CHARACTERIZATION OF ISOMORPHIC GROUP-SUBGROUP SUBFACTORS

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Abstract. We give a necessary and sufficient condition for two inclusions of finite groups to give rise to isomorphic subfactors via the crossed product construction.

1. Introduction

A subfactor is called a group-subgroup subfactor if it has the following form:

$$R \rtimes_\alpha G \supset R \rtimes_\alpha H,$$

where R is a factor, α is an outer action of a finite group G on R, and H is a subgroup of G. There have been several attempts in the literature to give combinatorial characterizations of group-subgroup subfactors [3], [4], [5], [6], and the categorical structure of the bimodules for them is completely described in terms of vector bundles over finite groups [12].

In [6], the author considered the problem of recovering the subfactor R and the action α from the group-subgroup subfactor, and formulated an equivalent problem, whose affirmative answer would be considered as a set theoretical analogues of the Doplicher-Roberts duality theorem [1]. However, the problem itself was left unsolved except for some special cases. In fact, V. Kodiyalam and V. S. Sunder showed, in the recent paper [10], that two pairs $\mathfrak{S}_4 \supset \langle(1234)\rangle$ and $\mathfrak{S}_4 \supset \langle(13),(24)\rangle$ give isomorphic subfactors, where $\mathfrak{S}_4$ is the permutation group of degree 4. This means that it is impossible to uniquely recover α from the group-subgroup subfactor.

The main purpose of the present paper is to give a necessary and sufficient condition for two pairs of finite groups to give isomorphic subfactors. With the condition at hand, we are able to provide a family of such examples, including Kodiyalam-Sunder's one as the first member. The key observation for our analysis is that a cocycle of the dual of the group G, introduced in [17], should naturally appear whenever two different pairs of groups give isomorphic subfactors. Hence, the present work is a natural continuation of [8].

Our basic reference for subfactors in this paper is [2]. For sectors, we use the notation in [7, Section 3].

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2. Main theorem

First, we recall a few results from [8]. Let G be a finite group and $\mathbb{C}G$ be the group algebra of G whose coproduct is denoted by δ_G. We identify $\mathbb{C}G$ with its image in the regular representation. We freely use the terminology and notation in [17] for cocycles of the dual of G. For a normalized cocycle $\omega \in Z^2(\widehat{G})$, we define $\delta_G^\omega = \mathrm{Ad}(\omega^*)\cdot\delta_G$. With the new coproduct δ_G^ω and the other Kac algebra operations remaining the same, $\mathbb{C}G$ becomes a new Kac algebra, which is denoted by $(\mathbb{C}G)^\omega$.

For a cocycle $\omega \in Z^2(K,\mathbb{T})$ of a finite abelian group K, we define a skew-symmetric bicharacter β_ω by $\beta(k,l)=\omega(k,l)\overline{\omega(l,k)}$, $k,l\in K$. ω is said to be non-degenerate if β_ω is a non-degenerate skew-symmetric bicharacter.

The following results are obtained in [8, Lemma 4.1, Lemma 4.3]:

Lemma 2.1. *Let G be a finite group and $\omega \in Z^2(\widehat{G})$ be a normalized cocycle.*

(1) *The following two conditions are equivalent:*
   (i) *δ_G^ω is cocommutative.*
   (ii) *There exist an abelian normal subgroup $N\subset G$ and a non-degenerate cocycle $\omega_1\in Z^2(\widehat{N})$ such that $[\omega]=[\omega_1]$ in $H^2(\widehat{G})$ and $[{}^g\omega_1]=[\omega_1]$ in $H^2(\widehat{N})$ for all $g\in G$, where ${}^g\omega_1=(g\otimes g)\omega_1(g^{-1}\otimes g^{-1})$. (Note that as $\mathbb{C}N$ is a Hopf subalgebra of $\mathbb{C}G$, $Z^2(\widehat{N})$ is regarded as a subset of $Z^2(\widehat{G})$.)*

(2) *Assume that N and $\omega=\omega_1$ satisfy the condition* (ii) *above. Then, there exist a map $\xi_g:\widehat{N}\longrightarrow\mathbb{T}$ for each $g\in G$ and a cocycle $\eta\in Z^2(G,N)$ such that for any pair $\sigma,\tau\in\widehat{N}$, we have*

$$\omega(\sigma^g,\tau^g)=\partial\xi_g(\sigma,\tau)\omega(\sigma,\tau),$$

$$\xi_{g_1}(\sigma)\xi_{g_2}(\sigma^{g_1})=\sigma(\eta(g_1,g_2))\xi_{g_1g_2}(\sigma),$$

where $\tau^g(n)=\tau(gng^{-1})$ for $n\in N$, $g\in G$, $\tau\in\widehat{N}$. We may take ξ_g so that $\xi_{gn}=\xi_g$ holds for all $g\in G$, $n\in N$, and η is regarded as an element in $Z^2(G/N,N)$. $G_\omega:=\{\xi_g g\}_{g\in G}$ is the set of group-like elements of $(\mathbb{C}G)^\omega$, and the group structure of G_ω is given by the extension

$$1\to N\to G_\omega\to G/N\to 1$$

whose difference from the extension G is given by η.

To state our main theorem, we give a variant of the above lemma involving a subgroup.

Lemma 2.2. *Let G and ω be as in Lemma 2.1. We further assume that H is a subgroup of G satisfying $\omega(\sigma,\tau)=1$ for all $\sigma,\tau\in K:=\widehat{N}\cap(H\cap N)^\perp$, where*

$$\widehat{N}\cap(H\cap N)^\perp=\{\sigma\in\widehat{N};\ \sigma(h)=1,\ \forall h\in H\cap N\}.$$

Then, we can choose ξ_g satisfying $\xi_h(\sigma)=1$ for all $h\in H$, $\sigma\in K$, and so $\sigma(\eta(h_1,h_2))=1$ holds for all $h_1,h_2\in H$, $\sigma\in K$. With this choice of ξ_h, $\eta(h_1,h_2)\in N\cap H$ holds for all $h_1,h_2\in H$, and the restriction of η to H can be regarded as a cocycle in $Z^2(H/(H\cap N),H\cap N)$.

Proof. Since $\omega(\sigma,\tau)=1$ for all $\sigma,\tau\in K$, we have $\partial\xi_h(\sigma,\tau)=1$ for all $h\in H$, $\sigma,\tau\in K$, and so the restriction of ξ_h to K is a homomorphism from K to $\mathbb{T}$. Since $\widehat{N}$ is abelian, the Pontryagin duality theorem implies that ξ_h extends to an homomorphism from $\widehat{N}$ to $\mathbb{T}$, say ξ'_h. Replacing ξ_h with $\xi_h\overline{\xi'_h}$ if necessary, we may assume $\xi_h(\sigma)=1$ for all $h\in H$, $\sigma\in K$. With such a choice of ξ_h, we have $\sigma(\eta(h_1,h_2))=1$ for all $h_1,h_2\in H$, $\sigma\in K$, and so the Pontryagin duality theorem again implies $\eta(h_1,h_2)\in H\cap N$. □

For H as above, we can consider an extension

$$1\to H\cap N\to H_\omega\to H/(H\cap N)\to 1$$

whose difference from the extension H is given by $\eta|_H$. Note that the pair $H_\omega\subset G_\omega$ depends only on the cohomology class $[\omega]\in H^2(\widehat{N},\mathbb{T})$, and it does not depend on the choice of ξ_g as far as the conditions of the above two lemmas are satisfied.

Strictly speaking, the conjugacy class of the group-subgroup subfactor $R\rtimes_\alpha G\supset R\rtimes_\alpha H$ depends on the factor R and the action α even though α is always assumed to be outer in what follows. However, the paragroup [2] (or the standard invariant [15]) for the group-subgroup subfactor is determined only by the pair $G\supset H$, and it does not depend on either R or α. In the sequel, whenever we state that two pairs of groups $G\supset H$ and $G'\supset H'$ give isomorphic subfactors, we mean that the paragroups of the resulting group-subgroup subfactors are isomorphic. (Another prescription for this not essential problem would be to assume that R is the hyperfinite II_1 factor, or II_∞ factor when R needs to be infinite, because these factors have only one conjugacy class of outer actions of G [9].)

Theorem 2.3. *Let $G\supset H$ be an inclusion of finite groups such that H does not contain any non-trivial normal subgroup of G. We assume the same condition for the inclusion $G'\supset H'$. Then, the following two conditions are equivalent:*

1. *There exist an abelian normal subgroup N and a cocycle $\omega\in Z^2(\widehat{N},\mathbb{T})$ satisfying the conditions of Lemma 2.1 and Lemma 2.2 such that there exists a group isomorphism of G' onto G_ω mapping H' onto H_ω.*
2. *$G\supset H$ and $G'\supset H'$ give isomorphic subfactors.*

Remark 2.4. We will actually show in the proof that there is one-to-one correspondence between the following two sets: (1) the set of inner conjugacy (either considered in $R\rtimes_\alpha G$ or considered in $R\times_\alpha H$) classes of subfactor $Q\subset R\rtimes_\alpha H$ such that $R\rtimes_\alpha G=Q\rtimes G'$ and $R\rtimes_\alpha H=Q\rtimes H'$ hold for some $G'\supset H'$ with the above condition, (2) the set of cohomology classes $[\omega]\in H^2(\widehat{G})$ satisfying the conditions of Lemma 2.1 and Lemma 2.2. This gives an answer to a problem posed in [6, Section 5].

The proof of Theorem 2.3 will be given in Section 3 by showing several lemmas.

3. Proof of Theorem 2.3

3.1. **Proof of (1) ⇒ (2).** Let G, N, and ω be as in Lemma 2.1. We choose a factor R_0 and an outer action $\alpha^{(0)}$ of G on R_0. We set $R=R_0\otimes\mathbb{B}(\ell^2(\widehat{N}))$, $\alpha_g=\alpha^{(0)}_g\otimes\mathrm{Id}$,

$M=R\rtimes_\alpha G=(R_0\rtimes_{\alpha^{(0)}}G)\otimes\mathbb{B}(\ell^2(\widehat{N}))$. We denote by $\lambda(g)$ the implementing unitary of $\alpha^{(0)}_g$ in $R_0\rtimes_{\alpha^{(0)}}G$.

Let $\{e_\sigma\}_{\sigma\in\widehat{N}}$ be the canonical orthonormal basis of $\ell^2(\widehat{N})$. We introduce an N-valued cocycle representation u of G (actually of G/N) and a projective representation v of $\widehat{N}$ on $\ell^2(\widehat{N})$ by

$$u(g)e_\tau=\overline{\xi_g(\tau^{g^{-1}})}e_{\tau^{g^{-1}}},\quad g\in G,\ \tau\in\widehat{N},$$

$$v(\sigma)e_\tau=\overline{\omega(\sigma,\tau)}e_{\sigma\tau},\quad \sigma,\tau\in\widehat{N}.$$

Then, the following hold:

$$\begin{aligned}
u(g_1)u(g_2) &= \nu(\eta(g_1,g_2)^{-1})u(g_1g_2),\quad g_1,g_2\in G,\\
v(\sigma_1)v(\sigma_2) &= \overline{\omega(\sigma_1,\sigma_2)}v(\sigma_1\sigma_2),\quad \sigma_1,\sigma_2\in\widehat{N},\\
u(g)v(\sigma^g) &= \overline{\xi_g(\sigma)}v(\sigma)u(g),\quad g\in G,\ \sigma\in\widehat{N},\\
v(\sigma)\nu(n) &= \overline{\sigma(n)}\nu(n)v(\sigma),\quad n\in N,\ \sigma\in\widehat{N},
\end{aligned}$$

where ν is the natural representation of N on $\ell^2(\widehat{N})$, namely $\nu(n)e_\sigma=\sigma(n)e_\sigma$.

Let $\theta^{(0)}$ be the dual action of the restriction of $\alpha^{(0)}$ to N acting on $R_0\rtimes_{\alpha^{(0)}}N$, and $\theta_\sigma=\theta^{(0)}_\sigma\otimes\mathrm{Ad}(v(\sigma))$, $\sigma\in\widehat{N}$. We set P to be the fixed point subalgebra of $R\rtimes_\alpha N$ under θ.

Lemma 3.1. *Let the notation be as above. Then, $M\supset P$ is an irreducible inclusion satisfying $M=P\rtimes G_\omega$.*

Proof. It is an easy exercise of the Frobenius reciprocity to show that $M\supset P$ is an irreducible inclusion of depth 2. Let $\mathcal{N}_M(P)$ be the normalizers of P in M, and $U(P)$ the unitary group of P. We show $\mathcal{N}_M(P)/U(P)\cong G_\omega$, which will prove the statement by the standard argument (see [11] and [14] for example). Setting $\mu(g)=\lambda(g)\otimes u(g)$, $g\in G$, we claim that the following holds on $R\rtimes_\alpha N$:

$$\mathrm{Ad}(\mu(g))\cdot\theta_{\sigma^g}=\theta_\sigma\cdot\mathrm{Ad}(\mu(g)),\quad g\in G,\ \sigma\in\widehat{N}.$$

Indeed, for $n\in N$, $x\in R_0$, and $y\in\mathbb{B}(\ell^2(\widehat{N}))$, we have

$$\begin{aligned}
\theta_\sigma\cdot\mathrm{Ad}(\mu(g))(x\lambda(n)\otimes y) &= \theta_\sigma(\alpha^{(0)}_g(x)\lambda(n^{g^{-1}})\otimes u(g)yu(g)^*)\\
&= \theta^{(0)}_\sigma(\alpha^{(0)}_g(x)\lambda(n^{g^{-1}}))\otimes v(\sigma)u(g)yu(g)^*v(\sigma)^*\\
&= \sigma^g(n)\alpha^{(0)}_g(x)\lambda(n^{g^{-1}})\otimes u(g)v(\sigma^g)yv(\sigma^g)^*u(g)^*\\
&= \mathrm{Ad}(\mu(g))\cdot\theta_{\sigma^g}(x\lambda(n)\otimes y).
\end{aligned}$$

Therefore, $\mu(g)$ normalizes P for all $g\in G$. Note that $(\lambda(n)\otimes\nu(n))\in P$ and $u(ng)=u(g)$ for all $n\in N$ and $g\in G$. Thus, for $g_1,g_2\in G$, we get

$$\begin{aligned}
\mu(g_1)\mu(g_2) &= (1\otimes\nu(\eta(g_1,g_2)^{-1}))\mu(g_1g_2)\\
&= (\lambda(\eta(g_1,g_2))\otimes\nu(\eta(g_1,g_2)))^*\mu(\eta(g_1,g_2)g_1g_2).
\end{aligned}$$

Since $\mu(g)$ does not belong to P for $g\neq e$ and the index $[M:P]$ is the same as the order of G_ω, we get $\mathcal{N}_M(P)/U(P)\cong G_\omega$. ∎

Now, we assume that H is a subgroup of G satisfying the condition of Lemma 2.2, and use the notations $L = R \rtimes_\alpha H$, $K = \widehat{N} \cap (H \cap N)^\perp$. In order to show that the pair $G_\omega \supset H_\omega$ gives the same subfactor as the one given by $G \supset H$, we have to find a unitary $w \in R \rtimes_\alpha N$ satisfying $wPw^* \subset L$. For $\sigma \in \widehat{N}$, we set

$$p_\sigma = \frac{1}{\# N} \sum_{n \in N} \overline{\sigma(n)} \lambda(n),$$

which satisfies $\theta^{(0)}_\tau(p_\sigma) = p_{\sigma\tau^{-1}}$. Choosing a transversal J of $\widehat{N}/K$, we introduce a unitary $w \in R \rtimes_\alpha N$ by

$$w = \sum_{\sigma \in K} \sum_{\tau \in J} p_{\sigma\tau} \otimes v(\sigma).$$

Since the restriction of v to K is a genuine representation, w satisfies

$$\theta_\sigma(w) = (1 \otimes v(\sigma))w = w(1 \otimes v(\sigma)), \quad \sigma \in K.$$

Let $\vartheta = \mathrm{Ad}(w) \cdot \theta \cdot \mathrm{Ad}(w^*)$. Then, the above relation shows that $\vartheta_\sigma = (\theta^{(0)}_\sigma \otimes \mathrm{Id})$ holds for all $\sigma \in K$. As $\tilde{P} := wPw^*$ is the fixed point algebra of ϑ, we have the inclusion relation

$$\tilde{P} \subset (R \rtimes_\alpha N)^{\vartheta|_K} = R \rtimes_\alpha (N \cap H).$$

This implies that $R \rtimes_\alpha H$ is an intermediate subfactor of $\tilde{P} \subset M = \tilde{P} \rtimes G_\omega$, and so the Galois correspondence implies that $R \rtimes_\alpha H$ is the crossed product of $\tilde{P}$ by a subgroup of G_ω. Note that the proof of Lemma 3.1 actually shows that $\{w\mu(g)w^*\}_{g \in G}$ is a transversal of $\mathcal{N}_M(\tilde{P})/U(\tilde{P})$. Therefore, the following lemma implies that $G \supset H$ and $G_\omega \supset H_\omega$ give isomorphic (in fact the same) subfactors.

Lemma 3.2. $\{w\mu(h)w^*\}_{h \in H} \subset R \rtimes_\alpha H$.

Proof. We show $(\lambda(h)^* \otimes 1)w\mu(h)w^* \in R \rtimes_\alpha (N \cap H)$ for $h \in H$, which will imply $w\mu(h)w^* \in R \rtimes_\alpha H$. Indeed, we have

$$\begin{aligned}(\lambda(h)^* \otimes 1)w\mu(h) &= \sum_{\sigma \in K, \tau \in J} (p_{\sigma^h \tau^h} \otimes \xi_h(\sigma)u(h)v(\sigma^h)) \\ &= \sum_{\sigma \in K, \tau \in J} (p_{\sigma\tau^h} \otimes u(h)v(\sigma)),\end{aligned}$$

and so $(\lambda(h)^* \otimes 1)w\mu(h) \in R \rtimes_\alpha N$ satisfies

$$(\theta^{(0)}_\sigma \otimes \mathrm{Id})((\lambda(h)^* \otimes 1)w\mu(h)) = (\lambda(h)^* \otimes 1)w\mu(h)(1 \otimes v(\sigma)), \quad \sigma \in K.$$

Therefore, we get

$$(\lambda(h)^* \otimes 1)w\mu(h)w^* \in (R \rtimes_\alpha N)^{(\theta^{(0)} \otimes \mathrm{Id})|_K} = R \rtimes_\alpha (N \cap H),$$

and so $w\mu(h)w^* \in R \rtimes_\alpha H$. ∎

3.2. **Proof of (2) $\Rightarrow$ (1).** Let R be an infinite factor and α be an outer action of G on R. We set $M = R \rtimes_\alpha G$, $L = R \rtimes_\alpha H$. We assume that there exist a subfactor $Q \subset L$ and an outer action α' of G' on Q such that $M = Q \rtimes_{\alpha'} G'$ and $L = Q \rtimes_{\alpha'} H'$. We denote by γ the canonical endomorphism for $M \supset L$ (see [13] for the definition).

Lemma 3.3. *Let γ_R and γ_Q be the canonical endomorphisms for $M \supset R$ and $M \supset Q$ respectively. Then, $[\gamma_R] = [\gamma_Q]$ in* $\mathrm{Sect}(M)$*.*

Proof. Let $\{[\rho_\lambda]\}_{\lambda\in\Lambda} \subset \mathrm{Sect}(M)$ be the set of irreducible sectors contained in the powers of $[\gamma]$. We claim that

$$[\gamma_R] = \bigoplus_{\lambda\in\Lambda} d(\rho_\lambda)[\rho_\lambda]$$

holds. Let $\hat{\alpha}$ be the dual action of α considered as a Roberts action [16]. Then, it is well-known that $[\gamma_R] = [\hat{\alpha}_{\pi_G}]$ holds, where π_G is the regular representation. On the other hand, [12] shows that we have $[\gamma] = [\hat{\alpha}_{\pi_{G/H}}]$, where $\pi_{G/H}$ is the quasi-regular representation of G on $\ell^2(G/H)$. Since H does not contain any non-trivial normal subgroup of G, $\pi_{G/H}$ is a faithful representation. Therefore, every irreducible representation of G is contained in some tensor power of $\pi_{G/H}$ as G is a finite group, which shows the claim. The same argument applied to $M \supset Q$ shows $[\gamma_P] = [\gamma_Q]$. □

In [8, Theorem 3.3], it is shown that there exists a one-to-one correspondence between $H^2(\widehat{G})$ and the set of inner conjugacy classes of subfactors of M whose canonical endomorphism is equivalent to γ_R. We denote by ω_Q the cocycle corresponding to Q. Furthermore, since $M = Q \rtimes_{\alpha'} G'$, $\delta_G^{\omega_Q}$ is cocommutative and ω_Q satisfies the condition of Lemma 2.1. We denote by N the abelian normal subgroup of G determined by Lemma 2.1. Then, we may assume $\omega_Q \in Z^2(\widehat{N}, \mathbb{T})$ and may identify G' with G_{ω_Q}. We set $K = \widehat{N} \cap (H \cap N)^\perp$.

Lemma 3.4. $\beta_{\omega_Q}(\sigma,\tau) = 1$ *for all* $\sigma, \tau \in K$*.*

Proof. Let $\mathcal{E} = \mathcal{E}(\gamma,\gamma)$ and $\mathcal{E}^{\omega_Q} = \mathcal{E}^{\omega_Q}(\gamma,\gamma)$ be the permutation symmetries coming from those of $\hat{\alpha}$ and $\hat{\alpha'}$ respectively [1]. We claim that for an isometry $W \in (\mathrm{Id}, \gamma|_L)$,

$$\mathcal{E}W = \mathcal{E}^{\omega_Q}W = W$$

holds. Indeed, we choose a Hilbert space $\mathcal{H}$ in $M \rtimes_{\hat{\alpha}} \widehat{G}$ with left support 1 such that the restriction of the second dual action $\hat{\hat{\alpha}}$ to $\mathcal{H}$ is equivalent to $\pi_{G/H}$. We set $\dot{g} = gH$ for $g \in G$ and choose a transversal Ω of G/H with $e \in \Omega$. Let $\{S_{\dot{g}}\}_{g\in\Omega}$ be an orthonormal basis of $\mathcal{H}$ satisfying $\hat{\hat{\alpha}}_g(S_{\dot{k}}) = S_{\dot{gk}}$. Then, since $[\hat{\alpha}_{G/H}] = [\gamma]$, we may assume that γ and $\mathcal{E}$ are given by

$$\gamma(x) = \sum_{g\in\Omega} S_{\dot{g}} x S_{\dot{g}}^*, \quad x \in M,$$

$$\mathcal{E} = \sum_{g,k\in\Omega} S_{\dot{g}} S_{\dot{k}} S_{\dot{g}}^* S_{\dot{k}}^*.$$

Note that $\dim(\mathrm{Id}, \gamma|_L) = 1$ and W is uniquely determined by up to a phase factor. Thus, thanks to [6], we may assume

$$W = \sum_{g\in\Omega} S_{\dot{g}} S_{\dot{g}} S_{\dot{g}}^*,$$

and so, $\mathcal{E}W = W$ holds. Applying the same argument to $\hat{\alpha}'$, we get the claim.

The relationship between $\mathcal{E}$ and $\mathcal{E}^{\omega_Q}$ is obtained in [8, Remark 4.2], and we have

$$\mathcal{E}^{\omega_Q} = (\pi_{G/H} \otimes \pi_{G/H})(\beta_{\omega_Q})\mathcal{E},$$

where $\pi_{G/H}$ is regarded as a representation of the group algebra $\mathbb{C}G$, and we use the usual tensor product notation inside the Cuntz algebra $C^*\{S_{\dot{g}}\}_{g\in\Omega}$. The above expression makes sense as we have

$$\beta_{\omega_Q} \in \ell^\infty(\widehat{N}) \otimes \ell^\infty(\widehat{N}) = \mathbb{C}N \otimes \mathbb{C}N \subset \mathbb{C}G \otimes \mathbb{C}G.$$

Therefore, we get,

$$\begin{aligned}
1 &= S_{\dot{e}}^* S_{\dot{e}}^* W S_{\dot{e}} = S_{\dot{e}}^* S_{\dot{e}}^* \mathcal{E}^{\omega_Q} W S_{\dot{e}} = S_{\dot{e}}^* S_{\dot{e}}^* (\pi_{G/H} \otimes \pi_{G/H})(\beta_{\omega_Q})\mathcal{E} W S_{\dot{e}} \\
&= S_{\dot{e}}^* S_{\dot{e}}^* (\pi_{G/H} \otimes \pi_{G/H})(\beta_{\omega_Q}) S_{\dot{e}} S_{\dot{e}} \\
&= \frac{1}{\#N^2} \sum_{\sigma_1,\sigma_2\in\widehat{N}} \sum_{n_1,n_2\in N} \sigma_1(n_1)\sigma_2(n_2)\beta_{\omega_Q}(\sigma_1,\sigma_2) S_{\dot{e}}^* \pi_{G/H}(n_1) S_{\dot{e}} S_{\dot{e}}^* \pi_{G/H}(n_2) S_{\dot{e}} \\
&= \frac{1}{\#N^2} \sum_{\sigma_1,\sigma_2\in\widehat{N}} \sum_{n_1,n_2\in H\cap N} \sigma_1(n_1)\sigma_2(n_2)\beta_{\omega_Q}(\sigma_1,\sigma_2) \\
&= \frac{\#(H\cap N)^2}{\#N^2} \sum_{\sigma_1,\sigma_2\in K} \beta_{\omega_Q}(\sigma_1,\sigma_2) \\
&= \frac{1}{\#K^2} \sum_{\sigma_1,\sigma_2\in K} \beta_{\omega_Q}(\sigma_1,\sigma_2),
\end{aligned}$$

which shows $\beta_{\omega_Q}(\sigma_1, \sigma_2) = 1$ for all $\sigma_1, \sigma_2 \in K$. □

The above lemma shows that the restriction of ω_Q to K is a coboundary, and we may and do assume $\omega_Q(\sigma_1, \sigma_2) = 1$ for all $\sigma_1, \sigma_2 \in K$.

Choosing a type I factor isomorphic to $\mathbb{B}(\ell^2(\widehat{N}))$ in the fixed point algebra R^α, we may assume $R = R_0 \otimes \mathbb{B}(\ell^2(\mathbb{N}))$ and $\alpha = \alpha^{(0)} \otimes \mathrm{Id}$ as before. Thus, we can use the same notation, with identification $\omega = \omega_Q$, as in the first half of this section

Lemma 3.5. *Q is inner conjugate to P in M.*

Proof. Let Γ be the dual action of α, which is, however, regarded as the Hopf algebra action of $\mathbb{C}G$ this time. Namely, Γ is a homomorphism from M into $M \otimes \mathbb{C}G$ given by

$$\Gamma(x) = x \otimes 1, \quad x \in R,$$

$$\Gamma(\lambda(g) \otimes 1) = \lambda(g) \otimes 1 \otimes g, \quad g \in G.$$

We define a projection $e(\sigma) \in \mathbb{C}G$ for $\sigma \in \widehat{N}$ by

$$e(\sigma) = \frac{1}{\#N} \sum_{n \in N} \overline{\sigma(n)} n,$$

and set

$$Z = \sum_{\sigma \in \widehat{N}} v(\sigma) \otimes e(\sigma) \in M \otimes \mathbb{C}G.$$

Then, Z is a unitary in $M \otimes \mathbb{C}G$ satisfying

$$(1 \otimes \delta_G)(Z) = (1 \otimes \omega_Q)(Z \otimes 1)(\Gamma \otimes \mathrm{Id})(Z).$$

It is a routine work to show that P is the fixed point algebra of M under the action $\mathrm{Ad}(Z) \cdot \Gamma$ of $(\mathbb{C}G)^\omega$. Therefore, [8, Corollary 3.6] shows that Q and P are inner conjugate in M. $\square$

Now, we are ready to finish the proof. We denote by ι_1, ι_2, and ι_3 the inclusion maps $Q \hookrightarrow L$, $\tilde{P} \hookrightarrow L$, and $L \hookrightarrow M$ respectively. Then, thanks to the above lemma there exists an isomorphism φ of Q onto $\tilde{P}$ such that $[\iota_3\iota_1] = [\iota_3\iota_2\varphi]$ holds in $\mathrm{Sect}(M, Q)$. Thus, the Frobenius reciprocity implies that $\overline{\iota_3}\iota_3\iota_1$ contains $\iota_2\varphi$. On the other hand, since $\overline{\iota_3}\iota_3\iota_1$ is contained in $\iota_1\overline{\iota_1}\overline{\iota_3}\iota_3\iota_1$ and

$$[\iota_1\overline{\iota_1}\overline{\iota_3}\iota_3\iota_1] = \bigoplus_{g \in G'} [\iota_1\alpha'_g],$$

we have $[\iota_2\varphi\alpha'_g] = [\iota_1]$ for some $g \in G'$. This means that $\tilde{P}$ and Q are inner conjugate in L, and so we get

$$(\tilde{P} \subset L \subset M) \cong (Q \subset L \subset M).$$

This shows that there exists a group isomorphism of G' onto G_{ω_Q} mapping H' onto H_{ω_Q}.

4. A series of examples

We take an example considered in [8, Section 4], which also fits into our situation. Let m be a power of 2. We set $N = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, $G = N \rtimes SL(2, \mathbb{Z}/m\mathbb{Z})$, $H = \langle p, q \rangle \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ with $p = (1, 0) \in N$ and

$$q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z}/m\mathbb{Z}).$$

Our convention here is that for $n \in N$ and

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}/m\mathbb{Z}),$$

${}^t(n^g) = g^{-1} \cdot {}^t n$. We identify $\widehat{N}$ with $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ via the paring

$$\langle (n_1, n_2), (\tau_1, \tau_2) \rangle = \zeta_m^{n_1\tau_1 + n_2\tau_2},$$

where $\zeta_m = \exp(\frac{2\pi i}{m})$. Therefore, for $\tau = (\tau_1, \tau_2) \in \widehat{N}$, we have $\tau^g = \tau \cdot g = (a\tau_1 + c\tau_2, b\tau_1 + d\tau_2)$. We define a cocycle $\omega \in Z^2(\widehat{N}, \mathbb{T})$ by

$$\omega((x_1, x_2), (y_1, y_2)) = \zeta_m^{x_1 y_2}.$$

Then, since β_ω is invariant under the $SL(2,\mathbb{Z}/m\mathbb{Z})$-action, G, N, and ω satisfy the assumption of Lemma 2.1. Since we have $\widehat{N}\cap(H\cap N)^\perp=\langle(0,1)\rangle$, H satisfies the condition of Lemma 2.2. It is a routine work to show the following [8, Lemma 4.5]:

Lemma 4.1. *For $(s,t)\in\widehat{N}$ and*

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z}/m\mathbb{Z}),$$

a general solution of the equation ${}^g\omega=\partial\xi_g\omega$ in Lemma 2.1 is given by

$$\xi_g(s,t)=\xi_g(1,0)^s\xi_g(0,1)^t\zeta_m^{-[\frac{abs(s-1)}{2}+\frac{cdt(t-1)}{2}+bcst]},$$

where $\xi_g(1,0)$ and $\xi_g(0,1)$ satisfy $\xi_g(1,0)^m=(-1)^{ab}$, $\xi_g(0,1)^m=(-1)^{cd}$. In particular, the following is a solution: $\xi_g(s,t)=\zeta_{2m}^{-(abs^2+cdt^2+2bcst)}$ where $0\leq a,b,c,d<m$.

Simple computation using the above lemma shows $H_\omega\cong\mathbb{Z}_{2m}\times\mathbb{Z}_{m/2}$, which is not isomorphic to H. When $m=4$, it is also shown in [8, Remark 4.7] that G_ω is not isomorphic to G.

When $n=2$, G is isomorphic to the symmetric group $\mathfrak{S}_4$ and the above example gives Kodiyalam-Sunder's one. To see $\mathfrak{S}_4\cong\mathfrak{S}_{4\omega}$, it is more convenient to use another expression of the cocycle. Let $\widehat{N}\setminus\{e\}=\{\sigma_1,\sigma_2,\sigma_3\}$. Then, $SL(2,\mathbb{Z}/2\mathbb{Z})$ acts on $\widehat{N}$ as the permutations of the above three elements, and we identify $SL(2,\mathbb{Z}/2\mathbb{Z})$ with the symmetric group $\mathfrak{S}_3$. We define $\omega_1\in Z^2(\widehat{N},\mathbb{T})$ by

$$\omega_1=(e,e)=\omega_1(e,\sigma_j)=\omega_1(\sigma_j,e)=\omega_1(\sigma_j,\sigma_j)=1,\quad j=1,2,3$$

$$\omega_1(\sigma_j,\sigma_{j+1})=-\omega_1(\sigma_{j+1},\sigma_j)=i,\quad j=1,2,3,$$

where j is understood modulo 3. Then, ω_1 represents a unique non-trivial element in $H^2(\widehat{N},\mathbb{T})$. For every odd permutation $g\in\mathfrak{S}_3$, we have ${}^g\omega_1=\overline{\omega_1}$ while every even permutation in $\mathfrak{S}_3$ leaves ω_1 invariant. We define an $\mathfrak{S}_3$-invariant function ξ on $\widehat{N}$ by

$$\xi(e)=1,\quad \xi(\sigma_1)=\xi(\sigma_2)=\xi(\sigma_3)=-1.$$

Then, we have $\omega_1^2=\partial\xi$. Therefore, we can set $\xi_g=1$ for every even permutation g and $\xi_g=\xi$ for every odd g. Therefore, we get $\eta=1$, and so $\mathfrak{S}_4\cong\mathfrak{S}_{4\omega_1}$.

In a sense, this isomorphism should be considered as just a coincidence. Indeed, the above computation shows that the map $\mathbb{C}\mathfrak{S}_4\ni g\mapsto\xi_g g\in(\mathbb{C}\mathfrak{S}_4)^{\omega_1}$ extends to a Hopf algebra isomorphism, say Φ. A little more careful examination shows that the restriction of Φ to the center of $\mathbb{C}\mathfrak{S}_4$ exchanges two minimal projections of $\mathbb{C}G$. (Note that the underlying algebra of $(\mathbb{C}\mathfrak{S}_4)^{\omega_1}$ is the *same* as $\mathbb{C}\mathfrak{S}_4$ by definition.) Let $\psi:\mathrm{Rep}(\mathfrak{S}_4)\to\mathrm{Rep}(\mathfrak{S}_{4\omega_1})$ be a tensor category isomorphism given by [8, Corollary 3,6, (4)], and $\Phi^*:\mathrm{Rep}(\mathfrak{S}_{4\omega_1})\to\mathrm{Rep}(\mathfrak{S}_4)$ be the isomorphism induced by Φ. Then, the above observation shows that the composition $\Phi^*\cdot\psi$ is an automorphism of the tensor category $\mathrm{Rep}(\mathfrak{S}_4)$ that exchanges two inequivalent irreducibles. As $\mathrm{Aut}(\mathfrak{S}_4)=\mathrm{Inn}(\mathfrak{S}_4)$, we can conclude the following:

Corollary 4.2. *There exists an automorphism of the tensor category* $\mathrm{Rep}(\mathfrak{S}_4)$ *that does not come from a group automorphism of* $\mathfrak{S}_4$.

This can be seen also from the fact that two quasi-regular representations $\pi_{\mathfrak{S}_4/\langle(1234)\rangle}$ and $\pi_{\mathfrak{S}_4/\langle(13),(24)\rangle}$, which are shown to be inequivalent by simple computation, correspond to the same canonical endomorphism γ in Section 3.

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