha_h=\mathrm{Ad}(u(g,h))\cdot\alpha_{gh},\quad g,h\in G,$$

$$u(g,h)u(gh,k)=\alpha_g(u(h,k))u(g,hk),\quad g,h,k\in G.$$

Two cocycle actions (α,u) and (β,v) on A are said to be *equivalent* if there exists a continuous map v from G into $U(A)$ such that the following hold:

$$\alpha_g=\mathrm{Ad}(v(g))\cdot\beta_g,\quad g\in G,$$

$$u(g,h)=\alpha_g(v(h))v(g)w(g,h)v(gh)^*,\quad g,h\in G.$$

When (α,u) is equivalent to a genuine action, i.e. there exists a continuous map v from G into $U(A)$ such that

$$u(g,h)=\alpha_g(v(h))v(g)v(gh)^*,\quad g,h\in G$$

holds, u is said to be a *coboundary*.

For a homomorphism φ from a C^*-algebra A to another C^*-algebra B, we denote by $K_i(\varphi)$, $i=0,1$, the homomorphism from $K_i(A)$ to $K_i(B)$ induced by φ.

For a C^*-algebra A, we set

$$c_0(A)=\{(a_n)\in\ell^\infty(\mathbb{N},A);\ \lim_{n\to\infty}||a_n||=0\},$$

$$A^\infty=\ell^\infty(\mathbb{N},A)/c_0(A).$$

We identify A with the C^*-subalgebra of A^∞ consisting of the equivalence classes of constant sequences, and set

$$A_\infty=A^\infty\cap A'.$$

The usual perturbation argument shows that every unitary (respectively projection) in A^∞ has a representing sequence in $\ell^\infty(\mathbb{N},A)$ consisting of unitaries (respectively projections). For an automorphism $\alpha\in\mathrm{Aut}(A)$, we denote by α^∞ and α_∞ the natural automorphisms of A^∞ and A_∞ induced by α respectively.

For a free ultrafilter $\omega\in\beta\mathbb{N}\setminus\mathbb{N}$, we employ similar notations:

$$c_\omega(A)=\{(a_n)\in\ell^\infty(\mathbb{N},A);\ \lim_{n\to\omega}||a_n||=0\},$$

$$A^\omega=\ell^\infty(\mathbb{N},A)/c_\omega(A).$$

$$A_\omega=A^\omega\cap A'.$$

α^ω (respectively α_ω) denotes the automorphisms of A^ω (respectively A_ω) induced by α as before. For a (cocycle) group action α, we also use the notations α^∞, α_∞, α^ω, α_ω in an obvious sense. Note that even when α is a cocycle action, α_∞ and α_ω are genuine actions as far as the group is discrete.

We denote by $\mathbb{M}_n$, $\mathbb{K}$, $\mathbb{M}_{n^\infty}$ $(n\in\mathbb{N})$, and $\mathcal{O}_n$ $(n=2,3,\cdots,\infty)$ the n by n matrix algebra, the C^*-algebra of all compact operators on $\ell^2(\mathbb{N})$, the UHF algebra of type n^∞, and the Cuntz algebra with the canonical generators $\{S_i\}_{i=1}^n$, [14] respectively.

For a natural number n, the cyclic group of order n is denoted by $\mathbb{Z}_n$. For a $\mathbb{Z}_n$-action α, we abuse the notation and denote α_1 by α for simplicity. An additive group Γ is said to be *uniquely n-divisible* if multiplying by n is a group automorphism of Γ. For a finite group G, $|G|$ denotes the order of G.

2.2. **Purely infinite C^*-algebras.** A simple (not necessarily unital) C^*-algebra A is said to be *purely infinite* if for every pair of non-zero elements $x, y \in A$, there exist $a, b \in A$ such that $y = axb$. There are a number of important K-theoretical properties fulfilled by purely infinite C^*-algebras (proven in [15]), and we use them freely without particularly mentioning them. A purely infinite simple unital C^*-algebra is said to be *in the Cuntz standard form* if $[1] = 0$ in $K_0(A)$.

Pure infiniteness is a property preserved by passing to the crossed product and the fixed-point subalgebra for a finite group action [45], [38], or more generally, it is a property inherited through an inclusion with a finite index [36].

Lemma 2.1. *Let α be an outer action of a finite group G on a purely infinite simple C^*-algebra A. Then, the fixed-point algebra A^α and the crossed product $A \rtimes_\alpha G$ are purely infinite simple.*

The following statements play important roles in our analysis of finite group actions on the Cuntz algebra $\mathcal{O}_2$, whose proofs are found in [39, Proposition 3.4] and [51, Lemma 2].

Proposition 2.2. *Let A be a separable purely infinite simple unital nuclear C^*-algebra, and $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. Then,*

(1) *A_ω is purely infinite simple.*
(2) *$\alpha \in \mathrm{Aut}(A)$ is outer if and only if α_ω is outer.*

(1) of the above proposition immediately implies the following lemma (note that B need not be separable, cf. [5, Theorem 1,4, (a)]):

Lemma 2.3. *Let A be a separable purely infinite simple nuclear C^*-algebra and B be a simple C^*-algebra. Then, the tensor product $A \otimes B$ is purely infinite.*

2.3. **1-cocycles.** The following lemma is probably a folklore among specialists (a proof in the finite group case is found in [32, Lemma II. 1]).

Lemma 2.4. *Let A be a unital C^*-algebra, and α be a continuous action of a compact group G on A. If u and v are α-cocycles satisfying*

$$||u(g) - v(g)|| < 1, \quad \forall g \in G,$$

then u and v are equivalent.

The next result gives a K-theoretical description of the first cohomology for a compact group action.

Proposition 2.5. *Let A be a purely infinite simple unital C^*-algebra, and α be a faithful continuous action of a compact group G on A. We assume that the crossed product $A \rtimes_\alpha G$ is purely infinite simple (this is always the case if G is a finite group* *and α is outer thanks to Lemma 2.1). Then, there exists a one-to-one correspondence between $H^1_\alpha(G, U(A))$ and the following set*

$$\mathrm{Ker}(K_0(\iota)) + [1] = \{x \in K_0(A^\alpha);\ K_0(\iota)(x) = [1]\},$$

where ι is the inclusion map of A^α into A. In consequence, $K_0(\iota)$ is injective if and only if α is stable.

Proof. Let $x \in \mathrm{Ker}(K_0(\iota)) + [1]$, and e a non-zero projection in A^α representing x. Since $[e] = [1]$ in $K_0(A)$, there exists an isometry $w \in A$ satisfying $ww^* = e$. We set $u(g) = w^*\alpha_g(w)$. Then, u is an α-cocycle. We show that the map sending x to the cohomology class of u is well-defined and that it gives the desired one-to-one correspondence.

First, we claim that for every α-cocycle u, there exists an isometry $w \in A$ such that $ww^* \in A^\alpha$ and $u(g) = w^*\alpha_g(w)$. Following A. Connes' argument [12], we introduce an action θ of G on $B := A \otimes \mathbb{M}_2$ by

$$\theta_g = \mathrm{Ad}\big(\begin{pmatrix} 1 & 0 \\ 0 & u(g) \end{pmatrix}\big) \cdot (\alpha_g \otimes \mathrm{Id}).$$

Since a cocycle perturbation does not change the isomorphism class of the crossed products, $B \rtimes_\theta G$ is isomorphic to $(A \rtimes_\alpha G) \otimes \mathbb{M}_2$, which is purely infinite simple by assumption. Being a corner of a purely infinite simple C^*-algebra, B^θ is purely infinite simple as well. Thus, there exists a partial isometry $V \in B^\theta$ such that

$$V^*V \leq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$VV^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The above relations imply that V is of the form

$$V = \begin{pmatrix} 0 & 0 \\ w^* & 0 \end{pmatrix},$$

and w has the desired property. Note that ww^* gives an element in $\mathrm{Ker}(K_0(\iota)) + [1]$.

Let $w_1, w_2 \in A$ be isometries such that $w_1w_1^*, w_2w_2^* \in A^\alpha$. We denote by u_i, $i = 1, 2$, the α-cocycles given by

$$u_i(g) = w_i^*\alpha_g(w_i), \quad g \in G,\ i = 1, 2.$$

To prove the statement, it suffices to show that $[w_1w_1^*] = [w_2w_2^*]$ in $K_0(A^\alpha)$ if and only if $[u_1] = [u_2]$ in $H^1_\alpha(G, U(A))$. Assume $[w_1w_1^*] = [w_2w_2^*]$ in $K_0(A^\alpha)$ first. Since A^α is purely infinite simple, there exists a partial isometry $v \in A^\alpha$ such that

$$v^*v = w_1w_1^*, \quad vv^* = w_2w_2^*.$$

Let $u = w_1^*v^*w_2$, which is a unitary in A. Then, we get

$$u^*w_1^*\alpha_g(w_1u) = w_2^*vw_1w_1^*\alpha_g(w_1w_1^*v^*w_2) = w_2^*\alpha_g(w_2).$$

Now, we assume that u_1 and u_2 are equivalent. Then, there exists a unitary $u \in A$ such that $u^*w_1^*\alpha_g(w_1u) = w_2^*\alpha_g(w_2)$, which implies $w_1uw_2^* \in A^\alpha$. Thus, we get $[w_1w_1^*] = [w_2w_2^*]$ in $K_0(A^\alpha)$, which finishes the proof. □

Remark 2.6. Let $\lambda(g)$ be the implementing unitary of α_g in the multiplier algebra $M(A \rtimes_\alpha G)$. For an α-cocycle and u, we define projections $e_\alpha, e_\alpha^u \in A \rtimes_\alpha G$ by

$$e_\alpha = \int_G \lambda(g)dg,$$
$$e_\alpha^u = \int_G u(g)\lambda(g)dg,$$

where dg stands for the normalized Haar measure. Under identification of $K_0(A^\alpha)$ with $K_0(A \rtimes_\alpha G)$, the one-to-one correspondence in the above statement is nothing but the one associating $[e_u] \in K_0(A \rtimes_\alpha G)$ with $[u] \in H^1_\alpha(G, U(A))$. Indeed, we define a homomorphism π by

$$\pi : A^\alpha \ni a \mapsto ae_\alpha \in A \rtimes_\alpha G.$$

Since $A \rtimes_\alpha G$ is simple, $K_0(\pi)$ is an isomorphism from $K_0(A^\alpha)$ onto $K_0(A \rtimes_\alpha G)$. Let w be the isometry in A with $ww^* \in A^\alpha$ corresponding to u as in the proof. Then

$$[e^u_\alpha] = [w^* e_\alpha w] = [e_\alpha ww^* e_\alpha] = [ww^* e_\alpha] = K_0(\pi)([ww^*]).$$

Remark 2.7. Even if we drop the condition that A and $A \rtimes_\alpha G$ are purely infinite in Proposition 2.5, we still have the following statement: for every α-cocycle u, there exists a partial isometry $w \in A \otimes \mathbb{K}$ such that $ww^* \in A^\alpha \otimes \mathbb{K}$ and

$$w^*(\alpha_g \otimes \mathrm{Id})(w) = u(g) \otimes f, \quad \forall g \in G,$$

where $f \in \mathbb{K}$ is a minimal projection. Indeed, the same argument as above using [10, Theorem 4.23] implies that there exists a unitary W in the multiplier algebra $M(A \otimes \mathbb{K})$ such that

$$u(g) \otimes 1 = W^*(\alpha_g \otimes \mathrm{Id})(W).$$

Then, $w = W(1 \otimes f) \in A \otimes \mathbb{K}$ satisfies the desired property. Assuming some more condition on A and $A \rtimes_\alpha G$, we could get a statement similar to Proposition 2.4 by using w (cf. [8, Section 4]).

2.4. **McDuff's type results.** The celebrated McDuff theorem [49] asserts that if a II_1 factor with a separable predual has a unital copy of matrix algebra in its central sequence algebra, it is stable under taking tensor product with the hyperfinite II_1 factor. There are three types of C^*-counterparts known so far [5, Proposition 2.12], [39, Section 3].

Proposition 2.8. *Let A be a separable unital simple C^*-algebra. Then,*

1. *A_∞ has a C^*-subalgebra containing the unit of A_∞ that is isomorphic to $\mathbb{M}_n$ if and only if A is isomorphic to $A \otimes \mathbb{M}_{n^\infty}$.*
2. *A_∞ has a C^*-subalgebra containing the unit of A_∞ that is isomorphic to $\mathcal{O}_\infty$ if and only if A is isomorphic to $A \otimes \mathcal{O}_\infty$.*
3. *A_∞ has a C^*-subalgebra containing the unit of A_∞ that is isomorphic to $\mathcal{O}_2$ if and only if A is isomorphic to $A \otimes \mathcal{O}_2$.*

In [1], the notion of relative central sequences for an inclusion of II_1 factors was introduced and a relative version of the McDuff theorem was obtained. Relative versions of the above statements can be proven by essentially the same argument as in the single algebra case taking a little account of subalgebras:

Lemma 2.9. *Let $A \supset B$ be an inclusion of separable unital simple C^*-algebra with a common unit. Then,*

(1) *$B^\infty \cap A'$ has a unital C^*-subalgebra containing the unit of $B^\infty \cap A'$ that is isomorphic to $\mathbb{M}_n$ if and only if*
$$(A \supset B) \cong (A \otimes \mathbb{M}_{n^\infty} \supset B \otimes \mathbb{M}_{n^\infty}),$$
where $B^\infty \cap A'$ is considered in A^∞.

(2) *$B^\infty \cap A'$ has a unital C^*-subalgebra containing the unit of $B^\infty \cap A'$ that is isomorphic to $\mathcal{O}_\infty$ if and only if*
$$(A \supset B) \cong (A \otimes \mathcal{O}_\infty \supset B \otimes \mathcal{O}_\infty).$$

(3) *$B^\infty \cap A'$ has a unital C^*-subalgebra containing the unit of $B^\infty \cap A'$ that is isomorphic to $\mathcal{O}_2$ if and only if*
$$(A \supset B) \cong (A \otimes \mathcal{O}_2 \supset B \otimes \mathcal{O}_2).$$

As one of the consequences, we get the following:

Corollary 2.10. *Let A be a separable unital simple C^*-algebra and α be an outer action of a finite group G on A. Then,*

(1) *$(A^\alpha)^\infty \cap A'$ has a unital C^*-subalgebra containing the unit of $(A^\alpha)^\infty \cap A'$ that is isomorphic to $\mathbb{M}_n$ if and only if*
$$(A, \alpha) \cong (A \otimes \mathbb{M}_{|G|^\infty}, \alpha \otimes \mathrm{Id}),$$
where $\cong$ stands for conjugacy.

(2) *$(A^\alpha)^\infty \cap A'$ has a unital C^*-subalgebra containing the unit of $(A^\alpha)^\infty \cap A'$ that is isomorphic to $\mathcal{O}_\infty$ if and only if*
$$(A, \alpha) \cong (A \otimes \mathcal{O}_\infty, \alpha \otimes \mathrm{Id}).$$

(3) *$(A^\alpha)^\infty \cap A'$ has a unital C^*-subalgebra containing the unit of $(A^\alpha)^\infty \cap A'$ that is isomorphic to $\mathcal{O}_2$ if and only if*
$$(A, \alpha) \cong (A \otimes \mathcal{O}_2, \alpha \otimes \mathrm{Id}).$$

Proof. The statements follow from Proposition 2.9 and consideration of the Galois group [7, Theorem 1] (see also [22, Theorem 3.4], [41]). □

Corollary 2.11. *Let A be a separable purely infinite simple unital nuclear C^*-algebra, and α be an outer action of a finite group G on A. Then,*
$$(A, \alpha) \cong (A \otimes \mathcal{O}_\infty, \alpha \otimes \mathrm{Id}).$$

Proof. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. Then, $(A^\alpha)^\omega \cap A'$ is nothing but the fixed-point subalgebra of A_ω under α_ω. Thus, Lemma 2.1 and Lemma 2.2 imply that when A is a separable purely infinite simple unital nuclear C^*-algebra, $(A^\alpha)^\omega \cap A'$ is purely infinite simple as well. Therefore, $(A^\alpha)^\omega \cap A'$ contains a unital copy of $\mathcal{O}_\infty$. □

3. The Rohlin property

In this section, we deduce basic properties of finite group actions with the Rohlin property on simple unital C^*-algebras.

3.1. **Intertwining argument.** First, we introduce the Rohlin property for finite groups. The same condition has already appeared in [40], [31], [32] with a different name, but we adopt the term "Rohlin property" here following the single automorphism case [11], [9].

Definition 3.1. Let α be a (cocycle) action of a finite group G on a unital C^*-algebra A. α is said to have the *Rohlin property* if there exists a partition of unity $\{e_g\}_{g\in G} \subset A_\infty$ consisting of projections satisfying

$$\alpha_{g\infty}(e_h) = e_{gh}, \quad g, h \in G.$$

Example 3.2. Let G be a finite group, and λ be the left regular representation of G. We identify $\mathbb{B}(\ell^2(G))$ with $\mathbb{M}_{|G|}$ and introduce an action of G on $\mathbb{M}_{|G|^\infty}$ by

$$\mu_g^G = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(\lambda(g)), \quad g \in G.$$

Then, μ^G has the Rohlin property. Indeed, let $e_g \in \mathbb{B}(\ell^2(G))$ be the projection onto $\mathbb{C}\delta_g$, $g \in G$. We set $e_g(n) = \iota_n(e_g)$ and $E_g = (e_g(n)) + c_0(A) \in A^\infty$ where $A = \mathbb{M}_{|G|^\infty}$ and ι_n is the inclusion map from $\mathbb{M}_{|G|}$ into the n-th tensor factor of $\mathbb{M}_{|G|^\infty}$. Then, $\{E_g\}_{g\in G}$ give the Rohlin projections. μ^G plays the role of a model action in [12], [37]. It is essentially shown in [40] that when G is abelian, the dual action of μ^G is conjugate to μ^G itself.

In [40], A. Kishimoto showed, among other results, that there exists at most one conjugacy class of finite cyclic group actions of product type on a UHF algebra with the Rohlin property. In [31], [32], R. H. Herman and V. F. R. Jones investigated finite group actions on AF algebras with the Rohlin property and generalized Kishimoto's result. Since they closely followed the model action splitting argument developed in the case of von Neumann algebras [11], [12], [37], their main concern is the case where α_g is approximately inner for all $g \in G$ and A is stable under taking tensor product with the UHF algebra $\mathbb{M}_{|G|^\infty}$, which seems to be too strong an assumption from the K-theoretical point of view now. Indeed, in [23, Theorem 4.1], D. E. Evans and A. Kishimoto developed an intertwining argument for automorphisms, an equivariant version of G. A. Elliott's celebrated argument in classification of C^*-algebras [19], which enables us to avoid using the model action and to compare two actions directly. We present here a variant of their argument modified for the case of finite group actions.

Lemma 3.3. *Let A be a unital C^*-algebra and α and β be actions of a finite group G on A with the Rohlin property. Assume that α_g and β_g are approximately unitary equivalent for all $g \in G$. Then, for every finite set $F \subset A$ and a positive number ϵ, there exists a unitary $u \in A$ such that the following hold for all $x \in F$:*

$$||\beta_g(x) - \mathrm{Ad}(u) \cdot \alpha_g \cdot \mathrm{Ad}(u^*)(x)|| < \epsilon, \quad g \in G,$$

$$||[u, x]|| < \epsilon + \sup_{g\in G}\{||\beta_g(x) - \alpha_g(x)||\}.$$

Proof. We set

$$F^1 = \bigcup_{g\in G}(\alpha_g(F) \cup \beta_g(F)),$$

and take unitaries $v_g \in A$, $g \in G$ such that

$$||\beta_g(y) - \mathrm{Ad}(v_g)\cdot\alpha_g(y)|| < \epsilon/2, \quad \forall y \in F^1.$$

We choose Rohlin projections $\{e_g\}_{g\in G} \subset A_\infty$ for α and set

$$w = \sum_{g\in G} e_g v_g \in U(A^\infty).$$

Then, for $x \in F$, we have

$$\begin{aligned}
\mathrm{Ad}(w)\cdot\alpha_g^\infty\cdot\mathrm{Ad}(w^*)(x) &= \sum_{h\in G}\mathrm{Ad}(w)\cdot\alpha_g^\infty(e_h v_h^* x v_h) \\
&= \sum_{h\in G} e_{gh}\mathrm{Ad}(v_{gh})\cdot\alpha_g\cdot\mathrm{Ad}(v_h^*)(x) \\
&= \sum_{h\in G} e_{gh}\mathrm{Ad}(v_{gh})\cdot\alpha_{gh}\cdot(\mathrm{Ad}(v_h)\cdot\alpha_h)^{-1}(x).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&||\beta_g(x) - \mathrm{Ad}(w)\cdot\alpha_g^\infty\cdot\mathrm{Ad}(w^*)(x)|| \\
&= ||\sum_{h\in G} e_{gh}\big(\beta_g(x) - \mathrm{Ad}(v_{gh})\cdot\alpha_{gh}\cdot(\mathrm{Ad}(v_h)\cdot\alpha_h)^{-1}(x)\big)|| \\
&\leq \sup_{h\in G}\{||\beta_g(x) - \mathrm{Ad}(v_{gh})\cdot\alpha_{gh}\cdot(\mathrm{Ad}(v_h)\cdot\alpha_h)^{-1}(x)||\} \\
&= \sup_{h\in G}\{||\beta_{gh}(\beta_{h^{-1}}(x)) - \mathrm{Ad}(v_{gh})\cdot\alpha_{gh}\cdot(\mathrm{Ad}(v_h)\cdot\alpha_h)^{-1}\cdot\beta_h(\beta_{h^{-1}}(x))||\}.
\end{aligned}$$

For $y \in F^1$, we have

$$\begin{aligned}
&||\beta_{gh}(y) - \mathrm{Ad}(v_{gh})\cdot\alpha_{gh}\cdot(\mathrm{Ad}(v_h)\cdot\alpha_h)^{-1}\cdot\beta_h(y)|| \\
&\leq ||\beta_{gh}(y) - \mathrm{Ad}(v_{gh})\cdot\alpha_{gh}(y)|| + ||y - (\mathrm{Ad}(v_h)\cdot\alpha_h)^{-1}\cdot\beta_h(y)|| \\
&= ||\beta_{gh}(y) - \mathrm{Ad}(v_{gh})\cdot\alpha_{gh}(y)|| + ||\mathrm{Ad}(v_h)\cdot\alpha_h(y) - \beta_h(y)|| \\
&< \epsilon,
\end{aligned}$$

which implies

$$||\beta_g(x) - \mathrm{Ad}(w)\cdot\alpha_g^\infty\cdot\mathrm{Ad}(w^*)(x)|| < \epsilon.$$

For $x \in F$, we have

$$\begin{aligned}
||[w,x]|| &= ||wxw^* - x|| = ||\sum_{g\in G} e_g(v_g x v_g^* - x)|| \\
&\leq \sup_{g\in G}\{||\mathrm{Ad}(v_g)\cdot\alpha_g(\alpha_{g^{-1}}(x)) - x||\} \\
&\leq \sup_{g\in G}\{||\mathrm{Ad}(v_g)\cdot\alpha_g(\alpha_{g^{-1}}(x)) - \beta_g(\alpha_{g^{-1}}(x))||\} + \sup_{g\in G}\{||\beta_g(\alpha_{g^{-1}}(x)) - x||\} \\
&< \epsilon + \sup_{g\in G}\{||\alpha_{g^{-1}}(x) - \beta_{g^{-1}}(x)||\}.
\end{aligned}$$

We take a representing sequence $(w(n))$ of w consisting of unitaries, and set $u = w(n)$ for sufficiently large n. Then, u satisfies the desired property. □

It is easy to show the following:

Lemma 3.4. *Let A be a unital C^*-algebra and $\{u_n\}_{n=1}^\infty$ be a sequence of unitaries in A. If $\{u_n x u_n^*\}_{n=1}^\infty$ converges for all x in a dense subset of A, then there exists a unital faithful endomorphism ϕ of A such that for all $x \in A$, $\{u_n x u_n^*\}_{n=1}^\infty$ converges to $\phi(x)$.*

Theorem 3.5. *Let A be a separable unital C^*-algebra and α and β be actions of a finite group G on A with the Rohlin property. Assume that α_g and β_g are approximately unitary equivalent for all $g \in G$. Then, there exists approximately inner automorphism θ such that*

$$\theta \cdot \alpha_g \cdot \theta^{-1} = \beta_g, \quad g \in G.$$

Proof. We choose an increasing family of finite subsets F_n, $n = 0, 1, \cdots$ of A whose union is dense in A. By induction using Lemma 3.3, we can construct a family of finite subsets F_n^1, $n = 0, 1, \cdots$ of A, a sequence of unitaries $\{u_n\}_{n=0}^\infty \subset A$, and a family of G-actions $\alpha^{(2n)}$, $\beta^{(2n+1)}$, $n = 0, 1, 2, \cdots$ satisfying $\alpha^{(0)} = \alpha$, $\beta^{(1)} = \beta$,

$$\alpha_g^{(2n+2)} = \mathrm{Ad}(u_{2n}) \cdot \alpha_g^{(2n)} \cdot \mathrm{Ad}(u_{2n}^*), \quad n \geq 0,$$

$$\beta_g^{(2n+1)} = \mathrm{Ad}(u_{2n-1}) \cdot \beta_g^{(2n-1)} \cdot \mathrm{Ad}(u_{2n-1}^*), \quad n \geq 1,$$

$$F_0^1 = \bigcup_{g \in G} (\alpha_g(F_0) \cup \beta_g(F_0)),$$

$$\begin{aligned} F_{2n+1}^1 &= \bigcup_{g\in G} \alpha_g^{(2n+2)}(F_{2n+1} \cup F_{2n}^1 \cup \mathrm{Ad}(u_{2n}u_{2n-2}\cdots u_0)(F_{2n}^1)) \\ &\cup \bigcup_{g\in G} \beta_g^{(2n+1)}(F_{2n+1} \cup F_{2n}^1 \cup \mathrm{Ad}(u_{2n}u_{2n-2}\cdots u_0)(F_{2n}^1)), \quad n \geq 0, \end{aligned} \tag{3.1}$$

$$\begin{aligned} F_{2n+2}^1 &= \bigcup_{g\in G} \alpha_g^{(2n+2)}(F_{2n+2} \cup F_{2n+1}^1 \cup \mathrm{Ad}(u_{2n+1}u_{2n-1}\cdots u_1)(F_{2n+1}^1)) \\ &\cup \bigcup_{g\in G} \beta_g^{(2n+3)}(F_{2n+2} \cup F_{2n+1}^1 \cup \mathrm{Ad}(u_{2n+1}u_{2n-2}\cdots u_1)(F_{2n+1}^1)), \\ & n \geq 0, \end{aligned} \tag{3.2}$$

$$||\beta_g^{(2n+1)}(x) - \alpha_g^{(2n+2)}(x)|| < \frac{1}{2^{2n}}, \quad x \in F_{2n}^1,\ g \in G,\ n \geq 0 \tag{3.3}$$

$$||\beta_g^{(2n+3)}(x) - \alpha_g^{(2n+2)}(x)|| < \frac{1}{2^{2n+1}}, \quad x \in F_{2n+1}^1,\ g \in G,\ n \geq 0, \tag{3.4}$$

$$\begin{aligned} ||[u_{2n+1}, x]|| &< \frac{1}{2^{2n+1}} + \sup_{x\in F_{2n}^1} \{||\beta_g^{(2n+1)}(x) - \alpha_g^{(2n+2)}(x)||\} \\ &< \frac{3}{2^{2n+1}}, \quad x \in F_{2n}^1,\ n \geq 0, \end{aligned} \tag{3.5}$$

$$(3.6)\qquad \begin{aligned} ||[u_{2n+2},x]|| &< \frac{1}{2^{2n+2}}+\sup_{x\in F^1_{2n+1}}\{||\beta_g^{(2n+3)}(x)-\alpha_g^{(2n+2)}(x)||\}, \\ &< \frac{3}{2^{2n+2}}, \quad x\in F^1_{2n+1},\ n\geq 0. \end{aligned}$$

Let $\theta_n=\mathrm{Ad}(u_{2n}u_{2n-2}\cdots u_0)$. We claim that there exists an automorphism $\theta\in \mathrm{Aut}(A)$ such that for every $x\in A$ the following hold:

$$\lim_{n\to\infty}\theta_n(x)=\theta(x),$$
$$\lim_{n\to\infty}\theta_n^{-1}(x)=\theta^{-1}(x).$$

Indeed, for $x\in F_k$ and $k<n$, (3.1) and (3.6) implies

$$||\theta_{n+1}(x)-\theta_n(x)||=||[u_{2n+2},\mathrm{Ad}(u_{2n}u_{2n-2}\cdots u_0)(x)]||<\frac{3}{2^{2n+2}}.$$

Therefore, there exists a faithful endomorphism θ such that $\{\theta_n(x)\}_{n=1}^\infty$ converging to $\theta(x)$ for all $x\in A$ thanks to Lemma 3.4. On the other hand, for $x\in F_k$ and $k<n$, (3.6) implies

$$||\theta_{n+1}^{-1}(x)-\theta_n^{-1}(x)||=||[u_{2n+2},x]||<\frac{3}{2^{2n+2}},$$

which shows that $\{\theta_n^{-1}\}_{n=1}^\infty$ also converges to a faithful endomorphism, say ϑ, in the point-norm topology. We show that θ is an automorphism and that ϑ is the inverse of θ. For this, it suffices to show $\theta\cdot\vartheta=\mathrm{Id}$ because θ is faithful. For $x\in F_k$ and $k<n<m$, we have

$$\begin{aligned} ||x-\theta_m\cdot\theta_n^{-1}(x)|| &= ||[u_{2m}u_{2m-2}\cdots u_{2n+2},x]||\leq\sum_{l=n}^{m-1}||[u_{2l+2},x]|| \\ &\leq \sum_{l=n}^{m-1}\frac{3}{2^{2l+2}}<\frac{1}{4^n}. \end{aligned}$$

Thus,

$$\theta(\vartheta(x))=\lim_{n\to\infty}\theta(\theta_n^{-1}(x))=\lim_{n\to\infty}\lim_{m\to\infty}\theta_m(\theta_n^{-1}(x))=x,$$

which shows the claim.

In the same way, we can show that there exists an automorphism $\nu\in\mathrm{Aut}(A)$ such that for all $x\in A$, we have

$$\lim_{n\to\infty}\mathrm{Ad}(u_{2n+1}u_{2n-1}\cdots u_1)(x)=\nu(x),$$
$$\lim_{n\to\infty}\mathrm{Ad}(u_1^*u_3^*\cdots u_{2n+1}^*)(x)=\nu^{-1}(x).$$

Thus, (3.3) and (3.4) imply

$$\nu\cdot\beta_g\cdot\nu^{-1}=\theta\cdot\alpha_g\cdot\theta^{-1},\quad g\in G.$$

□

Note that the above theorem immediately implies the main results of [31] and [32]. In fact, we get a fairly stronger classification result in the second part.

3.2. **Approximately representable actions.** One might suspect that the defining condition of the Rohlin property is so demanding that there are not so many examples satisfying the condition except for those coming from Example 3.2. It might be true in the sense that the Rohlin property gives rise to several K-theoretical constraints as we will see below. For example, it is easy to show that there is no $\mathbb{Z}_2$-action with the Rohlin property on $\mathbb{M}_{3^\infty}$. Nevertheless, actions with the Rohlin property are not very rare in the sense that there are large classes of well-known finite abelian group actions whose dual actions have the Rohlin property. Our next task is to show this fact. For this purpose, we introduce two new classes of finite abelian group actions.

Definition 3.6. Let A be a unital C^*-algebra, and α be an action of a finite abelian group G on A.

(1) α is said to be *strongly approximately inner* if there exist unitaries $u(g) \in (A^\alpha)^\infty$, $g \in G$ such that
$$\alpha_g(x) = u(g)xu(g)^*, \quad x \in A,\ g \in G.$$
(2) α is said to be *approximately representable* if α is strongly approximately inner and the unitaries $\{u(g)\}_{g\in G}$ above can be taken so that they form a representation of G in $(A^\alpha)^\infty$.

Remark 3.7. Although the above definition makes sense for a general finite group case, probably the right definition of an approximate representable action in the general case should be the following: There exists a unitary representation u of G in A^∞ such that
$$\alpha_g(x) = u(g)xu(g)^*, \quad x \in A,\ g \in G,$$
$$\alpha_g^\infty(u(h)) = u(ghg^{-1}), \quad g, h \in G.$$

The following is the precise relationship between the Rohlin property and approximate representability.

Lemma 3.8. *Let A be a separable C^*-algebra, and α be an action of a finite abelian group G on A. We denote by $\widehat{\alpha}$ the dual action of α on $A \rtimes_\alpha G$. Then,*

(1) *α has the Rohlin property if and only if $\widehat{\alpha}$ is approximately representable.*
(2) *α is approximately representable if and only if $\widehat{\alpha}$ has the Rohlin property.*

Proof. (1). Note that when G is abelian, the Rohlin property of α is equivalent to the following condition: there exists a unitary representation u of the dual group $\widehat{G}$ in A_∞ such that
$$\alpha_{g\infty}(u(\tau)) = \tau(g)u(\tau), \quad g \in G,\ \tau \in \widehat{G}.$$
Let λ be the implementing unitary representation of G in $A \rtimes_\alpha G$ for α. Then, the above relation is equivalent to
$$u(\tau)^*\lambda(g)u(\tau) = \tau(g)\lambda(g), \quad g \in G,\ \tau \in \widehat{G}.$$
Therefore, if α has the Rohlin property, $\widehat{\alpha}$ is implemented by $\mathrm{Ad}(u(\tau)^*)$, and so it is approximately representable. The same computation shows that the converse also holds as we have $(A \rtimes_\alpha G)^{\widehat{\alpha}} = A$.

(2). Assume that α is approximately representable and take a unitary representation u of G in $A^{\alpha\infty}$ as in Definition 3.6. Then, $u(g)^*\lambda(g) = \lambda(g)u(g)^*$ gives a representation of G in $(A \rtimes_\alpha G)_\infty$ satisfying

$$\widehat{\alpha}_{\tau\infty}(u(g)^*\lambda(g)) = \tau(g)u(g)^*\lambda(g), \quad g \in G,\ \tau \in \widehat{G},$$

which implies that $\widehat{\alpha}$ has the Rohlin property. The converse follows from the same formula. □

Let α be an action of a compact group G on a C^*-algebra A. α is said to be *locally representable* if there exists an α-invariant increasing sequence of C^*-subalgebras $\{A_n\}_{n=1}^\infty$ whose union is dense in A, and a unitary representations $u^{(n)}$ of G in A_n such that

$$\alpha_g(x) = \mathrm{Ad}(u^{(n)}(g))(x), \quad x \in A_n,\ g \in G.$$

When G is a finite abelian group, every locally representable action is approximately representable in our sense. There are several classification results of locally representable actions in terms of K-theory [27], [28], [29], [30], [42], [6]. When G is finite abelian, Theorem 3.5 and Lemma 3.8 with the Takai duality theorem [62] give a simple proof of these classification results.

Corollary 3.9. *Let α and β be outer actions of a finite abelian group G on a separable simple unital C^*-algebra A, and e_α and e_β be projections in $A \rtimes_\alpha G$ and $A \rtimes_\beta G$ defined in Remark 2.6 respectively. We assume that α and β are approximately representable. Then, α and β are conjugate if and only if there exists an isomorphism θ from $A \rtimes_\alpha G$ onto $A \rtimes_\beta G$ such that $\theta \cdot \widehat{\beta}_\tau \cdot \theta^{-1}$ and $\widehat{\alpha}_\tau$ are approximately unitary equivalent for all $\tau \in \widehat{G}$ and $\theta(e_\alpha)$ is Murray-von Neumann equivalent to e_β.*

Proof. Assume that there exists an isomorphism θ from $A \rtimes_\alpha G$ onto $A \rtimes_\beta G$ as above. Then, thanks to Lemma 3.8 and Theorem 3.5, we can replace θ with θ_1 such that θ_1 intertwines $\widehat{\alpha}$ and $\widehat{\beta}$, and $\theta_1(e_\alpha)$ is Murray-von Neumann equivalent to e_β. Let $e \in \mathbb{B}(\ell^2(G))$ be the projection onto the constant functions, and ρ be the right regular representation of G. Then, the Takai duality theorem [62] implies that there exists an automorphism Θ of $A \otimes \mathbb{B}(\ell^2(G))$ such that $\Theta(1 \otimes e)$ is Murray-von Neumann equivalent to $(1 \otimes e)$ in $(A \otimes \mathbb{B}(\ell^2(G)))^{\beta\otimes\mathrm{Ad}(\rho)}$ and Θ intertwines $\alpha \otimes \mathrm{Ad}(\rho)$ and $\beta \otimes \mathrm{Ad}(\rho)$. Thus, we get

$$\begin{aligned}
(A,\alpha) &\cong ((1\otimes e)(A\otimes\mathbb{B}(\ell^2(G)))(1\otimes e), (\alpha\otimes\mathrm{Ad}(\rho))|_{(1\otimes e)(A\otimes\mathbb{B}(\ell^2(G)))(1\otimes e)}) \\
&\cong (\Theta(1\otimes e)(A\otimes\mathbb{B}(\ell^2(G)))\Theta(1\otimes e), (\beta\otimes\mathrm{Ad}(\rho))|_{\Theta(1\otimes e)(A\otimes\mathbb{B}(\ell^2(G)))\Theta(1\otimes e)}) \\
&\cong ((1\otimes e)(A\otimes\mathbb{B}(\ell^2(G)))(1\otimes e), (\beta\otimes\mathrm{Ad}(\rho))|_{(1\otimes e)(A\otimes\mathbb{B}(\ell^2(G)))(1\otimes e)}) \\
&\cong (A,\beta).
\end{aligned}$$

The converse is obvious. □

In practice, it is rather hard to show directly that a given action is approximately representable unless it is an inductive limit action. However, in some cases as we will see below, there is a good chance to show strong approximate innerness. In order to deduce the former from the latter, we give the following criterion:

Lemma 3.10. *Let A be a separable unital C^*-algebra, and α be a strongly approximately inner action of a finite group G on A. If $(A^\alpha)^\infty \cap A'$ has a C^*-subalgebra containing the unit of $(A^\alpha)^\infty \cap A'$ that is isomorphic to $\mathbb{M}_{|G|}$, then α is approximately representable (and consequently, the dual action $\widehat{\alpha}$ has the Rohlin property thanks to Lemma 3.8).*

Proof. For each $g \in G$, we take a unitary $u(g) \in (A^\alpha)^\infty$ satisfying

$$u(g)xu(g)^* = \alpha_g(x), \quad x \in A,\ g \in G.$$

Since $u(g)$ normalizes A, $\mathrm{Ad}(u(g))$ globally preserves $(A^\alpha)^\infty \cap A'$, and we set β_g to be the restriction of $\mathrm{Ad}(u(g))$ to $(A^\alpha)^\infty \cap A'$. β is a cocycle action of G on $(A^\alpha)^\infty \cap A'$ with a cocycle c given by

$$c(g,h) = u(g)u(h)u(gh)^* \in (A^\alpha)^\infty \cap A', \quad g,h \in G.$$

We claim that there exists a C^*-subalgebra of $((A^\alpha)^\infty \cap A')^\beta$ containing the unit of $((A^\alpha)^\infty \cap A')^\beta$ that is isomorphic to $\mathbb{M}_{|G|}$. Indeed, by assumotion there exists a system of matrix units $\{e_{ij}\}_{i,j=1}^{|G|}$ in $(A^\alpha)^\infty \cap A'$ such that

$$\sum_{i=1}^{|G|} e_{ii} = 1.$$

We take a representing sequence $(e_{ij}(n))$ for each e_{ij}, which satisfies

$$\lim_{n\to\infty} ||[e_{ij}(n), x]|| = 0, \quad \forall x \in A.$$

Therefore, choosing an appropriate subsequence $\{n_k\}_{k=1}^\infty$ and setting $f_{ij}(k) := (e_{ij}(n_k))$,

$$f_{ij} := (f_{ij}(k)) + c_0(A) \in (A^\alpha)^\infty \cap A',$$

we get a system of matrix units $\{f_{ij}\}_{i,j=1}^n$ of $(A^\alpha)^\infty \cap A'$ commuting with $u(g)$ such that

$$\sum_{i=1}^{|G|} f_{ii} = 1.$$

As $\beta_g(f_{ij}) = f_{ij}$ holds for all $g \in G$, we get the claim.

The above claim and [32, Theorem 1.4], [61] (see the proof of Lemma 3.12 below for essentially the same argument) imply that c is a coboundary, that is, there exist unitaries $w(g) \in (A^\alpha)^\infty \cap A'$, $g \in G$ such that

$$c(g,h) = \beta_g(w(h))w(g)w(gh)^*, \quad g,h \in G,$$

which implies

$$w(g)^*u(g)w(h)^*u(h) = w(gh)^*u(gh), \quad g,h \in G.$$

As $\mathrm{Ad}(w(g)^*u(g))(x) = \alpha_g(x)$ holds for all $x \in A$ and $g \in G$, α is approximately representable. ∎

3.3. **Cohomology vanishing lemmas.** The Rohlin property is basically a means to deduce cohomology vanishing lemmas for group actions. We present two of them here, whose proofs are rather standard [37], [32].

Lemma 3.11. *Let α be an action of a finite group G on a unital C^*-algebra A with the Rohlin property. Then, α is stable.*

Proof. This is [32, Lemma III.1], which follows from a similar argument in the proof of Lemma 3.3 using Lemma 2.4. ☐

Lemma 3.12. *Let A be a unital C^*-algebra, and α be a cocycle action of a finite group G on A with a cocycle u. If α has the Rohlin property, then u is a coboundary.*

Proof. First, we claim that if α has the Rohlin property, for every $\epsilon > 0$, there exist unitaries $\{v(g)\}_{g\in G} \in A$ such that

$$||\alpha_g(v(h))v(g)v(gh)^* - u(g,h)|| < \epsilon, \quad g,h \in G,$$
$$||v(g) - 1|| < \epsilon + \sup_{h\in G}\{||1 - u(g,h)||\}, \quad g \in G.$$

Indeed, we take Rohlin projections $\{e_g\}_{g\in G} \subset A_\infty$ and set

$$w(g) = \sum_{k\in G} u(g, g^{-1}k)e_k \in A^\infty, \quad g \in G.$$

Then, for $g, h \in G$ we have

$$\begin{aligned}
\alpha_g^\infty(w(h))w(g)w(gh)^* &= \sum_{k\in G} \alpha_g(u(h,h^{-1}k))e_{gk}w(g)w(gh)^* \\
&= \sum_{k\in G} \alpha_g(u(h,h^{-1}k))u(g,k)u(gh,h^{-1}k)^*e_{gk} \\
&= \sum_{k\in G} \alpha_g(u(h,k))u(g,hk)u(gh,k)^*e_{ghk} \\
&= \sum_{k\in G} u(g,h)e_{ghk} \\
&= u(g,h),
\end{aligned}$$
$$||1 - w(g)|| = \sup_{k\in G}\{||1 - u(g,k)||\},$$

Therefore, we get the claim.

Now, by induction using the above claim, we construct sequences of unitaries $\{v^k(g)\}_{k=0}^\infty \subset U(A)$, $g \in G$, and a family of cocycle G-actions $\alpha^{(k)}$ with cocycles u^k, $k = 0, 1, \cdots$ satisfying $\alpha^{(0)} = \alpha$, $u^0 = u$, and

$$\alpha_g^{(k+1)} = \mathrm{Ad}(v^k(g)^*) \cdot \alpha_g^{(k)}, \quad g \in G,\ k \geq 0,$$
$$u^{k+1}(g,h) = v^k(g)^*\alpha_g^{(k)}(v^k(h)^*)u^k(g,h)v^k(gh), \quad g,h \in G,\ k \geq 0,$$
$$||1 - u^k(g,h)|| < \frac{1}{2^k}, \quad g,h \in G,\ k \geq 1,$$
$$||1 - v^k(g)|| < \frac{1}{2^{k-1}}, \quad g \in G,\ k \geq 1.$$

Let
$$w^k(g)=v^k(g)v^{k-1}(g)\cdots v^0(g),\quad g\in G.$$
Then, by construction we have
$$u^k(g,h)=w^{k-1}(g)^*\alpha_g(w^{k-1}(h)^*)u(g,h)w^{k-1}(gh),\quad g,h\in G,\ k\geq 1.$$
Since we have $||w^{k+1}(g)-w^k(g)||<1/2^k$, $\{w^k(g)\}_{k=1}^n$ converges, say to $w(g)$, for all $g\in G$. Thus, we get
$$1=w(g)^*\alpha_g(w(h)^*)u(g,h)w(gh),\quad g,h\in G,$$
which shows the statement. □

3.4. **A K-theoretical constraint.** As an application of Proposition 2.5 and the above cohomology vanishing lemma, we get the following K-theoretical constraint of actions with the Rohlin property.

Theorem 3.13. *Let A be a simple unital C^*-algebra, and α be an action of a finite group G on A with the Rohlin property. Let ι be the inclusion map of A^α into A. Then, $K_i(\iota)$, $i=0,1$, are injective. Moreover,*
$$\mathrm{Im}(K_i(\iota))=\mathrm{Im}(\sum_{g\in G}K_i(\alpha_g))=\bigcap_{g\in G}\mathrm{Ker}(1-K_i(\alpha_g)).$$

Proof. First, we show that the general case is reduced to the case where A is purely infinite and $i=0$. Indeed, considering $(A\otimes\mathcal{O}_\infty,\alpha\otimes\mathrm{Id})$ instead of (A,α), we may assume that A is purely infinite thanks to Lemma 2.3. Let $\mathcal{P}_\infty$ be a (unique up to isomorphism) separable purely infinite simple unital nuclear C^*-algebra satisfying the UCT [52] such that
$$K_0(\mathcal{P}_\infty)=\{0\},\quad K_1(\mathcal{P}_\infty)\cong\mathbb{Z}.$$
Then, the $i=1$ case of the statement follows from the $i=0$ case of $(A\otimes\mathcal{P}_\infty,\alpha\otimes\mathrm{Id})$.

We assume that A is purely infinite in what follows. Then, Proposition 2.5 and Lemma 3.11 imply that $K_0(\iota)$ is injective. It is easy to show
$$\mathrm{Im}(K_0(\iota))\subset\mathrm{Im}(\sum_{g\in G}K_0(\alpha_g))\subset\bigcap_{g\in G}\mathrm{Ker}(1-K_0(\alpha_g)).$$
Indeed, the second inclusion is trivial. Let $e\in A^\alpha$ be a projection and ϵ a sufficiently small positive number. Using the Rohlin projections, one can construct projections $\{f_g\}_{g\in G}\subset A$ such that
$$||1-\sum_{g\in G}f_g||<\epsilon,$$
$$||f_ge-ef_g||<\epsilon,\quad g\in G,$$
$$||\alpha_g(f_he)-f_{gh}e||<\epsilon,\quad g,h\in G.$$
As f_ee is almost a projection, using polar decomposition and functional calculus, one can construct a projection e_1 close to f_ee. Therefore, a usual perturbation argument implies
$$\sum_{g\in G}K_0(\alpha_g)([e_1])=K_0(\iota)([e]),$$

which proves the first inclusion.

Now, we show
$$\bigcap_{g\in G}\mathrm{Ker}(1-K_0(\alpha_g))\subset \mathrm{Im}(K_0(\iota)).$$
Let $p\in A$ be a projection such that $\alpha_g(p)$ is Murray-von Neumann equivalent to p for all $g\in G$. For each $g\in G$, we choose a unitary $u(g)\in A$ satisfying $u(g)\alpha_g(p)u(g)^*=p$. Note that $u(gh)\alpha_g(u(h)^*)u(g)^*$ commutes with p for all $g,h\in G$. We choose Rohlin projections $\{e_g\}_{g\in G}\subset A_\infty$ for α and define a unitary $w\in A^\infty$ by
$$w=\sum_{g\in G}u(g)e_g.$$
Then,
$$\begin{aligned}\mathrm{Ad}(w)\cdot\alpha_g^\infty\cdot\mathrm{Ad}(w^*)(p) &= \sum_{h\in G}\mathrm{Ad}(w)\cdot\alpha_g^\infty(u(h)^*pu(h)e_h)\\ &= \sum_{h\in G}\mathrm{Ad}(u(gh)\alpha_g(u(h)^*))\cdot\alpha_g(p)e_{gh}\\ &= \sum_{h\in G}\mathrm{Ad}(u(gh)\alpha_g(u(h)^*)u(g)^*)(p)e_{gh}\\ &= \sum_{h\in G}pe_{gh}\\ &= p.\end{aligned}$$
Therefore, for any small positive number ϵ (less than $1/4$ for our purpose), there exists a unitary $w_1\in A$ such that $||\alpha_g(w_1^*pw_1)-w_1^*pw_1||<\epsilon$, $g\in G$. Let
$$a=\frac{1}{|G|}\sum_{g\in G}\alpha_g(w_1^*pw_1).$$
Then, a is a positive contraction in A^α satisfying $||a-w_1^*pw_1||<\epsilon$. Therefore, by functional calculus, we can get a projection $q\in A^\alpha$ satisfying
$$K_0(\iota)([q])=[p],$$
which finishes the proof. □

Remark 3.14. Since the above theorem holds for $(A\otimes\mathcal{O}_{n+1},\alpha\otimes\mathrm{Id})$ as well, the conclusion is true for the K-groups with coefficient $\mathbb{Z}_n$ [59].

4. Finite group actions on $\mathcal{O}_2$

In [39, Lemma 3.7], E. Kirchberg and N. C. Phillips showed the following remarkable characterization of the Cuntz algebra $\mathcal{O}_2$:

Theorem 4.1. *Let A be a separable simple unital nuclear C^*-algebra. Then, the following two conditions are equivalent:*

1. *A is isomorphic to $\mathcal{O}_2$.*
2. *There exists a C^*-subalgebra of A_∞ containing the unit of A_∞ that is isomorphic to $\mathcal{O}_2$.*

In particular, $B \otimes \mathcal{O}_2$ is always isomorphic to $\mathcal{O}_2$ if B is a separable simple unital nuclear C^*-algebra.

The purpose of this section is to give a variant of the above theorem in the case with finite group actions, and to show how the Rohlin property naturally arises in this context. We also give a classification result of some class of finite group actions on $\mathcal{O}_2$.

Theorem 4.2. *Let α be an outer action of a finite group G on $\mathcal{O}_2$. Then, the following conditions are equivalent:*
(1) *α has the Rohlin property.*
(2) *Let μ^G be the action of G on $\mathbb{M}_{|G|^\infty}$ constructed in Example 3.2. Then,*

$$(\mathcal{O}_2, \alpha) \cong (\mathcal{O}_2 \otimes \mathbb{M}_{|G|^\infty}, \mathrm{Id} \otimes \mu^G).$$

(3) *There exists a C^*-subalgebra of $(\mathcal{O}_2^\alpha)^\infty \cap \mathcal{O}_2'$ containing the unit of $(\mathcal{O}_2^\alpha)^\infty \cap \mathcal{O}_2'$ that is isomorphic to $\mathcal{O}_2$.*

Proof. (1) $\Rightarrow$ (2). Thanks to M. Rørdam's result [55, Theorem 8.1], we have

$$\mathrm{Aut}(\mathcal{O}_2) = \overline{\mathrm{Inn}(\mathcal{O}_2)}.$$

Thus, Theorem 3.5 implies that there is at most one conjugacy class of G-actions on $\mathcal{O}_2$ with the Rohlin property. On the other hand, Theorem 4.1 shows that $\mathcal{O}_2 \otimes \mathbb{M}_{|G|^\infty}$ is isomorphic to $\mathcal{O}_2$. Since $\mathrm{Id} \otimes \mu^G$ has the Rohlin property, we get this implication.

(2) $\Rightarrow$ (3). This follows from the fact that $\mathcal{O}_{2\infty}$ has a unital copy of $\mathcal{O}_2$.

(3) $\Rightarrow$ (1). Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. We claim that α_ω is stable. Indeed, thanks to Lemma 2.2, $\mathcal{O}_{2\omega}$ is purely infinite simple and α_ω is an outer action of G. Note that we have

$$(\mathcal{O}_{2\omega})^{\alpha_\omega} = (\mathcal{O}_2^\alpha)^\omega \cap \mathcal{O}_2',$$

which contains a unital copy of $\mathcal{O}_2$ by assumption. Let e be a projection in $(\mathcal{O}_2^\alpha)^\omega \cap \mathcal{O}_2'$. By the usual argument taking subsequences, we can find a unital copy of $\mathcal{O}_2$ in the relative commutant of e in $(\mathcal{O}_2^\alpha)^\omega \cap \mathcal{O}_2'$. Therefore, $[e] = [e] + [e]$ in $K_0((\mathcal{O}_2^\alpha)^\omega \cap \mathcal{O}_2')$, and so $[e] = 0$. Since e is arbitrary and $(\mathcal{O}_2^\alpha)^\omega \cap \mathcal{O}_2'$ is purely infinite simple thanks to Lemma 2.1, we actually have

$$K_0((\mathcal{O}_2^\alpha)^\omega \cap \mathcal{O}_2') = \{0\}.$$

Therefore, Proposition 2.5 implies that α_ω is stable.

Since $(\mathcal{O}_2^\alpha)^\omega \cap \mathcal{O}_2'$ contains a unital copy of $\mathcal{O}_2$ and $\mathcal{O}_2$ contains a unital copy of $\mathcal{O}_n$ for all $n \geq 2$, there exist isometries $\{S_g\}_{g \in G} \subset (\mathcal{O}_2^\alpha)^\omega \cap \mathcal{O}_2'$ satisfying the Cuntz algebra relation. For each $g \in G$, we define a unitary $u(g) \in (\mathcal{O}_2^\alpha)^\omega \cap \mathcal{O}_2'$ by

$$u(g) = \sum_{h \in G} S_{gh} S_h^*.$$

Then, u is an α_ω-cocycle, and there exists a unitary $v \in \mathcal{O}_{2\omega}$ satisfying

$$v^* \alpha_{g\omega}(v) = u(g), \quad g \in G,$$

and so

$$\alpha_{g\omega}(v S_h) = v S_{gh}, \quad g, h \in G.$$

Let $e_g = vS_gS_g^*v^*$, $g \in G$. Then, $\{e_g\}_{g\in G}$ is a partition of unity in $\mathcal{O}_{2\omega}$ satisfying

$$\alpha_{g_\omega}(e_h) = e_{gh}, \quad g, h \in G,$$

which shows that α has the Rohlin property. □

As in the case without actions, we have the following as a consequence of Theorem 4.2:

Corollary 4.3. *Let α be an outer action of a finite group G on a separable simple unital nuclear C^*-algebra A. Then,*

$$(\mathcal{O}_2 \otimes A, \mathrm{Id} \otimes \alpha) \cong (\mathcal{O}_2 \otimes \mathbb{M}_{|G|^\infty}, \mathrm{Id} \otimes \mu^G).$$

In consequence, $\mathrm{Id} \otimes \alpha$ *has the Rohlin property (and so does its dual action).*

In [3], B. Blackadar announced the following results on $\mathbb{Z}_2$-actions on $\mathcal{O}_2$ using his construction in [2]:

(1) Let α be an action of $\mathbb{Z}_2$ on $\mathcal{O}_2$. Then, $K_i(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2)$, $i = 0, 1$ are uniquely 2-divisible.
(2) For every pair of countable uniquely 2-divisible torsion groups Γ_i, $i = 0, 1$, there exists a $\mathbb{Z}_2$-action α on $\mathcal{O}_2$ such that

$$K_i(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2) \cong \Gamma_i, \quad i = 0, 1.$$

He also posed a question about classification of $\mathbb{Z}_2$-actions on $\mathcal{O}_2$ in terms of the K-groups of the crossed products. We give a proof of (a generalization of) the above statement and a partial answer to his question now.

Lemma 4.4. *Let p be a prime number and $q = p^m$, $m \in \mathbb{N}$. If α is an action of $\mathbb{Z}_q$ on a C^*-algebra A satisfying*

$$K_0(A) \cong K_1(A) \cong \{0\},$$

then, $K_i(A \rtimes_\alpha \mathbb{Z}_q)$, $i = 0, 1$ are uniquely p-divisible.

Proof. Let $B = A \rtimes_\alpha \mathbb{Z}_q$, and $\widehat{\alpha}$ be the dual action of α. Then, [4, Theorem 10.7.1] implies that the following sequence is exact:

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{1-K_0(\widehat{\alpha})^{-1}} & K_0(B) & \longrightarrow & \{0\} \\ {\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle\delta} \\ \{0\} & \longleftarrow & K_1(B) & \xleftarrow{1-K_1(\widehat{\alpha})^{-1}} & K_1(B) \end{array}.$$

Thus, $\psi_i := 1 - K_i(\widehat{\alpha})$ is a group automorphism of $K_i(B)$ for $= 0, 1$. Since $K_i(\widehat{\alpha})$ has a period q, we get

$$1 = K_i(\widehat{\alpha})^q = (1 - \psi_i)^q = \sum_{k=0}^{q} \left(\begin{array}{c} q \\ k \end{array} \right) (-\psi_i)^k,$$

where $\left(\begin{array}{c} q \\ k \end{array}\right)$ is the binomial coefficient. So,

$$-(-\psi_i)^q = \sum_{k=1}^{q-1} \left(\begin{array}{c} q \\ k \end{array}\right) (-\psi_i)^k.$$

Since the left-hand side is an automorphism and $\left(\begin{array}{c} q \\ k \end{array}\right)$ is a multiple of p for $k \neq 0, q$, $K_i(A)$ is uniquely p-divisible. □

Lemma 4.5. *Let α and β be actions of a compact group G on $\mathcal{O}_2$, and ρ be the right regular representation of G. Let $\widehat{\widehat{\alpha}}$ and $\widehat{\widehat{\beta}}$ be actions of G on $\mathcal{O}_2 \otimes \mathbb{K}(L^2(G))$ defined by*

$$\widehat{\widehat{\alpha}}_g = \alpha_g \otimes \mathrm{Ad}(\rho(g)), \quad \widehat{\widehat{\beta}}_g = \beta_g \otimes \mathrm{Ad}(\rho(g)), \quad g \in G.$$

Then, α and β are cocycle conjugate if and only if $\widehat{\widehat{\alpha}}$ and $\widehat{\widehat{\beta}}$ are conjugate.

Proof. The duality theorem of crossed products [62], [34] implies that in general, if α and β are cocycle conjugate, $\widehat{\widehat{\alpha}}$ and $\widehat{\widehat{\beta}}$ are conjugate.

Assume that $\widehat{\widehat{\alpha}}$ and $\widehat{\widehat{\beta}}$ are conjugate now. Then, there exists an automorphism θ of $\mathcal{O}_2 \otimes \mathbb{K}(L^2(G))$ such that

$$\theta \cdot \alpha_g \otimes \mathrm{Ad}(\rho(g)) = \beta_g \otimes \mathrm{Ad}(\rho(g)) \cdot \theta, \quad g \in G.$$

Let $e \in \mathbb{K}(L^2(G))$ be the projection onto the constant functions, and $f = \theta(1 \otimes e)$. Then,

$$(\mathcal{O}_2, \alpha) \cong (f(\mathcal{O}_2 \otimes \mathbb{K}(L^2(G)))f, \widehat{\widehat{\beta}}|_{f(\mathcal{O}_2 \otimes \mathbb{K}(L^2(G)))f}).$$

Since every pair of non-zero projections in $\mathcal{O}_2 \otimes \mathbb{K}(L^2(G))$ are equivalent, there exists a partial isometry $v \in \mathcal{O}_2 \otimes \mathbb{K}(L^2(G))$ such that

$$vv^* = f, \quad v^*v = (1 \otimes e).$$

We introduce a β-cocycle u by the relation

$$u(g) \otimes e = v^* \widehat{\widehat{\beta}}_g(v), \quad g \in G.$$

Then, it is easy to show

$$(f(\mathcal{O}_2 \otimes \mathbb{K}(L^2(G)))f, \widehat{\widehat{\beta}}|_{f(\mathcal{O}_2 \otimes \mathbb{K}(L^2(G)))f}) \cong (\mathcal{O}_2, \beta^u),$$

which finishes the proof. □

Theorem 4.6. *Let p be a prime number and $q = p^m$, $m \in \mathbb{N}$. Let α and β be outer actions of $\mathbb{Z}_q$ on $\mathcal{O}_2$. We assume that α and β are strongly approximately inner. Then,*

1. *α and β are approximately representable.*
2. *α and β are conjugate if and only if there exists an isomorphism θ from $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_q$ onto $\mathcal{O}_2 \rtimes_\beta \mathbb{Z}_q$ such that $\theta \cdot \widehat{\alpha}_\tau \cdot \theta^{-1}$ and $\widehat{\beta}_\tau$ are approximately unitary equivalent for all $\tau \in \widehat{\mathbb{Z}_q}$, and $K_0(\theta)([e_\alpha]) = [e_\beta]$.*

(3) *α and β are cocycle conjugate if and only if there exists an isomorphism θ from $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_q$ onto $\mathcal{O}_2 \rtimes_\beta \mathbb{Z}_q$ such that $\theta \cdot \widehat{\alpha}_\tau \cdot \theta^{-1}$ and $\widehat{\beta}_\tau$ are approximately unitary equivalent for all $\tau \in \widehat{\mathbb{Z}_q}$.*

Proof. Once we show that α and β are approximately representable, (2) and (3) follow from Theorem 3.5, Lemma 3.8, Corollary 3.9, and Lemma 4.5. Thus, it suffices to prove (1). Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. The same argument as in the proof of Theorem 4.2 implies $K_0(\mathcal{O}_{2\omega}) \cong K_1(\mathcal{O}_{2\omega}) \cong \{0\}$. Thanks to Lemma 2.2, α_ω is an outer action of $\mathbb{Z}_q$ on the simple C^*-algebra $\mathcal{O}_{2\omega}$. Thus, $\mathcal{O}_{2\omega} \rtimes_{\alpha_\omega} \mathbb{Z}_q$ and $(\mathcal{O}_{2\omega})^{\alpha_\omega} = (\mathcal{O}_2^\alpha)^\omega \cap \mathcal{O}_2'$ are strongly Morita equivalent and they have isomorphic K-groups. Therefore, Lemma 4.4 implies that $K_0((\mathcal{O}_2^\alpha)^\omega \cap \mathcal{O}_2')$ is uniquely p-divisible. Thanks to Lemma 2.1, $(\mathcal{O}_2^\alpha)^\omega \cap \mathcal{O}_2'$ is purely infinite. Since $(\mathcal{O}_2^\alpha)^\omega \cap \mathcal{O}_2'$ is a unital purely infinite simple C^*-algebra with the uniquely p-divisible K_0-group, it has a C^*-subalgebra containing the unit of $(\mathcal{O}_2^\alpha)^\omega \cap \mathcal{O}_2'$ that is isomorphic to $\mathbb{M}_q$. Therefore, Lemma 3.10 implies that α is approximately representable. □

When $q = 2$, we can get a more satisfactory classification result (though not complete yet). Before stating it, we construct a model action.

Let $\mathcal{O}_\infty^{\mathrm{st}}$ be a C^*-algebra that is strongly Morita equivalent to the Cuntz algebra $\mathcal{O}_\infty$ and is in the Cuntz standard form. Then, we have

$$K_0(\mathcal{O}_\infty^{\mathrm{st}}) \cong \mathbb{Z}, \quad K_1(\mathcal{O}_\infty^{\mathrm{st}}) \cong \{0\}.$$

Since $\mathcal{O}_\infty^{\mathrm{st}}$ is in the Cuntz standard form, it contains a unital copy of $\mathcal{O}_2$, and Theorem 4.1 implies that the infinite tensor product of $\mathcal{O}_\infty^{\mathrm{st}}$ is isomorphic to $\mathcal{O}_2$. We fix such an isomorphism and identify $\mathcal{O}_2$ with the infinite tensor product of $\mathcal{O}_\infty^{\mathrm{st}}$. We choose a projection $e \in \mathcal{O}_\infty^{\mathrm{st}}$ whose class generates $K_0(\mathcal{O}_\infty^{\mathrm{st}})$. We set $u = 2e - 1$ and introduce a period two automorphism of $\mathcal{O}_2$ by

$$\nu = \bigotimes_{k=1}^{\infty} \mathrm{Ad}(u).$$

We regard ν as an outer action of $\mathbb{Z}_2$.

Lemma 4.7. *Let ν be as above. Then, $\mathcal{O}_2 \rtimes_\nu \mathbb{Z}_2$ is a separable purely infinite simple nuclear unital C^*-algebra satisfying the UCT such that*

$$(K_0(\mathcal{O}_2 \rtimes_\nu \mathbb{Z}_2), [1]) \cong (\mathbb{Z}[\frac{1}{2}], 1),$$
$$K_1(\mathcal{O}_2 \rtimes_\nu \mathbb{Z}_2) \cong \{0\}.$$

Proof. Since ν is outer, Lemma 2.1 implies that $\mathcal{O}_2 \rtimes_\nu \mathbb{Z}_2$ is separable purely infinite simple nuclear. Let λ be the implementing unitary of ν in $\mathcal{O}_2 \rtimes \mathbb{Z}_2$. We set

$$A_n = \bigotimes_{k=1}^{n} \mathcal{O}_\infty^{\mathrm{st}},$$

$$u_n = \bigotimes_{k=1}^{n} \mathrm{Ad}(u),$$

$$e_n = \bigotimes_{k=1}^{n} e,$$

and $\nu_n = \mathrm{Ad}(u_n)$. We denote by B_n the C^*-algebra generated by A_n and λ, which is identified with $A_n \rtimes_{\nu_n} \mathbb{Z}_2$. Let $v_n = u_n\lambda \in B_n$ and set

$$f_n = \frac{1+v_n}{2}, \quad g_n = \frac{1-v_n}{2}.$$

Then, f_n and g_n are central projection of B_n, and we have

$$B_n = B_n f_n \oplus B_n g_n \cong \mathcal{O}_\infty^{\mathrm{st}} \oplus \mathcal{O}_\infty^{\mathrm{st}}.$$

Since $\mathcal{O}_2 \rtimes_\nu \mathbb{Z}_2$ is the inductive limit of $\{B_n\}_{n=1}^\infty$, it satisfies the UCT. Note that $K_0(B_n)$ is isomorphic to $\mathbb{Z}^2$ with generators $[e_n f_n]$, $[e_n g_n]$, and $K_1(B_n)$ is trivial. Let ι_n be the inclusion map of B_n into B_{n+1}. Then,

$$\begin{aligned} K_0(\iota_n)([e_n f_n]) &= [\frac{e_n \otimes 1 + (e_n \otimes u)v_{n+1}}{2}] \\ &= [e_{n+1}f_{n+1} + (e_n \otimes (1-e))g_{n+1}] \\ &= [e_{n+1}f_{n+1}] - [e_{n+1}g_{n+1}], \end{aligned}$$

and so, $K_0(\mathcal{O}_2 \rtimes_\nu \mathbb{Z}_2)$ is isomorphic to the inductive limit of $\mathbb{Z}^2$ with the stationary connecting maps given by the matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Therefore, we get the statement. $\square$

Theorem 4.8. *Let α and β be outer actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$. We assume that α and β are strongly approximately inner. Then,*

(1) *The following two conditions are equivalent:*
  (i) *α and β are conjugate.*
  (ii) *There exists an isomorphism θ from $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2$ onto $\mathcal{O}_2 \rtimes_\beta \mathbb{Z}_2$ such that $K_0(\theta)([e_\alpha]) = [e_\beta]$.*
  (iii) *$\mathcal{O}_2^\alpha$ and $\mathcal{O}_2^\beta$ are isomorphic.*

(2) *The following conditions are equivalent:*
  (i) *α and β are cocycle conjugate.*
  (ii) *$\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2$ and $\mathcal{O}_2 \rtimes_\beta \mathbb{Z}_2$ are isomorphic.*

(3) *For every pair of countable abelian uniquely 2-divisible groups Γ_i, $i = 0, 1$, there exists an approximately representable outer action γ of $\mathbb{Z}_2$ such that*

$$K_i(\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_2) \cong \Gamma_i, \quad i = 0, 1.$$

γ can be taken so that $\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_2$ satisfies the UCT.

Proof. (1), (2). Note that if two automorphisms of a separable purely infinite simple unital nuclear C^*-algebra give the same class in the KK-group, then they are asymptotically unitary equivalent [39], and so approximately unitary equivalent as well. Therefore, to prove the statement applying Theorem 4.6, it suffices to show that if α is an outer action of $\mathbb{Z}_2$ on $\mathcal{O}_2$, then $[\widehat{\alpha}] = -[\mathrm{Id}]$ in $KK(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2, \mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2)$.

Let $A = \mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2$. We denote by ι_1 the inclusion map of A into $A \rtimes_{\widehat{\alpha}} \mathbb{Z}_2$, and by ι_2 the inclusion map of $A \rtimes_{\widehat{\alpha}} \mathbb{Z}_2$ into $A \otimes \mathbb{M}_2$ given by the crossed product by $\widehat{\widehat{\alpha}}$. Then, the Takai duality theorem shows that we have

$$\iota_2 \cdot \iota_1(x) = \begin{pmatrix} x & 0 \\ 0 & \widehat{\alpha}(x) \end{pmatrix}, \quad x \in A.$$

On the other hand, since $A \rtimes_{\widehat{\alpha}} \mathbb{Z}_2$ is isomorphic to $\mathcal{O}_2$, $[\iota_1]$ and $[\iota_2]$ are trivial in the KK-groups, and so we get $[\mathrm{Id}] + [\widehat{\alpha}] = 0$ in $KK(A, A)$.

(3). Let B be a separable purely infinite simple unital nuclear C^*-algebra such that $K_i(B) \cong \Gamma_i$, $i = 0, 1$, [20, Theorem 5.6]. Thanks to Theorem 4.1, $\mathcal{O}_2 \otimes B$ is isomorphic to $\mathcal{O}_2$. We define a $\mathbb{Z}_2$-action γ on $\mathcal{O}_2 \otimes B$ by $\gamma = \nu \otimes \mathrm{Id}$, where ν is as in Lemma 4.7. By construction, γ is approximately representable. We show that γ has the desired property under identification of $\mathcal{O}_2$ and $\mathcal{O}_2 \otimes B$. Indeed, we have

$$(\mathcal{O}_2 \otimes B) \rtimes_\gamma \mathbb{Z}_2 = (\mathcal{O}_2 \rtimes_\nu \mathbb{Z}_2) \otimes B.$$

Since Γ_i is uniquely 2-divisible, Lemma 4.7 and the Künneth formula [58] imply

$$K_i((\mathcal{O}_2 \rtimes_\nu \mathbb{Z}_2) \otimes B) \cong \Gamma_i, \quad i = 0, 1,$$

which finishes the proof. □

Remark 4.9. (1) It is also possible to construct model actions for $\mathbb{Z}_q$-actions with a prime power $q = p^m$. As in the $\mathbb{Z}_2$ case, one can deduce a necessary condition for the K-groups of the crossed product as follows: $K_i(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_q)$, $i = 0, 1$ are uniquely p-divisible, and

$$\sum_{i=0}^{q-1} [\widehat{\alpha}^i] = 0$$

in $KK(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_q, \mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_q)$. It turns out that this is indeed sufficient. In the second part, we will construct a $\mathbb{Z}_q$-action on $\mathcal{O}_2$ such that the K-groups of the crossed product is isomorphic to a given pair of countable $\mathbb{Z}_q$-modules fulfilling the above constraint.

(2) To the best knowledge of the author, all the known examples of $\mathbb{Z}_2$-actions on $\mathcal{O}_2$ so far are strongly approximately inner, as we will see partly below.

5. Examples

For the Cuntz algebra $\mathcal{O}_n$, $n < \infty$, we set $\mathcal{H}_n$ to be the linear span of the canonical generators $\{S_i\}_{i=1}^n$. $\mathcal{H}_n$ has natural Hilbert space structure and $\mathcal{H}_n \mathcal{H}_n^*$ is identified with $\mathbb{B}(\mathcal{H}_n)$. An action α of a topological group G on $\mathcal{O}_n$ is said to be *quasi-free* if α_g globally preserves $\mathcal{H}_n$ for each $g \in G$. For a quasi-free action α, we set

$$\pi_\alpha(g) = \sum_{i=1}^n \alpha_g(S_i) S_i^*, \quad g \in G.$$

Then, π_α is a continuous unitary representation of G in $U(\mathcal{H}) \subset \mathcal{O}_n$. The map $\alpha \mapsto \pi_\alpha$ gives a one-to-one correspondence between the set of unitary representation of G in $U(\mathcal{H})$ and the set of quasi-free actions of G on $\mathcal{O}_n$.

First, we recall M. H. Mann, I. Raeburn, and C. E. Sutherland's result on the K-theory of the fixed-point algebra for a quasi-free action of a finite group [48].

Lemma 5.1. *Let α be a quasi-free action of a finite group G on $\mathcal{O}_n$, $n \neq \infty$, and $\widehat{G}$ be the set of equivalence classes of irreducible representations of G. We define a matrix $M = (M_{\pi,\sigma})_{\pi,\sigma\in\widehat{G}}$ by*

$$M_{\pi,\sigma} = \dim \mathrm{Hom}(\pi_\alpha \otimes \pi, \sigma), \quad \pi, \sigma \in \widehat{G}.$$

Then, the fixed-point algebra $\mathcal{O}_n^\alpha$ is stably isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_M$ [17]. In consequence, we have

$$K_0(\mathcal{O}_n^\alpha) \cong \mathbb{Z}^n/(I-M)\mathbb{Z}^n,$$
$$K_1(\mathcal{O}_n^\alpha) \cong \mathrm{Ker}((I-M) : \mathbb{Z}^n \longrightarrow \mathbb{Z}^n).$$

As an application of Theorem 4.2, we show the Rohlin property for some class of finite group actions, which is not at all clear from the definition.

Example 5.2. Let p be a prime number. We set

$$A = \bigotimes_{k=1}^{p} \mathcal{O}_2,$$

and α to be the cyclic permutation of tensor components. Then, A is isomorphic to $\mathcal{O}_2$ and α defines an outer action of $\mathbb{Z}_p$ on A. We show that α has the Rohlin property. We denote by $\alpha^{(n)}$ the cyclic permutation of the tensor components of

$$A_n = \bigotimes_{k=1}^{p} \mathcal{O}_n.$$

Then, since there exists a unital embedding of $\mathcal{O}_n$ into $\mathcal{O}_2$ for all $n \geq 2$ and $\mathcal{O}_2$ is isomorphic to the infinite tensor product of itself, $(A^\alpha)^\infty \cap A'$ contains a unital copy of

$$B = \bigotimes_{n=2}^{\infty} A_n^{\alpha^{(n)}}.$$

We show that B contains a unital copy of $\mathcal{O}_2$, which will imply that α has the Rohlin property thanks to Theorem 4.2. Let $\{S_i^{(n)}\}_{i=1}^n$ be the canonical generator of $\mathcal{O}_n$. We set

$$T^{(n)}_{i_1 i_2 \cdots i_p} = \bigotimes_{k=1}^{p} S^{(n)}_{i_k}.$$

Then, the C^*-algebra generated by $\{T^{(n)}_{i_1 i_2 \cdots i_p}\}$ is isomorphic to $\mathcal{O}_{n^p}$ and the restriction of $\alpha^{(n)}$ to $C^*\{T^{(n)}_{i_1 i_2 \cdots i_p}\}$ is quasi-free. We denote by B_n the fixed-point subalgebra of $C^*\{T^{(n)}_{i_1 i_2 \cdots i_p}\}$ under this quasi-free action. Lemma 2.1 and Lemma 5.1 imply that B_n is a separable purely infinite simple unital nuclear C^*-algebra satisfying the UCT. We compute the K-groups of B_n using Lemma 5.1. Simple counting argument shows that the representation of $\mathbb{Z}_p$ on $\bigotimes_{k=1}^{p} \mathbb{C}^n$ induced by the cyclic permutation of the tensor components is equivalent to the direct sum of the trivial representation with multiplicity n and the regular representation with multiplicity $(n^p-n)/p$. Let J be the p by p matrix whose entries are all 1. Then, Lemma 5.1 implies that

$$K_0(B_n)\cong \mathbb{Z}^p/(I-M_n)\mathbb{Z}^p,$$
$$K_1(B_n)\cong \mathrm{Ker}((I-M_n):\mathbb{Z}^p\longrightarrow \mathbb{Z}^p),$$

where

$$M_n=nI+\frac{n^p-n}{p}J.$$

We set $Q=J/p$, which is a projection of rank one. Then,

$$I-M_n=(1-n)(I-Q)+(1-n^p)Q,$$

and so,

$$\det(I-M_n)=(1-n)^{(p-1)}(1-n^p)\neq 0.$$

Thus, $K_1(B_n)$ is trivial. Let $2^p-1=q_1^{c_1}q_2^{c_2}\cdots q_k^{c_k}$ be the decomposition of 2^p-1 into distinct odd prime powers. Then, we have

$$K_0(B_2)=G_1\oplus G_2\oplus\cdots\oplus G_k,$$

where G_i is a q_i-group for every i. We set

$$D=B_2\otimes\bigotimes_{i=1}^k B_{q_i}.$$

As we have $\det(I-M_{q_i})\equiv 1 \mod q_i$, the Künneth formula [58] implies

$$K_0(D)\cong K_1(D)\cong 0,$$

which shows that D is isomorphic to $\mathcal{O}_2$ thanks to the classification theorem [39]. Therefore, α has the Rohlin property.

In a similar way, we give an example of an action of a non-commutative group on $\mathcal{O}_2$ with the Rohlin property.

Example 5.3. Let $A=\mathcal{O}_2\otimes\mathcal{O}_2\otimes\mathcal{O}_2$, which is isomorphic to $\mathcal{O}_2$. Then, the symmetric group $\mathfrak{S}_3$ acts on A as permutations of the tensor components, which is denoted by α. We show that α has the Rohlin property. We define B_n in the same way as above replacing $\mathbb{Z}_p$ with $\mathfrak{S}_3$, and compute the K-groups of B_n. Let π_n be the representation of $\mathfrak{S}_3$ on $\mathbb{C}^n\otimes\mathbb{C}^n\otimes\mathbb{C}^n$ coming from the permutations of the tensor components. Then, a simple counting argument yields that π_n is equivalent to the direct sum of trivial representation with multiplicity n, the representation induced by the natural action of $\mathfrak{S}_3$ on the three point set with multiplicity $n(n-1)$, and the regular representation with multiplicity $n(n-1)(n-2)/6$. We denote by 1, χ, and σ the trivial representation, the signature representation, and the unique 2-dimensional irreducible representation of $\mathfrak{S}_3$ respectively. Then,

$$\pi_n\cong\frac{n(n+1)(n+2)}{6}\cdot 1\oplus\frac{n(n-1)(n-2)}{6}\cdot\chi\oplus\frac{n(n-1)(n+1)}{3}\cdot\sigma.$$

Thus, we get

$$K_0(B_n)\cong\mathbb{Z}^3/(I-M_n)\mathbb{Z}^3,$$
$$K_1(B_n)\cong\mathrm{Ker}((I-M_n):\mathbb{Z}^3\longrightarrow\mathbb{Z}^3),$$

where

$$M_n = \begin{pmatrix} a & b & c \\ b & a & c \\ c & c & a+b+c \end{pmatrix},$$

$$a = \frac{n(n+1)(n+2)}{6}, \quad b = \frac{n(n-1)(n-2)}{6}, \quad c = \frac{n(n-1)(n+1)}{3}.$$

Direct computation implies

$$\det(I - M_n) = -(n-1)(n^2-1)(n^3-1),$$

and in particular, we get

$$\det(I - M_2) = -3 \cdot 7, \quad \det(I - M_3) = -2^5 \cdot 13.$$

Thus,

$$K_0(B_2 \otimes B_3) \cong K_1(B_2 \otimes B_3) \cong \{0\},$$

which shows that $B_2 \otimes B_3$ are isomorphic to $\mathcal{O}_2$, and α has the Rohlin property.

Remark 5.4. Let A be the n-fold tensor product of $\mathcal{O}_2$ and α be the action of the symmetric group $\mathfrak{S}_n$ on A obtained as in Example 5.3. Then, it is possible to generalize the above result and to show the Rohlin property of α for general n without using the representation theory. Indeed, as we saw in the above argument, to show the Rohlin property using Theorem 4.2 in this particular case, it suffices to prove that the fixed-point algebra A^α contains a unital copy $\mathcal{O}_2$. For this, it further suffices to show $K_0(A \rtimes_\alpha \mathfrak{S}_n) = \{0\}$, as A^α is strongly Morita equivalent to $A \rtimes_\alpha \mathfrak{S}_n$. Indeed, one can show by a purely KK-theoretical argument that $A \rtimes_\alpha \mathfrak{S}_n$ is KK-equivalent to $\mathcal{O}_2$ (and so, the classification theorem tells that A^α is isomorphic to $\mathcal{O}_2$). We leave details here (and discuss them elsewhere).

Next, we analyze quasi-free actions themselves. In order to show that some class of quasi-free actions are approximately representable, we need some preparation.

Lemma 5.5. *Let π be a unitary representation of a finite group G in $\mathbb{C}^n$. We set*

$$B = \bigotimes_{i=-\infty}^{\infty} \mathbb{M}_n,$$

$$\gamma_g = \bigotimes_{i=-\infty}^{\infty} \mathrm{Ad}(\pi(g)), \quad g \in G.$$

Let σ be the shift automorphism of B and σ_G be the restriction of σ to B^γ. Then, σ_G has the Rohlin property as a single automorphism in the sense of [43].

Proof. Let τ be the unique normalized trace on B. Since γ is the infinite tensor product of identical actions of a finite group, B^γ is an AF algebra with stationary inclusion matrices. Thus, B^γ has a unique normalized trace [18], which is the restriction of τ. Therefore, in order to show the Rohlin property of σ_G applying [43, Theorem 2.1], it suffices to show that σ_G is approximately inner and the weakly continuous extension of σ_G to the weak closure of B^γ in the GNS representation of the unique trace is aperiodic.

Since B^γ is AF, to prove that σ_G is approximately inner, it suffices to show that it acts on $K_0(B^\gamma)$ trivially. Let

$$B_m=\bigotimes_{k=-m}^{m}\mathbb{M}_n,$$

and $\gamma^{(m)}$ be the restriction of γ to B_m. Then, the fixed-point algebra $B_m^{\gamma^{(m)}}$ is nothing but the commutant of the representation

$$\pi_m:=\bigotimes_{k=-m}^{m}\pi.$$

For a projection $e\in B_m^{\gamma^{(m)}}$, we define the representation π_m^e by the subrepresentation of π_m in the image of e. Then, two projections $e,f\in B_m^{\gamma^{(m)}}$ are equivalent in $B_m^{\gamma^{(m)}}$ if and only if the corresponding two representations π_m^e and π_m^f are equivalent. Since

$$\pi\otimes\pi_m^e\otimes\pi\cong\pi\otimes\pi\otimes\pi_m^e,$$

we get $[\sigma_G(e)]=[e]$ in $K_0(B_{m+1}^{\gamma^{(m+1)}})$ for all projection $e\in B_m^{\gamma^{(m)}}$. As B^γ is the inductive limit of the system $\{B_m^{\gamma^{(m)}}\}_{m=1}^\infty$, we obtain $K_0(\sigma_G)=1$.

Let R be the weak closure of B in the GNS representation of τ. We denote by $\overline{\sigma}$ and $\overline{\gamma}$ the weakly continuous extension of σ and γ respectively. To prove that the weakly continuous extension of σ_G in the GNS representation of the unique trace is aperiodic, it suffices to show that the restriction of $\overline{\sigma}$ to $R^{\overline{\gamma}}$ is aperiodic. Note that for all $x\in R$, $\{\overline{\sigma}^n(x)\}_{n=1}^\infty$ converges to $\tau(x)1$ in the weak operator topology [53, Chapter 7, Section 13]. Suppose that there exist an integer $n\neq 0$ and a unitary $u\in R^{\overline{\gamma}}$ satisfying

$$\overline{\sigma}^n(x)=uxu^*,\quad x\in R^{\overline{\gamma}}.$$

Then, we would have

$$u=\text{w-}\lim_{k\to\infty}\overline{\sigma}^{nk}(u)=\tau(u)1\in\mathbb{C}1.$$

However, this implies that σ_G^n would be trivial, which is a contradiction. Thus, σ_G has the Rohlin property in the sense of [43]. □

Proposition 5.6. *Let α be a quasi-free action of a finite group G on $\mathcal{O}_n$, $n\neq\infty$. Then,*

1. *There exists a sequence of unital endomorphisms $\{\rho_k\}_{k=1}^\infty$ of $\mathcal{O}_n$ such that each ρ_k commutes with α_g, $g\in G$, and*
$$\lim_{k\to\infty}[x,\rho_k(y)]=0,\quad x,y\in\mathcal{O}_n.$$
2. *If G is abelian, α is strongly approximately inner.*

Proof. We follow the arguments in [9] and [55]. Let $\mathcal{F}_n$ be the UHF part of $\mathcal{O}_n$ and $\mathcal{F}_n^\alpha=\mathcal{F}_n\cap\mathcal{O}_n^\alpha$. We define the canonical endomorphism σ_0 of $\mathcal{O}_n$ by

$$\sigma_0(x)=\sum_{i=1}^n S_ixS_i^*,\quad x\in\mathcal{O}_n.$$

We claim that for every unitary $u \in \mathcal{F}_n^\alpha$ and a positive number ϵ, there exists a unitary $v \in \mathcal{F}_n^\alpha$ such that $||u - v\sigma_0(v^*)|| < \epsilon$. Indeed, the claim follows from Lemma 5.5 and the same argument as in [33], [55, Section 4].

(1). Let $\mathcal{E}$ be the permutation operator, which is a self-adjoint unitary defined by

$$\mathcal{E} = \sum_{i,j=1}^{n} S_i S_j S_i^* S_j^*.$$

Then, we have $\sigma_0(S_i) = \mathcal{E}S_i$, $i = 1, 2, \cdots, n$. Applying the claim to $\mathcal{E}$, we get a sequence of unitaries $\{u_k\}_{k=1}^\infty \subset \mathcal{F}_n^\alpha$ such that

$$\lim_{k\to\infty} ||\mathcal{E} - u_k^*\sigma_0(u_k)|| = 0.$$

We define ρ_k by $\rho_k(S_i) = u_n S_i$, $i = 1, 2, \cdots, n$. Then,

$$\begin{aligned} ||\sigma_0(\rho_k(S_i)) - \rho_k(S_i)|| &= ||\sigma_0(u_k)\sigma_0(S_i) - u_k S_i|| = ||(\mathcal{E} - \sigma_0(u_k^*)u_k)S_i|| \\ &\leq ||\mathcal{E} - u_k^*\sigma_0(u_k)||, \quad i = 1, 2, \cdots, n. \end{aligned}$$

Thus, we get

$$\lim_{k\to\infty} ||[\rho_k(S_i), S_j]|| = \lim_{k\to\infty} ||[\rho_k(S_i), S_j^*]|| = 0, \quad i, j = 1, 2, \cdots n,$$

which shows that $\{\rho_k(x)\}_{k=1}^\infty$ is a central sequence for all $x \in \mathcal{O}_n$. Since u_k is fixed by α, ρ_k commutes with α_g for $g \in G$.

(2). When G is abelian, $\pi_\alpha(g) \in \mathcal{F}_n^\alpha$ for $g \in G$. Thus, applying the claim to $\pi_\alpha(g)$, we get a sequence of unitaries $\{v(g)_k\}_{k=1}^\infty \subset \mathcal{F}_n^\alpha$ such that

$$\lim_{k\to\infty} ||\pi_\alpha(g) - v(g)_k\sigma_0(v(g)_k^*)||.$$

This implies

$$\begin{aligned} 0 &= \lim_{k\to\infty} ||(\pi_\alpha(g) - v(g)_k\sigma_0(v(g)_k^*))S_i|| \\ &= \lim_{k\to\infty} ||\alpha_g(S_i) - \mathrm{Ad}(v(g)_k)(S_i)||, \quad i = 1, 2, \cdots, n, \end{aligned}$$

which shows that α is strongly approximately inner. □

Example 5.7. Let G be a finite group and $\{S_g\}_{g\in G}$ be the canonical generators of the Cuntz algebra $\mathcal{O}_n$ where $n = |G|$. We introduce a quasi-free action α of G on $\mathcal{O}_n$ by

$$\alpha_g(S_h) = S_{gh}, \quad g, h \in G.$$

Then, Proposition 5.6, (1), implies that α has the Rohlin property. Thanks to [55], [39], we have $\mathrm{Aut}(\mathcal{O}_n) = \overline{\mathrm{Inn}(\mathcal{O}_n)}$ and $\mathcal{O}_n \cong \mathcal{O}_n \otimes \mathbb{M}_{n^\infty}$. Therefore, Theorem 4.2 implies

$$(\mathcal{O}_n, \alpha) \cong (\mathcal{O}_n \otimes \mathbb{M}_{n^\infty}, \mathrm{Id} \otimes \mu^G).$$

Example 5.8. There are two essentially different quasi-free $\mathbb{Z}_2$-actions on $\mathcal{O}_2$ defined by

$$\alpha(S_1) = S_1, \quad \alpha(S_2) = -S_2,$$
$$\beta(S_1) = -S_1, \quad \beta(S_2) = -S_2.$$

Thanks to Theorem 4.8 and Proposition 5.6, α and β are approximately representable and they are classified by the K-theoretical data of the crossed products. Since α is conjugate to the one in Example 5.7 for $G = \mathbb{Z}_2$, α has the Rohlin property. It is easy to show that $\mathcal{O}_2^\beta$ is isomorphic to $\mathcal{O}_4$ and

$$(K_0(\mathcal{O}_2^\beta), K_1(\mathcal{O}_2^\beta), [1]) \cong (\mathbb{Z}_3, \{0\}, 1).$$

Therefore, Theorem 4.8, (1) and the classification theorems [55, Theorem 7.1], [39] with direct computation of the K-theory of the fixed-point algebras imply

$$(\mathcal{O}_2, \beta) \cong (\mathcal{O}_4 \otimes \mathcal{O}_2, \mathrm{Id} \otimes \nu) \cong (\mathcal{O}_4 \otimes \mathbb{M}_{3^\infty}, \mathrm{Id} \otimes \bigotimes_{k=1}^{\infty} \mathrm{Ad}(w)),$$

where ν is as in Lemma 4.7 and

$$w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Part of these results can be also obtained by using D. E. Evans and H. Su's classification result [24].

Now we discuss the quasi-free actions of a cyclic group of prime power order.

Theorem 5.9. *Let p be a prime number, and $q = p^m$, $m \in \mathbb{N}$. We assume that n is a natural number satisfying $n \not\equiv 1 \mod p$. If α is a quasi-free action of $\mathbb{Z}_q$ on the Cuntz algebra $\mathcal{O}_n$, then α is approximately representable.*

Proof. Proposition 5.6 implies that α is strongly approximately inner. Thus, to prove the statement it suffices to show that $(\mathcal{O}_n^\alpha)^\infty \cap \mathcal{O}_n'$ contains a unital copy of $\mathbb{M}_q$ thanks to Lemma 3.10. Proposition 5.6 implies that $(\mathcal{O}_n^\alpha)^\infty \cap \mathcal{O}_n'$ contains a unital copy of $\mathcal{O}_n^\alpha$. Thus, we show that if $n \not\equiv 1 \mod p$, $K_0(\mathcal{O}_n^\alpha)$ is p-divisible, which will imply that $\mathcal{O}_n^\alpha$ contains a unital copy of $\mathbb{M}_q$.

We choose a character χ of $\mathbb{Z}_q$ with order q, and denote by n_i the multiplicity of χ^i in π_α, $i = 0, 1, \cdots, q-1$. Let λ be the permutation matrix of order q, i.e.

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Then, Lemma 5.1 implies

$$K_0(\mathcal{O}_n^\alpha) \cong \mathbb{Z}^q/(I-M)\mathbb{Z}^q,$$
$$K_1(\mathcal{O}_n^\alpha) \cong \mathrm{Ker}((I-M) : \mathbb{Z}^q \longrightarrow \mathbb{Z}^q),$$

where

$$M = \sum_{i=0}^{q-1} n_i \lambda^i.$$

To prove that $K_0(\mathcal{O}_n^\alpha)$ is p-divisible, it suffices to show $\det(I-M)\not\equiv 0 \mod p$. Indeed, over the finite field $\mathbb{F}_p=\mathbb{Z}/p\mathbb{Z}$, the permutation matrix λ is similar to the following Jordan block:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Thus, we get

$$\det(1-M)\equiv(1-n)^q\equiv 1-n \mod p,$$

which finishes the proof. □

Example 5.10. Let m be a natural number and α be a quasi-free $\mathbb{Z}_m$-action on $\mathcal{O}_2$ defined by

$$\alpha(S_1)=S_1,\quad \alpha(S_2)=e^{2\pi i/m}S_2.$$

Lemma 5.1 implies

$$K_0(\mathcal{O}_2\rtimes_\alpha\mathbb{Z}_m)\cong K_1(\mathcal{O}_2\rtimes_\alpha\mathbb{Z}_m)\cong\{0\},$$

which shows that $\mathcal{O}_2\rtimes_\alpha\mathbb{Z}_q\cong\mathcal{O}_2$. The same argument as in the above proof shows that α is approximately representable. Thus, Lemma 3.8 implies that $\widehat{\alpha}$ has the Rohlin property and

$$(\mathcal{O}_2\rtimes_\alpha\mathbb{Z}_m,\widehat{\alpha})\cong(\mathcal{O}_2\otimes\mathbb{M}_{m^\infty},\mathrm{Id}\otimes\mu^{\mathbb{Z}_m}).$$

In particular, $\widehat{\alpha}$ is approximately representable, and so α itself has the Rohlin property.

Example 5.11. Let α be a quasi-free $\mathbb{Z}_2$-action on $\mathcal{O}_3$ defined by

$$\alpha(S_1)=S_1,\quad \alpha(S_2)=S_2,\quad \alpha(S_3)=-S_3.$$

Then, Lemma 5.1 (or [16]) implies

$$K_0(\mathcal{O}_3\rtimes_\alpha\mathbb{Z}_2)\cong K_1(\mathcal{O}_3\rtimes_\alpha\mathbb{Z}_2)\cong\mathbb{Z}.$$

On the other hand, we have [15]

$$K_0(\mathcal{O}_3)\cong\mathbb{Z}_2,\quad K_1(\mathcal{O}_3)=\{0\}.$$

Therefore, Theorem 3.13 implies that the dual action $\widehat{\alpha}$ does not have the Rohlin property, which shows that α is *not* approximately representable. This means that there exists essential difference between quasi-free $\mathbb{Z}_2$-actions on the even and odd Cuntz algebras. This kind of difference was already observed in [24], but it appears to be rather of technical nature there.

Remark 5.12. In the second part, we will more systematically pursue K-theoretical constraints coming from the Rohlin property, and show the following: Let p,q,χ,α, π_α, and n_i, $i=0,1,\cdots,q-1$ be as in the proof of Theorem 5.9. We assume that $n=p^al+1$, $a,l,m\in\mathbb{N}$ such that l is prime to p. If $n_i\not\equiv 0 \mod p^a$ for some $i\neq 0$, then α is not approximately representable.

We determine when a quasi-free $\mathbb{Z}_2$-action has the Rohlin property.

Proposition 5.13. *Let α be a non-trivial quasi-free $\mathbb{Z}_2$-action on $\mathcal{O}_n$. Then, α has the Rohlin property if and only if n is even and the multiplicity of the trivial representation in π_α is either $n/2$ or $n/2+1$.*

Proof. Assume that α has the Rohlin property. Let ι be the inclusion map of the fixed-point algebra $\mathcal{O}_n^\alpha$ into $\mathcal{O}_n$. Then, Theorem 3.13 implies that $\mathrm{Im}(K_i(\iota))$ is injective. Moreover, since every automorphism of $\mathcal{O}_n$ acts on the K-groups trivially, we have

$$\mathrm{Im}(K_i(\iota)) = K_i(\mathcal{O}_n) = 2K_i(\mathcal{O}_n), \quad i = 0, 1.$$

This implies that $K_i(\mathcal{O}_n)$ is 2-divisible, and so n is even. Easy computation with Lemma 5.1 shows that the multiplicity of the trivial representation in π_α is either $n/2$ or $n/2+1$.

Now, we assume conversely that n is even and the trivial representation in π_α is either $n/2$ or $n/2+1$. Then, Theorem 5.9 implies that α is approximately representable, and so Lemma 3.8 implies that the dual action $\widehat{\alpha}$ has the Rohlin property. Direct computation with Lemma 5.1 implies that the crossed product $\mathcal{O}_n \rtimes_\alpha \mathbb{Z}_2$ has the same K-theory as $\mathcal{O}_n$, and in fact it is isomorphic to $\mathcal{O}_n$ thanks to the classification theorem. Since every automorphism of $\mathcal{O}_n$ is approximately inner and $\mathcal{O}_n$ is isomorphic to $\mathcal{O}_n \otimes \mathbb{M}_{2^\infty}$ for even n [55], Theorem 3.5 implies

$$(\mathcal{O}_n \rtimes_\alpha \mathbb{Z}_2, \widehat{\alpha}) \cong (\mathcal{O}_n \otimes \mathbb{M}_{2^\infty}, \mathrm{Id} \otimes \mu^{\mathbb{Z}_2}).$$

In particular, $\widehat{\alpha}$ is approximately representable. Thus, Lemma 3.8 shows that α has the Rohlin property. □

Definition 5.14. Let p be a prime number, and $q = p^m$, $m \in \mathbb{N}$. A $\mathbb{Z}_q$-action α on a unital simple C^*-algebra A is said to be of *Evans-Su type* if (A, α) is the inductive limit of the following system $\{(A_k, \alpha^{(k)})\}_{k=1}^\infty$:

$$A_k = \bigoplus_{j=1}^{l_k} \mathcal{O}_{n_{k,j}} \otimes \mathbb{M}_{m_{k,j}},$$

$$\alpha^{(k)} = \bigoplus_{j=1}^{l_k} \alpha^{(k,j)} \otimes \mathrm{Ad}(u_{k,j}),$$

where $\alpha^{(k,j)}$ is a quasi-free action, $u_{k,j} \in \mathbb{M}_{m_{k,j}}$ is a unitary, and $n_{k,j} \not\equiv 1 \mod p$ for every k, j.

In [24], D. E. Evans and H. Su classified this class of actions when $q = 2$. Thanks to Theorem 5.9, every Evans-Su type action is approximately representable and the K_1-group of the crossed product is trivial. Thus we get the following from Theorem 3.5, [55, Theorem 5.2], and [39]:

Corollary 5.15. *Let p be a prime number, and $q = p^m$, $m \in \mathbb{N}$. Let α and β be $\mathbb{Z}_q$-actions of Evans-Su type on a unital simple C^*-algebra A. Then the following two conditions are equivalent:*

(1) *α and β are conjugate.*

(2) *There exists an isomorphism φ from $K_0(A \rtimes_\alpha \mathbb{Z}_q)$ onto $K_0(A \rtimes_\beta \mathbb{Z}_q)$ such that*

$$\varphi \cdot K_0(\widehat{\alpha}) = K_0(\widehat{\beta}) \cdot \varphi,$$
$$\varphi([1_{A \rtimes_\alpha \mathbb{Z}_q}]) = [1_{A \rtimes_\beta \mathbb{Z}_q}],$$
$$\varphi([e_\alpha]) = [e_\beta].$$

In other words, the quadruple $(K_0(A \rtimes_\alpha \mathbb{Z}_q), [1_{A \rtimes_\alpha \mathbb{Z}_q}], [e_\alpha], K_0(\widehat{\alpha}))$ is a complete invariant for the Evans-Su type actions up to conjugacy.

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