ANTICANONICAL DIVISORS OF A MODULI SPACE OF PARABOLIC VECTOR BUNDLES OF HALF WEIGHT ON $\mathbb{P}^1$

TAKESHI ABE

1. Introduction

Let $\mathcal{M}_0$ [resp. $\mathcal{M}_1$] be a coarse moduli space of rank 2 semistable vector bundles of even [resp. odd] degree with fixed determinant on a smooth projective curve X. The Picard group is infinite cyclic . Let L be the ample generator. The dimension of a vector space $\mathrm{H}^0(\mathcal{M}_i, L^m)$ $(i = 0, 1)$ is given by the Verlinde formula. For small $m > 0$, the meaning of this dimension can be explained in the framework of algebraic geometry. For example, we have

$$\dim \mathrm{H}^0(\mathcal{M}_0, L) = 2^g,$$

where g is the genus of X. On the otherhand, we have

$$\dim \mathrm{H}^0(\mathrm{Jac}(X), \mathcal{O}(2\Theta)) = 2^g.$$

In fact we have a natural isomorphism between these two vector spaces (See [1]). In [2], the meaning of the two equations

$$\begin{aligned} \dim \mathrm{H}^0(\mathcal{M}_0, L^2) &= 2^{g-1}(2^g + 1) \\ \dim \mathrm{H}^0(\mathcal{M}_1, L) &= 2^{g-1}(2^g - 1) \end{aligned}$$

are clarified. The above dimensions are the number of even or odd theta characterictics on X. Beauville associated to an even [resp. odd] theta characterictic κ a divisor D_κ on $\mathcal{M}_0$ [resp. $\mathcal{M}_1$] that can be described from a moduli-theoretic viewpoint, and proved that they form a basis of $\mathrm{H}^0(\mathcal{M}_0, L^2)$ [resp. $\mathrm{H}^0(\mathcal{M}_1, L)$]. In [13], two vector spaces $\mathrm{H}^0(\mathcal{M}_0, L^4)$ and $\mathrm{H}^0(\mathcal{M}_1, L^2)$ are considered.

The purpose of this paper is to carry out a similar study for a moduli space $\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)$ of rank 2 semistable parabolic vector bundles with half weights of degree zero on $\mathbb{P}^1$.

To be more precise, by Verlinde formula, we have

$$\dim \mathrm{H}^0(\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I), K^{-1}_{\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)}) = \frac{2^{2g+1} + 1}{3}$$

Department of Mathematics, Faculty of Science, Kyoto University,
Kyoto, 606-8502, Japan
email:abeken@kusm.kyoto-u.ac.jp.

for $I = \{x_1, \dots, x_{2g+2}\}$, and

$$\dim \mathrm{H}^0(\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I), K^{-1}_{\overline{\mathcal{M}}^{Par}(\mathbb{P}^1;I)}) = \frac{2^{2g}-1}{3}$$

for $I = \{x_1, \dots, x_{2g+1}\}$. Let us consider the case when $|I|$ is even. Let C be the hyperelliptic curve whose branch locus is I. By a result of Bhosle in [4], $\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)$ is isomorphic to a moduli space $\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}$ of rank 2 semistable vector bundles endowed with an involution action with trivial determinat. On $\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}$, we shall find as many effective anticanonical divisors as the dimension of $\mathrm{H}^0(\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I), K^{-1}_{\overline{\mathcal{M}}^{Par}(\mathbb{P}^1;I)})$ that are described moduli-theoretically, and shall prove that they form a basis of $\mathrm{H}^0(\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}, K^{-1}_{\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}})$ (Theorem3.4). In Section 4, we investigate a relation between these divisors and those divisors constructed by Beauville (Theorem4.1, 4.2). It would be desirable that the divisors constructed on $\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}$ in Section 3 are defined set-theoretically. So we shall prove the reducedness of these divisors in Section 5.

Notation.
- Otherwise mentioned, all schemes are of finite type over $\mathbb{C}$.
- If a group $G = \langle g | g^2 = e \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ acts on a coherent sheaf F on a scheme X, F^+ [resp. F^-] stands for the g-invariant [resp. $(-g)$-invariant] subsheaf of F. For a vector space V acted on by G, we define V^+ and V^- similarly. $\chi(F)^+$ means $\sum(-1)^i \mathrm{H}^i(X, F)^+$.
- If E is a locally free sheaf, $\mathcal{E}nd^\circ(E)$ is a sheaf of traceless endomorphisms.
- Otherwise mentioned, p_X, p_Y, p_Z etc. stand for projections to the factors X, Y, Z etc..

2. Preliminaries

Let X be an smooth irreducible projective curve over $\mathbb{C}$ and I a non-empty finite set of points of X. Let E be a vector bundle on X.

Definition 2.1. A parabolic structure on E is giving, at each point $x \in I$, a filtration $E_x = F_1(E)_x \supset \cdots \supset F_{n_x(E)}(E) \supset F_{n_x(E)+1}(E) = 0$ and a sequence of real numbers, called parabolic weights, $0 \le a_1(x) < \cdots < a_{n_x(E)}(x) < 1$. The parabolic degree of E, denoted by $\mathrm{pardeg}(E)$, is defined by

$$\mathrm{pardeg}(E) := \deg E + \sum_{x \in I} \sum_{i=1}^{n_x(E)} a_i(x) \Big(\dim F_i(E)_x - \dim F_{i+1}(E)_x \Big)$$

2.2. Let E' be a subbundle of a parabolic vector bundle E. We can equip E' with the canonical parabolic structure: the filtration of E' consists of the distinct ones of $\left\{ E'_x \cap F_i(E)_x \right\}$, and the parabolic weight $a'_j(x)$ of E' is given by $a'_j(x) = a_i(x)$, where i is the biggest integer satisfying $F_j(E')_x = E'_x \cap F_i(E)_x$.

Definition 2.3. A parabolic vector bundle E is said to be semistable [resp. stable] if for any subbundle E' of E with $0 < \mathrm{rank}E' < \mathrm{rank}E$, we have

$$\frac{\mathrm{pardeg}E'}{\mathrm{rank}E'} \leq \frac{\mathrm{pardeg}E}{\mathrm{rank}E} \quad [\mathrm{resp.}<],$$

where E' is equipped with the canonical parabolic structure.

2.4. In this paper, we are concerned with only rank 2 parabolic bundles of parabolic weights $(0, \frac{1}{2})$ at each $x \in I$. Let $\overline{\mathcal{M}}^{Par}(X; I)$ [resp.$\mathcal{M}^{Par}(X; I)$] be the coarse moduli space of semistable [resp. stable] rank 2 parabolic vector bundles with trivial determinant of parabolic weights $(0, \frac{1}{2})$ at each point $x \in I$

2.5. Given a family of parabolic rank 2 vector bundles with trivial determinant of weight $(0, \frac{1}{2})$ at each point $x \in I$ parametrized by a scheme S, i.e. a rank 2 vector bundle $\mathcal{E}$ on $S \times X$ such that $\det \mathcal{E}$ is a pull-back of a line bundle on S and a surjection $\mathcal{E} \mid_{S\times\{x\}} \to \mathcal{Q}_x$ for each $x \in I$, where $\mathcal{Q}_x$ is a line bundle on $S \times \{x\}$ such that, for any $s \in S$, $\mathcal{E} \mid_s$ is parabolic vector bundle of parabolic weight $(0, \frac{1}{2})$ at each point $x \in I$, we denote the line bundle $(\det Rp_{S*}\mathcal{E})^{-2} \otimes (\det Rp_{S*}\mathcal{E}')^{-2} \otimes (\det \mathcal{E} \mid_{S\times\{y\}})^{4(1-g)-|I|}$ by $\Xi_{\mathcal{E}}$, where g is the genus of X and y is a point of X and $\mathcal{E}' := \mathrm{Ker}(\mathcal{E} \to \oplus_{x\in I}\mathcal{E}|_{S\times\{x\}})$.

In [14] Pauly proved the following theorem.

Theorem 2.6. *There exists a unique line bundle Ξ on $\overline{\mathcal{M}}^{Par}(X; I)$ such that for any family of semistable rank 2 parabolic vector bundles $\mathcal{E}$ with trivial determinant of weight $(0, \frac{1}{2})$ at each point $x \in I$ parametrized by a scheme S, $f^*\Xi \simeq \Xi_{\mathcal{E}}$, where f is the natural morphism $f : S \to \overline{\mathcal{M}}^{Par}(X; I)$. Moreover Ξ is ample.*

Proposition 2.7. *Let Ξ be as above. Then $\Xi|_{\mathcal{M}^{Par}(X;I)} \simeq K^{-1}_{\mathcal{M}^{Par}(X;I)}$.*

Proof. Recall that $\mathcal{M}^{Par}(X; I)$ is constructed as a geometric quotient of a smooth variety R^s by a PGL-action, where R^s is an open subscheme of a $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$-bundle over an open subset of a quot scheme. Note that the quotient map $\pi : R^s \to \mathcal{M}^{Par}(X; I)$ is a principal PGL-bundle. In order to prove the proposition, we have to prove that $\pi^* K^{-1}_{\mathcal{M}^{Par}(X;I)}$ and $\Xi_{\mathcal{E}}$ are isomorphic as PGL-linearized line bundles on R^s, where $\mathcal{E}$ is the universal parabolic bundle on $R^s \times X$. By Lemma3.2 of [5], it suffices to prove that $\pi^* K^{-1}_{\mathcal{M}^{Par}(X;I)}$ and $\Xi_{\mathcal{E}}$ are isomorphic as line bundles. Since $\pi^* T_{\mathcal{M}^{Par}(X;I)} \simeq R^1 p_* \mathcal{E}nd^\circ(\mathcal{E}' \subset \mathcal{E})$, where $\mathcal{E}nd^\circ(\mathcal{E}' \subset \mathcal{E})$ is a sheaf of traceless endomorphism preserving $\mathcal{E}' := \mathrm{Ker}(\mathcal{E} \to \oplus_{x\in I}\mathcal{E}|_{R^s\times\{x\}})$, we need to prove that $\det R^1 p_* \mathcal{E}nd^\circ(\mathcal{E}' \subset \mathcal{E}) \simeq (\det R^1 p_* \mathcal{E}')^{-2} \otimes (\det R^1 p_* \mathcal{E})^{-2} \otimes (\det \mathcal{E}|_{S\times\{y\}})^{4(1-g)-|I|}$, where p is the projection $R^s \times C \to R^s$. A calculation using Riemann-Roch theorem implies that $\det R^1 p_* \mathcal{E}nd^\circ(\mathcal{E}' \subset \mathcal{E})$ and $(\det R^1 p_* \mathcal{E}')^{-2} \otimes$

$(\det R^1p_*\mathcal{E})^{-2} \otimes (\det \mathcal{E}|_{S\times\{y\}})^{4(1-g)-|I|}$ are isomorphic modulo torsion. Using Theorem2.3 of [5], we know $\det R^1p_*\mathcal{E}nd^\circ(\mathcal{E}' \subset \mathcal{E})$ also descends to $\overline{\mathcal{M}}^{Par}(X;I)$. Therefore the proof will be completed if we prove $\mathrm{Pic}(\overline{\mathcal{M}}^{Par}(X;I))$ is torsion-free. This follows from the next lemma. □

Lemma 2.8. $\mathrm{Pic}(\overline{\mathcal{M}}^{Par}(X;I))$ *is torsion-free.*

Proof. Since $\overline{\mathcal{M}}^{Par}(X;I)$ is a GIT quotient of a smooth variety, it has rational singularities by [10]. On top of that, the canonical divisor of $\overline{\mathcal{M}}^{Par}(X;I)$ is Cartier, hence it has canonical singularities by Corollary5.24 of [9]. Since $K^{-1}_{\overline{\mathcal{M}}^{Par}(X;I)}$ is ample, we have $\mathrm{H}^i(\overline{\mathcal{M}}^{Par}(X;I), \mathcal{O}_{\overline{\mathcal{M}}^{Par}(X;I)}) = 0$ for any $i > 0$ by Theorem1-2-5 in [8]. Given a finite étale morphism $f : Y \to \overline{\mathcal{M}}^{Par}(X;I)$, we also have $\mathrm{H}^i(Y, \mathcal{O}_Y) = 0$ for any $i > 0$. Since $1 = \mathrm{H}^0(Y, \mathcal{O}_Y) = \chi(\mathcal{O}_Y) = \deg f \chi(\mathcal{O}_{\overline{\mathcal{M}}^{Par}(X;I)}) = \deg f \dim \mathrm{H}^0(\overline{\mathcal{M}}^{Par}(X;I), \mathcal{O}_{\overline{\mathcal{M}}^{Par}(X;I)}) = \deg f$, f is an isomorphism. This proves the torsion-freeness of $\mathrm{Pic}(\overline{\mathcal{M}}^{Par}(X;I))$. □

Remark 2.9. Lemma2.8 also follows from the description of the Picard group of a moduli stack of quasi-parabolic bundles given in [11].

2.10. From now on throughout this paper, we treat the case when $X = \mathbb{P}^1$. When $I = \{x_1, \ldots, x_{2g+2}\}$, put $B := I$. When $I = \{x_1, \ldots, x_{2g+1}\}$, put $B := I \cup \{x_{2g+2}\}$, where x_{2g+2} is a point of $X - I$. Let $\pi : C \to X = \mathbb{P}^1$ be the hyperelliptic curve of genus g whose branch points are B. Let $i : C \to C$ be the involution and put $\{c_j\} := \pi^{-1}(x_j)$. These notations are used throughout this paper.

Definition 2.11. An involutional vector bundle on C is a vector bundle F on C endowed with an i-action ,i.e., an isomorphism $\alpha : i^*F \to F$ such that $\alpha \circ i^*\alpha$ is the identity. An involutional vector bundle F is said to be involutionally semistable [resp. involutionally stable] if for any i-invariant subbundle G with $0 < \mathrm{rank} G < \mathrm{rank} F$ the inequality $\deg G/\mathrm{rank} G \leq \deg F/\mathrm{rank} F$ [resp. $\deg G/\mathrm{rank} G < \deg F/\mathrm{rank} F$] holds.

Remark 2.12. An involutional vector bundle is involutionally semistable if and only if it is semistable as a vector bundle.

2.13. When I is even, we denote by $\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}$ [resp. $\mathcal{M}^{invo}_{C/\mathbb{P}^1}$] the coarse moduli space of rank 2 involutionally semistable [resp. involutionally stable] involutional vector bundles on C with trivial determinant such that the eigen values of the i-action on the fiber over c_j are 1 and -1 for $1 \leq j \leq 2g+2$.

When I is odd, we denote by $\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}$ [resp. $\mathcal{M}^{invo}_{C/\mathbb{P}^1}$] the coarse moduli space of rank 2 involutionally semistable [resp. involutionally stable] involutional vector bundles on C with determinant $\mathcal{O}(c_{2g+2})$ such that the eigen values of the $\imath$-action on the fiber over c_j are 1 and -1 for $1 \leq j \leq 2g+1$, and 1 with multiplicity two for $j = 2g+2$ and when g is odd, and -1 with multiplicity two for $j = 2g+2$ and when g is even.

Let $(E, \{E_{x_j} = F_1(E)_{x_j} \supset F_2(E)_{x_j}\})$ be an element of $\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)$. When $|I|$ is even [resp. odd], put $\tilde{E} := \mathrm{Ker}(\pi^* E \to \oplus_{j=1}^{2g+2} F_1(E)_{x_j}/F_2(E)_{x_j})$ [resp. $\tilde{E} := \mathrm{Ker}(\pi^* E \to \oplus_{j=1}^{2g+1} F_1(E)_{x_j}/F_2(E)_{x_j})$], where the morphism $\pi^* E \to F_1(E)_{x_j}/F_2(E)_{x_j}$ is given by $\pi^* E \to \pi^* E \otimes \mathbb{C}_{c_j} \simeq E \otimes \mathbb{C}_{c_j} \twoheadrightarrow F_1(E)_{x_j}/F_2(E)_{x_j}$. We endow $\tilde{E}$ with the natural $\imath$-action. Then by Proposition1.2 of [4], we obtain an isomorphism between $\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)$ and $\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}$, taking $(E, \{E_{x_j} = F_1(E)_{x_j} \supset F_2(E)_{x_j}\})$ to $\tilde{E} \otimes \mathcal{O}_C\big((g+1)c_{2g+2}\big)$, where $\mathcal{O}_C\big((g+1)c_{2g+2}\big)$ is given the natural $\imath$-action.

3. A basis of anticanonical sections

In the rest of this paper, we assume that $g \geq 2$ mainly because we want the codimension of the locus of non-stable bundles in the moduli to be greater than one.

3.1. In Section 2, we saw that, for $I = \{x_1, \ldots, x_{2g+2}\}$ or $\{x_1, \ldots, x_{2g+1}\}$, $\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)$ and $\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}$ are isomorphic. Since $\mathrm{codim}\big(\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I) \setminus \mathcal{M}^{Par}(\mathbb{P}^1; I), \overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)\big) \geq 2$, Proposition2.7 implies that the canonical divisor of $\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I)$ is a Cartier divisor and that $\Xi \simeq K^{-1}_{\overline{\mathcal{M}}^{Par}(\mathbb{P}^1;I)}$. We have isomorphisms of vector spaces

$$\begin{aligned} \mathrm{H}^0(\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}, K^{-1}_{\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}}) &\simeq \mathrm{H}^0(\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I), K^{-1}_{\overline{\mathcal{M}}^{Par}(\mathbb{P}^1;I)}) \\ &\simeq \mathrm{H}^0(\overline{\mathcal{M}}^{Par}(\mathbb{P}^1; I), \Xi). \end{aligned}$$

By Verlinde formula (cf.[3]), the dimension of these vector spaces is $\frac{2^{2g+1}+1}{3}$ if $|I|$ is even, and $\frac{2^{2g}-1}{3}$ if $|I|$ is odd. Our goal in this section is to find a basis of $\mathrm{H}^0(\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}, K^{-1}_{\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}})$. We define $\mathcal{S}^{\mathrm{even}}$ and $\mathcal{S}^{\mathrm{odd}}$ to be the sets given by

$$\mathcal{S}^{\mathrm{even}} := \left\{ (\mathfrak{A}, \lambda) \in 2^{\{c_1, \ldots, c_{2g+1}\}} \times \mathbb{Z} \;\middle|\; 4|\mathfrak{A}| + 3\lambda = 4g - 2 \right\}$$

and

$$\mathcal{S}^{\mathrm{odd}} := \left\{ (\mathfrak{A}, \lambda) \in 2^{\{c_1, \ldots, c_{2g+1}\}} \times \mathbb{Z} \;\middle|\; \begin{array}{l} |\mathfrak{A}| \leq g-1 \\ 4|\mathfrak{A}| + 3\lambda = 4g - 4 \end{array} \right\}.$$

3.2. Let $\mathcal{F}$ be a family of involutionally vector bundles on C parameterized by an irreducible scheme S. If $\chi(\mathcal{F}|_{\{s\}\times C})^+ = 0$ for any $s \in S$ and $\mathrm{H}^0(C, \mathcal{F}|_{\{s\}\times C})^+ = 0$ for general $s \in S$, then we can construct an effective divisor denoted by $\mathrm{Div}(R^1 p_{S*}(\mathcal{F})^+)$ whose support is $\{s \in S | \mathrm{H}^0(C, \mathcal{F}|_{\{s\}\times C})^+ \neq 0\}$: for any $s \in S$ we can find a resolution $0 \to \mathcal{O}_U^{\oplus m} \xrightarrow{(\varphi_{ij})} \mathcal{O}_U^{\oplus m} \to R^1 p_{S*}(\mathcal{F})^+ \to 0$ in a neighborhood U of s, then on U $\mathrm{Div}(R^1 p_{S*}(\mathcal{F})^+)$ is defined by the equation $\det(\varphi_{ij}) = 0$.

Recall that $\mathcal{M}^{invo}_{C/\mathbb{P}^1}$ is constructed as a geometric quotient of a smooth irreducible variety Z by an action of a reductive algebraic group G of the form $GL(a_1) \times GL(a_2)/\{(t\mathrm{Id}, t\mathrm{Id}) | t \in \mathbb{G}_{\mathrm{m}}\}$, where Z is an open subscheme of an equivariant quot scheme with trivial determinant. Let $\mathcal{O}_{Z\times C}(-N) \otimes V \to \mathcal{Q} \to 0$ be the universal quotient, where $\mathcal{O}_{Z\times C}(-N) \otimes V$ and $\mathcal{Q}$ have an i-action. For $(\mathfrak{A}, \lambda)$ in $\mathcal{S}^{\mathrm{even}}$ or $\mathcal{S}^{\mathrm{odd}}$, we have the divisor $\mathrm{Div}(R^1 p_{Z*}(\mathcal{E}nd^\circ(\mathcal{Q}) \otimes p_C^* \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+)$ on Z by 3.2. By construction, it is G-invariant, hence it is a pullback of a divisor, say $D^\circ_{(\mathfrak{A},\lambda)}$, on $\mathcal{M}^{invo}_{C/\mathbb{P}^1}$.

Lemma 3.3. *We have an isomorphism* $\mathcal{O}_{\mathcal{M}^{invo}_{C/\mathbb{P}^1}}(D^\circ_{(\mathfrak{A},\lambda)}) \simeq K^{-1}_{\mathcal{M}^{invo}_{C/\mathbb{P}^1}}$.

Proof. We have to prove that the two line bundles $\det\left(R^1 p_{Z*}\mathcal{E}nd^\circ(\mathcal{Q})^+\right)$ and $\mathcal{O}_Z\left(\mathrm{Div}(R^1 p_{Z*}(\mathcal{E}nd^\circ(\mathcal{Q}) \otimes p_C^*\mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+)\right)$ are isomorphic as G-linearized line bundles. We can check that

$$\begin{aligned}
&\mathcal{O}_Z\left(\mathrm{Div}(R^1 p_{Z*}(\mathcal{E}nd^\circ(\mathcal{Q}) \otimes p_C^*\mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+)\right) \\
&\simeq \det\left(R^1 p_{Z*}\mathcal{E}nd^\circ(\mathcal{Q})^+\right) \otimes \otimes_{c_j\in\mathfrak{A}} \left(\det \mathcal{E}nd^\circ(Q)|^-_{Z\times\{c_j\}}\right)^{-1} \\
&\otimes \left(\det\left(\mathcal{E}nd^\circ(Q)|^+_{Z\times\{c_{2g+2}\}}\right) \otimes \det\left(\mathcal{E}nd^\circ(Q)|^-_{Z\times\{c_{2g+2}\}}\right)\right)^{-\frac{\lambda}{2}}
\end{aligned}$$

as G-linearized line bundles. Since $\otimes_{c_j\in\mathfrak{A}} \left(\det \mathcal{E}nd^\circ(Q)|^-_{Z\times\{c_j\}}\right)^{-1} \otimes \left(\det\left(\mathcal{E}nd^\circ(Q)|^+_{Z\times\{c_{2g+2}\}}\right) \otimes \det\left(\mathcal{E}nd^\circ(Q)|^-_{Z\times\{c_{2g+2}\}}\right)\right)^{-\frac{\lambda}{2}}$ is trivial as a G-linearized line bundle, the lemma is proved. $\square$

Since $\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1} \simeq \overline{\mathcal{M}}^{Par}(X; I)$ and $K_{\overline{\mathcal{M}}^{Par}(X;I)}$ is a Cartier divisor by Proposition2.7, $K_{\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}}$ is also a Cartier divisor. Since $\mathrm{codim}(\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1} \setminus \mathcal{M}^{invo}_{C/\mathbb{P}^1}, \overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}) \geq 2$, $D^\circ_{(\mathfrak{A},\lambda)}$ extends uniquely to an effective divisor, denoted by $D_{(\mathfrak{A},\lambda)}$, on $\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}$. Now we can state the main theorem of this section.

Theorem 3.4. *If* $|I|$ *is even* [*resp. odd*], $\{D_{(\mathfrak{A},\lambda)}\}_{(\mathfrak{A},\lambda)\in\mathcal{S}^{\mathrm{even}}}$ [*resp.* $\{D_{(\mathfrak{A},\lambda)}\}_{(\mathfrak{A},\lambda)\in\mathcal{S}^{\mathrm{odd}}}$] *is a basis of* $\mathrm{H}^0(\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}, K^{-1}_{\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}})$.

We give a proof only for the case when $|I|$ is even. The following lemmas in this section are valid only for this case. The proof of the case when $|I|$ is odd is similar.

We follow closely the proof given in [2]

3.5. Let $\mathcal{F}$ be a family of involutionally stable involutional vector bundles with trivial determinant on C parameterized by a scheme by S. Then we have a natural map $\varphi : S \to \mathcal{M}^{invo}_{C/\mathbb{P}^1}$. We have $\chi(\mathcal{E}nd^\circ(\mathcal{F}|_{\{s\}\times C}) \otimes \mathcal{O}(\mathfrak{A}+\lambda c_{2g+2}))^+ = 0$ for any $s \in S$ and any $(\mathfrak{A}, \lambda) \in \mathcal{S}^{\mathrm{even}}$. Assume that S is irreducible and that $\mathrm{H}^0(\mathcal{E}nd^\circ(\mathcal{F}|_{\{s\}\times C}) \otimes \mathcal{O}(\mathfrak{A}+\lambda c_{2g+2}))^+ = 0$ for general $s \in S$. By the construction of $D_{(\mathfrak{A},\lambda)}$, we have $\mathrm{Div}(R^1 p_{S*}(\mathcal{E}nd^\circ(\mathcal{F}) \otimes \mathcal{O}(\mathfrak{A}+\lambda c_{2g+2}))^+) = \varphi^* D_{(\mathfrak{A},\lambda)}$ as effective divisors on S.

Let $\mathcal{P}$ be a Poincaré line bundle on $\mathrm{Jac}(C)\times C$. Since $\mathcal{P}\oplus(1\times i)^*\mathcal{P}$ has a natural i-action, it is a family of rank 2 involutional vector bundles with trivial determinant on C parameterized by $\mathrm{Jac}(C)$.

Lemma 3.6. (i) *$P \oplus i^*P$ is involutionally semistable for any $P \in \mathrm{Jac}(C)$*
(ii) *For $P \in \mathrm{Jac}(C)$, $P \oplus i^*P$ is not involutionally stable if and only if $P \in \mathrm{Jac}(C)[2]$, where $\mathrm{Jac}(C)[2]$ is the subgroup of $\mathrm{Jac}(C)$ consisting of 2-torsion points.*

Proof. (i) is obvious by Remark2.12. If $P \in \mathrm{Jac}(C)[2]$, P and i^*P are isomorphic. We can find an isomorphism $\alpha : P \to i^*P$ such that the composite of α and $i^*\alpha : i^*P \to i^*i^*P \simeq P$ is the identity. Then $(1,\alpha) : P \hookrightarrow P\oplus i^*P$ is an i-invariant line subbundle. Since $\deg P = 0$, $P \oplus i^*P$ is not involutionally stable. Conversely, if $P \oplus i^*P$ is not involutionally stable, we can find an i-invariant line subbundle A of $P \oplus i^*P$ with $\deg A = 0$. Since $A \simeq i^*A$, $A \in \mathrm{Jac}(C)[2]$. The inclusion $A \hookrightarrow P\oplus i^*P$ gives rise to an isomorphism $A \xrightarrow{\sim} P$ or $A \xrightarrow{\sim} i^*P$. Hence $P \in \mathrm{Jac}(C)[2]$. □

3.7. By Lemma3.6, $\mathcal{P} \oplus i^*\mathcal{P}$ gives rise to the natural morphism $\varphi : \mathrm{Jac}(C) \to \overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}$.

Lemma 3.8. *Take $(\mathfrak{A}, \lambda) \in \mathcal{S}^{\mathrm{even}}$.*
(i) *If $|\mathfrak{A}| \neq g+1$, then $\mathrm{H}^0(C, \mathcal{E}nd^\circ(L \oplus i^*L) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ \neq 0$ for any $L \in \mathrm{Jac}(C)$.*
(ii) *If $|\mathfrak{A}| = g+1$, then $\mathrm{H}^0(C, \mathcal{E}nd^\circ(L \oplus i^*L) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ = 0$ for general $L \in \mathrm{Jac}(C)$ and $\varphi^* D_{(\mathfrak{A},\lambda c_{2g+2})} = 2^*_{\mathrm{Jac}} T^*_{(\mathfrak{A},\lambda)}\Theta$, where Θ is the theta divisor on $\mathrm{Jac}(C)^{g-1}$ and $2_{\mathrm{Jac}} : \mathrm{Jac}(C) \to \mathrm{Jac}(C)$ is given by $L \mapsto L^{\otimes 2}$ and $T_{(\mathfrak{A},\lambda)} : \mathrm{Jac}(C) \to \mathrm{Jac}(C)^{g-1}$ is given by $L \mapsto L \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})$.*

Proof. For $E = L \oplus i^*L$, where $L \in \mathrm{Jac}(C)$ and E is endowed with the natural i-action, $\mathcal{E}nd^\circ(E)$ is isomorphic to $\mathcal{O}_C \oplus \big((L^{-1} \otimes i^*L) \oplus (i^*L^{-1} \otimes L)\big)$, where i acts on $\mathcal{O}_C$ by (-1) multiplication and on $(L^{-1} \otimes i^*L) \oplus (i^*L^{-1} \otimes L)$ by $i^*\left((L^{-1} \otimes i^*L) \oplus (i^*L^{-1} \otimes L)\right) \simeq (i^*L^{-1} \otimes L) \oplus (L^{-1} \otimes i^*L) \xrightarrow{\text{switch the factors}} (L^{-1} \otimes i^*L) \oplus (i^*L^{-1} \otimes L)$. Hence we have $\mathrm{H}^0(C, \mathcal{E}nd^\circ(L \oplus i^*L) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ \simeq \mathrm{H}^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^- \oplus$

$\mathrm{H}^0(C, i^*L^{-1} \otimes L \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))$. If $|\mathfrak{A}| < g+1$, then $\deg i^*L^{-1} \otimes L \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) > g-1$, hence $\mathrm{H}^0(C, i^*L^{-1} \otimes L \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})) \neq 0$. Since $\dim \mathrm{H}^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^- = \max\{\frac{-2-\lambda}{4}, 0\}$, if $|\mathfrak{A}| > g+1$ then $\dim \mathrm{H}^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^- \neq 0$. These prove (i).

If $|\mathfrak{A}| = g+1$, then $\dim \mathrm{H}^0(C, \mathcal{O}(\mathfrak{A}+\lambda c_{2g+2}))^- = 0$ and $\mathrm{H}^0(C, i^*L^{-1} \otimes L \otimes \mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})) = 0$ for general $L \in \mathrm{Jac}(C)$ because $\deg i^*L^{-1} \otimes L \otimes \mathcal{O}(\mathfrak{A}+\lambda c_{2g+2}) = g-1$. This implies the former statement of (ii). Since we are assuming that $g \geq 2$, in order to prove the latter statement of (ii), it suffices to see that $\varphi^* D_{(\mathfrak{A}, \lambda c_{2g+2})} = 2^*_{\mathrm{Jac}} T^*_{(\mathfrak{A},\lambda)} \Theta$ over $\mathrm{Jac}(C) \setminus \mathrm{Jac}(C)[2]$. Over $\mathrm{Jac}(C) \setminus \mathrm{Jac}(C)[2]$, $\mathcal{P} \oplus i^*\mathcal{P}$ gives a family of involutionally stable involutional vector bundles, hence we have $\varphi^* D_{(\mathfrak{A},\lambda)} = \mathrm{Div}(R^1 p_{\mathrm{Jac}(C)\setminus \mathrm{Jac}(C)[2]}(\mathcal{E}nd^\circ(\mathcal{P} \oplus i^*\mathcal{P}) \otimes p_C^*\mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+)$ by 3.5. As above $R^1 p_{\mathrm{Jac}(C)\setminus \mathrm{Jac}(C)[2]}(\mathcal{E}nd^\circ(\mathcal{P} \oplus i^*\mathcal{P}) \otimes p_C^*\mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ \simeq R^1 p_{\mathrm{Jac}(C)\setminus \mathrm{Jac}(C)[2]}(i^*\mathcal{P}^{-1} \otimes \mathcal{P} \otimes p_C^*\mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))$. The latter statement follows from this. $\square$

We define the set $\mathcal{T}$ by

$$\mathcal{T} := \left\{ \mathfrak{C} \subsetneq \{c_1, \dots, c_{2g+2}\} \;\middle|\; \begin{array}{l} |\mathfrak{C}| \text{ is even.} \\ \mathfrak{C} \neq \phi \end{array} \right\}.$$

Since $\mathcal{O}(\mathfrak{C} - |\mathfrak{C}| c_{2g+2}) \in \mathrm{Jac}(C)[2]$ for any $\mathfrak{C} \in \mathcal{T}$, from this we can construct an étale double cover $\pi_{\mathfrak{C}} : C_{\mathfrak{C}} \to C$. Moreover, since $\mathcal{O}(\mathfrak{C} - |\mathfrak{C}| c_{2g+2})$ has the natural i-action, we get an involution $i_{\mathfrak{C}} : C_{\mathfrak{C}} \to C_{\mathfrak{C}}$ such that $\pi_{\mathfrak{C}} \circ i_{\mathfrak{C}} = i \circ \pi_{\mathfrak{C}}$. By $h_{\mathfrak{C}} : C_{\mathfrak{C}} \to C_{\mathfrak{C}}/i_{\mathfrak{C}}$ we mean the quotient of $C_{\mathfrak{C}}$ by this involution. There is a unique map $q_{\mathfrak{C}} : C_{\mathfrak{C}}/i_{\mathfrak{C}} \to \mathbb{P}^1$ such that $q_{\mathfrak{C}} \circ h_{\mathfrak{C}} = p \circ \pi_{\mathfrak{C}}$.

$$\begin{tikzcd}
C_{\mathfrak{C}} \arrow[rr, "i_{\mathfrak{C}}"] \arrow[dr, "h_{\mathfrak{C}}"] \arrow[dd, "\pi_{\mathfrak{C}}"] & & C_{\mathfrak{C}} \arrow[dl, "h_{\mathfrak{C}}"'] \arrow[dd, "\pi_{\mathfrak{C}}"] \\
& C/i_{\mathfrak{C}} \arrow[dd, "q_{\mathfrak{C}}"] & \\
C \arrow[rr, "i"] \arrow[dr, "p"'] & & C \arrow[dl, "p"] \\
& \mathbb{P}^1 &
\end{tikzcd}$$

The branch locus of $q_{\mathfrak{C}}$ is $p(\mathfrak{C})$ and the genus of $C_{\mathfrak{C}}/i_{\mathfrak{C}}$ is $\frac{|\mathfrak{C}|}{2} - 1$. Put $\{c_j^{\mathfrak{C}}, c_j^{\dagger\mathfrak{C}}\} := \pi_{\mathfrak{C}}^{-1}(c_j)$. For $N \in \mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})$, the line bundle $h_{\mathfrak{C}}^*(N) \otimes \mathcal{O}(-\sum_{c_j \notin \mathfrak{C}} c_j^{\mathfrak{C}})$ on $C_{\mathfrak{C}}$ has the natural $i_{\mathfrak{C}}$-action, which induces an i-action on $\pi_{\mathfrak{C}*}(h_{\mathfrak{C}}^*(N) \otimes \mathcal{O}(-\sum_{c_j \notin \mathfrak{C}} c_j^{\mathfrak{C}})) \otimes \mathcal{O}(\frac{2g+2-|\mathfrak{C}|}{2} c_{2g+2})$. Note that $\det \left(\pi_{\mathfrak{C}*}(h_{\mathfrak{C}}^*(N) \otimes \mathcal{O}(-\sum_{c_j \notin \mathfrak{C}} c_j^{\mathfrak{C}})) \otimes \mathcal{O}(\frac{2g+2-|\mathfrak{C}|}{2} c_{2g+2}) \right) \simeq \mathcal{O}_C$ and the eigen values of the i-action on the fiber of $\pi_{\mathfrak{C}*}(h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}}))\otimes\mathcal{O}(\frac{2g+2-|\mathfrak{C}|}{2}c_{2g+2})$ over every c_j are 1 and -1.

Lemma 3.9. *The above involutional vector bundle* $\pi_{\mathfrak{C}*}(h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}}))\otimes\mathcal{O}(\frac{2g+2-|\mathfrak{C}|}{2}c_{2g+2})$ *is involutionally stable.*

Proof. Given a line subbundle $L \subset \pi_{\mathfrak{C}*}(h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}}))\otimes\mathcal{O}(\frac{2g+2-|\mathfrak{C}|}{2}c_{2g+2})$, we have a nonzero homomorphism $\pi_{\mathfrak{C}}^*L \to h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}})\otimes\mathcal{O}(\frac{2g+2-|\mathfrak{C}|}{2}(c_{2g+2}^{\mathfrak{C}}+c_{2g+2}^{\dagger\mathfrak{C}}))$. Since $\deg h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}})\otimes\mathcal{O}(\frac{2g+2-|\mathfrak{C}|}{2}(c_{2g+2}^{\mathfrak{C}}+c_{2g+2}^{\dagger\mathfrak{C}}))=0$, we have $\deg L\le 0$, which implies the semistability of $\pi_{\mathfrak{C}*}(h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}}))\otimes\mathcal{O}(\frac{2g+2-|\mathfrak{C}|}{2}c_{2g+2})$.

Suppose that the above L is i-invariant and of degree zero. Then there exists a subset U of $\{c_1,\dots,c_{2g+2}\}$ with $|U|$ even such that L is isomorphic to $\mathcal{O}(\sum_{c_j\in U} c_j-|U|c_{2g+2})$ as involutional bundles. Then $h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}})\otimes\mathcal{O}(\frac{2g+2-|\mathfrak{C}|+2|U|}{2}(c_{2g+2}^{\mathfrak{C}}+c_{2g+2}^{\dagger\mathfrak{C}}))\otimes\mathcal{O}_C(-\sum_{c_j\in U}(c_j^{\mathfrak{C}}+c_j^{\dagger\mathfrak{C}}))$ is a trivial line bundle with a trivial $i_{\mathfrak{C}}$-action. For $c_j\notin(\mathfrak{C}\cup\{c_{2g+2}\}$, however, either $c_j^{\mathfrak{C}}$ or $c_j^{\dagger\mathfrak{C}}$ appears with an odd multiplicity. This means that the above line bundle cannot be trivial as an involutional bundle. □

3.10. Let $\mathcal{P}_{\mathfrak{C}}$ be a Poincaré line bundle on $\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})\times C_{\mathfrak{C}}/i_{\mathfrak{C}}$. By Lemma3.9, $(1\times\pi_{\mathfrak{C}})_*\left((1\times h_{\mathfrak{C}})^*\mathcal{P}_{\mathfrak{C}}\otimes p_{C_{\mathfrak{C}}}^*\mathcal{O}(-\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}})\right)\otimes p_C^*\mathcal{O}(\frac{2g+2-|\mathfrak{C}|}{2}c_{2g+2})$ is a family of involutionally stable involutional vector bundles with trivial determinant, hence gives rise to a natural map $\varphi_{\mathfrak{C}}:\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})\to\mathcal{M}_{C/\mathbb{P}^1}^{invo}\subset\overline{\mathcal{M}}_{C/\mathbb{P}^1}^{invo}$.

Lemma 3.11. *Take* $(\mathfrak{A},\lambda)\in\mathcal{S}^{\mathrm{even}}$.

(i) *If* $2|\mathfrak{A}\cap\mathfrak{C}|+\lambda-|\mathfrak{C}|+2\neq 0$, *then*

$$\mathrm{H}^0\left(C,\mathcal{E}nd^{\circ}\left(\pi_{\mathfrak{C}*}\left(h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}})\right)\right)\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})\right)^+\neq 0$$

for any $N\in\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})$.

(ii) *If* $2|\mathfrak{A}\cap\mathfrak{C}|+\lambda-|\mathfrak{C}|+2=0$, *then*

$$\mathrm{H}^0\left(C,\mathcal{E}nd^{\circ}\left(\pi_{\mathfrak{C}*}\left(h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}})\right)\right)\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})\right)^+=0$$

for general $N\in\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})$, *and we have* $\varphi_{\mathfrak{C}}^*(D_{(\mathfrak{C},\lambda)})=2^*_{\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})}\left(T^*_{\tau((\mathfrak{A},\lambda);\mathfrak{C})}\Theta\right)$, *where* Θ *is the theta divisor on* $\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})^{\frac{|\mathfrak{C}|}{2}-2}$, $2_{\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})}:\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})\to\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})$ *is defined by* $L\mapsto L^{\otimes 2}$ *and* $T_{\tau((\mathfrak{A},\lambda);\mathfrak{C})}:\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})\to$

$\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})^{\frac{|\mathfrak{C}|}{2}-2}$ *by* $L \mapsto L \otimes \tau((\mathfrak{A},\lambda);\mathfrak{C})$, *here*

$$\tau((\mathfrak{A},\lambda),\mathfrak{C}) := \mathcal{O}\Big(2\textstyle\sum_{c_j\in\mathfrak{C}\cap\mathfrak{A}} h_{\mathfrak{C}}(c_j^{\mathfrak{C}}) + \sum_{c_j\in\mathfrak{A}\setminus\mathfrak{C}} h_{\mathfrak{C}}(c_j^{\dagger\mathfrak{C}}) + \sum_{c_j\in\mathfrak{C}\setminus\mathfrak{A}} h_{\mathfrak{C}}(c_j^{\mathfrak{C}}) - \sum_{c_j\notin\mathfrak{C}\cup\mathfrak{A}} h_{\mathfrak{C}}(c_j^{\mathfrak{C}}) + \frac{\lambda-|\mathfrak{C}|}{2}\Big(h_{\mathfrak{C}}(c_{2g+2}^{\mathfrak{C}}) + h_{\mathfrak{C}}(c_{2g+2}^{\dagger\mathfrak{C}})\Big)\Big).$$

Proof. On $C_{\mathfrak{C}}$, we have the exact sequence

$$0 \to h_{\mathfrak{C}}^*(N^{-1}) \otimes \mathcal{O}\Big(-\textstyle\sum_{c_j\in\mathfrak{C}}(c_j^{\mathfrak{C}}+c_j^{\dagger\mathfrak{C}}) - \sum_{c_j\notin\mathfrak{C}} c_j^{\dagger\mathfrak{C}} + |\mathfrak{C}|(c_{2g+2}^{\mathfrak{C}}+c_{2g+2}^{\dagger\mathfrak{C}})\Big) \to \pi_{\mathfrak{C}}^*\pi_{\mathfrak{C}*}\Big(h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\textstyle\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}})\Big) \to h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\textstyle\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}}) \to 0.$$

From this, we have the exact sequence on C,

$$\begin{aligned} &0 \to \mathcal{O}(\mathfrak{A}+\mathfrak{C}+(\lambda-|\mathfrak{C}|)c_{2g+2}) \\ &\to \mathcal{E}nd^{\circ}\Big(\pi_{\mathfrak{C}*}\Big(h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\textstyle\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}})\Big)\Big)\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2}) \\ &\to \pi_{\mathfrak{C}*}\left(h_{\mathfrak{C}}^*(N^2)\otimes Q\right)\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2}) \to 0, \end{aligned}$$

where

$$Q := \mathcal{O}_{C_{\mathfrak{C}}}\left(\sum_{c_j\in\mathfrak{C}}(c_j^{\mathfrak{C}}+c_j^{\dagger\mathfrak{C}}) + \sum_{c_j\notin\mathfrak{C}}(c_j^{\dagger\mathfrak{C}}-c_j^{\mathfrak{C}}) - |\mathfrak{C}|(c_{2g+2}^{\mathfrak{C}}+c_{2g+2}^{\dagger\mathfrak{C}})\right).$$

This gives rise to the long exact sequence

$$\begin{aligned} &0 \to \mathrm{H}^0\left(C,\mathcal{O}(\mathfrak{A}+\mathfrak{C}+(\lambda-|\mathfrak{C}|)c_{2g+2})\right)^+ \\ &\to \mathrm{H}^0\Big(C,\mathcal{E}nd^{\circ}\Big(\pi_{\mathfrak{C}*}\Big(h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\textstyle\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}})\Big)\Big)\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})\Big)^+ \\ &\to \mathrm{H}^0\left(C,\pi_{\mathfrak{C}*}\left(h_{\mathfrak{C}}^*(N^2)\otimes Q\right)\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})\right)^+ \\ &\to \mathrm{H}^1\left(C,\mathcal{O}(\mathfrak{A}+\mathfrak{C}+(\lambda-|\mathfrak{C}|)c_{2g+2})\right)^+ \to \cdots. \end{aligned}$$

Since $\dim\mathrm{H}^0\left(C,\mathcal{O}(\mathfrak{A}+\mathfrak{C}+(\lambda-|\mathfrak{C}|)c_{2g+2})\right)^+ = \max\{0,(2|\mathfrak{A}\cap\mathfrak{C}|+\lambda-|\mathfrak{C}|+2)/2\}$, we have

$$\mathrm{H}^0\left(C,\mathcal{E}nd^{\circ}\left(\pi_{\mathfrak{C}*}\left(h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}})\right)\right)\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})\right)^+ \neq 0$$

for any $N\in\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})$ if $2|\mathfrak{A}\cap\mathfrak{C}|+\lambda-|\mathfrak{C}|+2>0$. If $2|\mathfrak{A}\cap\mathfrak{C}|+\lambda-|\mathfrak{C}|+2<0$, applying the above argument to $K_C\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})^{-1}$ in place of $\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})$ and using Serre duality, we obtain

$$\mathrm{H}^1\left(C,\mathcal{E}nd^{\circ}\left(\pi_{\mathfrak{C}*}\left(h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}})\right)\right)\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})\right)^+ \neq 0$$

for any $N\in\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})$. Since

$$\chi\left(\mathcal{E}nd^{\circ}\left(\pi_{\mathfrak{C}*}\left(h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\sum_{c_j\notin\mathfrak{C}} c_j^{\mathfrak{C}})\right)\right)\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})\right)^+ = 0,$$

this proves (1).

Suppose that $2|\mathfrak{A}\cap\mathfrak{C}|+\lambda-|\mathfrak{C}|+2=0$. Then $\mathrm{H}^0(C,\mathcal{O}(\mathfrak{A}+\mathfrak{C}+(\lambda-|\mathfrak{C}|)c_{2g+2}))^+=0$. Since $\chi(\mathcal{O}(\mathfrak{A}+\mathfrak{C}+(\lambda-|\mathfrak{C}|)c_{2g+2}))^+=(2|\mathfrak{A}\cap\mathfrak{C}|+\lambda-|\mathfrak{C}|+2)/2=0$, $\mathrm{H}^1(C,\mathcal{O}(\mathfrak{A}+\mathfrak{C}+(\lambda-|\mathfrak{C}|)c_{2g+2}))^+$ also vanishes. We obtain isomorphisms

$$\begin{aligned}
&\mathrm{H}^0\left(C,\mathcal{E}nd^\circ\left(\pi_{\mathfrak{C}*}\left(h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\textstyle\sum_{c_j\notin\mathfrak{C}}c_j^{\mathfrak{C}})\right)\right)\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})\right)^+\\
&\simeq \mathrm{H}^0\left(C,\pi_{\mathfrak{C}*}\left(h_{\mathfrak{C}}^*(N^2)\otimes Q\right)\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})\right)^+\\
&\simeq \mathrm{H}^0\Big(C_{\mathfrak{C}},h_{\mathfrak{C}}^*(N^{\otimes 2})\otimes\mathcal{O}_{C_{\mathfrak{C}}}\Big(2\textstyle\sum_{c_j\in\mathfrak{C}\cap\mathfrak{A}}(c_j^{\mathfrak{C}}+c_j^{\dagger\mathfrak{C}})+2\sum_{c_j\in\mathfrak{A}\setminus\mathfrak{C}}c_j^{\dagger\mathfrak{C}}\\
&\quad+\textstyle\sum_{c_j\in\mathfrak{C}\setminus\mathfrak{A}}(c_j^{\mathfrak{C}}+c_j^{\dagger\mathfrak{C}})+\sum_{c_j\notin\mathfrak{C}\cup\mathfrak{A}}(c_j^{\dagger\mathfrak{C}}-c_j^{\mathfrak{C}})+(\lambda-|\mathfrak{C}|)(c_{2g+2}^{\mathfrak{C}}+c_{2g+2}^{\dagger\mathfrak{C}})\Big)\Big)^+\\
&\simeq \mathrm{H}^0\Big(C_{\mathfrak{C}}/i_{\mathfrak{C}},N^{\otimes 2}\otimes\mathcal{O}\Big(2\textstyle\sum_{c_j\in\mathfrak{C}\cap\mathfrak{A}}h_{\mathfrak{C}}(c_j^{\mathfrak{C}})+\sum_{c_j\in\mathfrak{A}\setminus\mathfrak{C}}h_{\mathfrak{C}}(c_j^{\dagger\mathfrak{C}})\\
&\quad+\textstyle\sum_{c_j\in\mathfrak{C}\setminus\mathfrak{A}}h_{\mathfrak{C}}(c_j^{\mathfrak{C}})-\sum_{c_j\notin\mathfrak{C}\cup\mathfrak{A}}h_{\mathfrak{C}}(c_j^{\mathfrak{C}})+\frac{\lambda-|\mathfrak{C}|}{2}\left(h_{\mathfrak{C}}(c_{2g+2}^{\mathfrak{C}})+h_{\mathfrak{C}}(c_{2g+2}^{\dagger\mathfrak{C}})\right)\Big)\Big).
\end{aligned}$$

Therefore, $\mathrm{H}^0\left(C,\mathcal{E}nd^\circ\left(\pi_{\mathfrak{C}*}\left(h_{\mathfrak{C}}^*(N)\otimes\mathcal{O}(-\sum_{c_j\notin\mathfrak{C}}c_j^{\mathfrak{C}})\right)\right)\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})\right)^+\neq 0$ if and only if $\mathrm{H}^0(C_{\mathfrak{C}}/i_{\mathfrak{C}},N^{\otimes 2}\otimes\tau((\mathfrak{A},\lambda);\mathfrak{C}))\neq 0$. This proves that, set-theoretically, $\varphi_{\mathfrak{C}}^*(D_{(\mathfrak{C},\lambda)})=2^*_{\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})}\left(T^*_{\tau((\mathfrak{A},\lambda);\mathfrak{C})}\Theta\right)$. The proof that this holds scheme-theoretically is similar to the counterpart in the proof of Lemma3.8. □

Proof of Theorem3.4. Since $|\mathcal{S}^{\mathrm{even}}|=\sum_{\substack{0\le k\le 2g+1\\ k\equiv g+1 \pmod 3}}\binom{g+1}{k}=\frac{2^{2g+1}+1}{3}=\dim\mathrm{H}^0\left(\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1},K^{-1}_{\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}}\right)$, we have only to prove that $D_{(\mathfrak{C},\lambda)}$ are linearly independent. Suppose that we have

$$(\clubsuit)\qquad \sum_{(\mathfrak{A},\lambda)\in\mathcal{S}^{\mathrm{even}}}a_{(\mathfrak{A},\lambda)}D_{(\mathfrak{C},\lambda)}=0$$

for $a_{(\mathfrak{A},\lambda)}\in\mathbb{C}$. Pulling back this equation to $\mathrm{Jac}(C)$ by φ in 3.7, we obtain $\sum_{|\mathfrak{A}|=g+1}a_{(\mathfrak{A},\lambda)}\left(2^*_{\mathrm{Jac}(C)}T^*_{(\mathfrak{A},\lambda)}\Theta\right)=0$ by Lemma3.8. $\left\{2^*_{\mathrm{Jac}(C)}T^*_\kappa\Theta\right\}_{\kappa\text{: theta characteristics}}$ are linearly independent by PropositionA.8 of [2]. $\{\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})\}_{|\mathfrak{A}|=g+1}$ is a subset of the set of theta characteristics of C by Lemma3 of Chapter VIII of [6]. Thus $a_{(\mathfrak{A},\lambda)}=0$ if $|\mathfrak{A}|=g+1$. Fix $(\mathfrak{A}_0,\lambda_0)\in\mathcal{S}^{\mathrm{even}}$ with $|\mathfrak{A}_0|<g+1$. We choose an element $\mathfrak{C}\in\mathcal{T}$ such that $\mathfrak{A}_0\subset\mathfrak{C}$ and $c_{2g+2}\in\mathfrak{C}$ and $|\mathfrak{C}|=2|\mathfrak{A}_0|+\lambda_0+2$. Pulling back ($\clubsuit$) to $\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})$ by $\varphi_{\mathfrak{C}}$ of 3.10, we obtain

$$\sum_{2|\mathfrak{A}\cap\mathfrak{C}|+\lambda+2=|\mathfrak{C}|}a_{(\mathfrak{A},\lambda)}2^*_{\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})}T^*_{\tau((\mathfrak{A},\lambda);\mathfrak{C})}\Theta=0$$

by Lemma3.11. For$(\mathfrak{A},\lambda)$, $(\mathfrak{A}',\lambda')\in\mathcal{S}^{\mathrm{even}}$ such that $2|\mathfrak{A}\cap\mathfrak{C}|+\lambda+2=|\mathfrak{C}|$ and $2|\mathfrak{A}'\cap\mathfrak{C}|+\lambda'+2=|\mathfrak{C}|$,

$$\tau((\mathfrak{A},\lambda),\mathfrak{C})\otimes\tau((\mathfrak{A}',\lambda'),\mathfrak{C})^{-1}\simeq\mathcal{O}_{C_{\mathfrak{C}}/i_{\mathfrak{C}}}\Big(\sum_{c_j\in\mathfrak{C}\cap\mathfrak{A}}h_{\mathfrak{C}}(c_j^{\mathfrak{C}})-\sum_{c_j\in\mathfrak{C}\cap\mathfrak{A}'}h_{\mathfrak{C}}(c_j^{\mathfrak{C}})$$
$$+\big(\lambda-\lambda'+2|\mathfrak{A}\setminus(\mathfrak{C}\cup\mathfrak{A}')|-2|\mathfrak{A}'\setminus(\mathfrak{C}\cup\mathfrak{A})|\big)h_{\mathfrak{C}}(c_{2g+2}^{\mathfrak{C}})\Big)$$

Claim. If $\tau((\mathfrak{A}_0,\lambda_0);\mathfrak{C})\simeq\tau((\mathfrak{A},\lambda);\mathfrak{C})$, then $(\mathfrak{A}_0,\lambda_0)=(\mathfrak{A},\lambda)$.

Proof of Claim. If $\tau((\mathfrak{A}_0,\lambda_0);\mathfrak{C})\simeq\tau((\mathfrak{A},\lambda);\mathfrak{C})$, we have $\mathfrak{C}\cap\mathfrak{A}_0=\mathfrak{C}\cap\mathfrak{A}$ or $\mathfrak{C}\cap\mathfrak{A}_0=\mathfrak{C}\setminus(\mathfrak{A}\cup\{c_{2g+2}\})$. If $\mathfrak{C}\cap\mathfrak{A}_0=\mathfrak{C}\cap\mathfrak{A}$, we get $\lambda=\lambda_0$. Hence $|\mathfrak{A}_0|=|\mathfrak{A}|$. Since $\mathfrak{A}_0=\mathfrak{C}\cap\mathfrak{A}_0=\mathfrak{C}\cap\mathfrak{A}\subset\mathfrak{A}$, we have $\mathfrak{A}_0=\mathfrak{A}$. Let us prove that $\mathfrak{C}\cap\mathfrak{A}_0=\mathfrak{C}\setminus(\mathfrak{A}\cup\{c_{2g+2}\})$ does not occur. If it did occur, we have $\lambda=-\lambda_0-2$. Since $\lambda+\lambda_0\equiv 0(\mathrm{mod} 4)$, we obtain a contradiction. □

Taking account of the above claim and the fact that $\tau((\mathfrak{A},\lambda),\mathfrak{C})\otimes\tau((\mathfrak{A}',\lambda'),\mathfrak{C})^{-1}$ is a 2-torsion point of $\mathrm{Jac}(C_{\mathfrak{C}}/i_{\mathfrak{C}})$, we have $a_{(\mathfrak{A}_0,\lambda_0)}=0$ again by PropositionA.8 of [2]. For $(\mathfrak{A}_0,\lambda_0)\in\mathcal{S}^{\mathrm{even}}$ with $|\mathfrak{A}_0|>g+1$, we choose an element $\mathfrak{C}$ of $\mathcal{T}$ such that $\{c_1,\dots,c_{2g+1}\}\setminus\mathfrak{A}_0\subset\mathfrak{C}$ and $c_{2g+2}\in\mathfrak{C}$ and $|\mathfrak{C}|=4g+2-\lambda_0-2|\mathfrak{A}_0|$. Then a similar argument as above implies that $a_{(\mathfrak{A}_0,\lambda_0)}=0$.

□

4. Relation with Beauville's basis

By Remark2.12, we have a natural homomorphism $f:\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}\to\overline{\mathcal{M}}(2,\mathcal{O}_C)$ when $|I|$ is even, and $f:\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}\to\overline{\mathcal{M}}(2,\mathcal{O}_C(c_{2g+2}))$ when $|I|$ is odd, where $\overline{\mathcal{M}}(2,\mathcal{O}_C)$ [resp. $\overline{\mathcal{M}}(2,\mathcal{O}_C(c_{2g+2}))$] is the coarse moduli space of rank 2 semistable vector bundles with the determinant $\mathcal{O}_C$ [resp. $\mathcal{O}_C(c_{2g+2})$]. In this section, we compare the basis $\{D_{(\mathfrak{A},\lambda)}\}_{(\mathfrak{A},\lambda)\in\mathcal{S}^{\mathrm{even}}}$ [resp. $\{D_{(\mathfrak{A},\lambda)}\}_{(\mathfrak{A},\lambda)\in\mathcal{S}^{\mathrm{odd}}}$] of $\mathrm{H}^0(\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1},K^{-1}_{\overline{\mathcal{M}}^{invo}_{C/\mathbb{P}^1}})$ with the basis $\{D_\kappa\}_{\kappa:\text{even theta characteristic of }C}$ of $\mathrm{H}^0(\overline{M}(2,\mathcal{O}_C),K^{-\frac{1}{2}}_{\overline{M}(2,\mathcal{O}_C)})$ [resp. $\{D_\kappa\}_{\kappa:\text{odd theta characteristic of }C}$ of $\mathrm{H}^0(\overline{M}(2,\mathcal{O}_C(c_{2g+2})),K^{-\frac{1}{2}}_{\overline{M}(2,\mathcal{O}_C(c_{2g+2}))})$]. Here D_κ is the divisor constructed by Beauville in [2]. It is the unique effective divisor that satisfies the relation $2D_\kappa=\mathrm{Div}(Rp_*(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*\kappa))$ over the stable locus of the moduli space, where $\mathcal{E}$ is the universal vector bundle and p is the projection to the moduli space. (See [2] for details.)

Put

$$\mathcal{V}:=\left\{V\subset\{c_1,\dots,c_{2g+1}\}\;\middle|\;|V|\equiv g+1(\mathrm{mod}\ 2)\right\}.$$

The map from $\mathcal{V}$ to the set of theta characteristics of C that sends V to $\mathcal{O}_C(V+(g-1-|V|)c_{2g+2})$ is a bijection. We have

$$\dim\mathrm{H}^0\left(C,\mathcal{O}_C(V+(g-1-|V|)c_{2g+2})\right)=\frac{\big|g+1-|V|\big|}{2}.$$

Theorem 4.1. *Assume that $|I|$ is even. Let κ be an even theta characteristic of C, say $\mathcal{O}_C(V+(g-1-|V|)c_{2g+2})$.*

(i) *If $\mathrm{H}^0(C,\kappa)=0$, i.e. $|V|=g+1$, then $f^*D_\kappa = D_{(V,-2)}$.*
(ii) *If $\mathrm{H}^0(C,\kappa)\neq 0$, i.e. $|V|\neq g+1$, then $f^*D_\kappa = 0$.*

Theorem 4.2. *Assume that $|I|$ is odd. Let κ be an odd theta characteristic of C, say $\mathcal{O}_C(V+(g-1-|V|)c_{2g+2})$.*

(i) *If $|V|=g-1$, then $f^*D_\kappa = D_{(V,0)}$.*
(ii) *If $|V|\neq g-1$, then $f^*D_\kappa = 0$.*

We give a proof only for Theorem4.1.

Proof of Theorem4.1. Suppose that $\mathrm{H}^0(C,\kappa)=0$. Then $f^*(2D_\kappa) = \mathrm{Div}\left(R^1p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*\kappa\right)\right)$ over $\mathcal{M}^{invo}_{C/\mathbb{P}^1}$, where $\mathcal{E}$ is the universal involutional bundle on $\mathcal{M}^{invo}_{C/\mathbb{P}^1}\times C$ and p is the projection $\mathcal{M}^{invo}_{C/\mathbb{P}^1}\times C \to \mathcal{M}^{invo}_{C/\mathbb{P}^1}$.

Claim. $\mathrm{Div}\left(R^1p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*\kappa\right)^+\right) = \mathrm{Div}\left(R^1p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*\kappa\right)^-\right)$.

Proof of Claim. Grothendieck duality (cf. Chapter VII Theorem3.3 in [7] says that

$$\begin{aligned}&\mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathcal{M}^{invo}_{C/\mathbb{P}^1}}}\left(\mathbb{R}p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*\kappa\right),\mathcal{O}_{\mathcal{M}^{invo}_{C/\mathbb{P}^1}}\right)\\ &\qquad\to \mathbb{R}p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*(\kappa^{-1}\otimes K_C)\right)[1]\end{aligned}$$

is a quasi-isomorphism. Taking cohomologies, we obtain an isomorphism

$$\begin{aligned}&\mathcal{E}xt^1_{\mathcal{O}_{\mathcal{M}^{invo}_{C/\mathbb{P}^1}}}\left(R^1p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*\kappa\right),\mathcal{O}_{\mathcal{M}^{invo}_{C/\mathbb{P}^1}}\right)\\ &\qquad\to R^1p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*(\kappa^{-1}\otimes K_C)\right).\end{aligned}$$

Since this isomorphism is compatible with the i-action, we get an isomorphism

$$\begin{aligned}&\mathcal{E}xt^1_{\mathcal{O}_{\mathcal{M}^{invo}_{C/\mathbb{P}^1}}}\left(R^1p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*\kappa\right)^+,\mathcal{O}_{\mathcal{M}^{invo}_{C/\mathbb{P}^1}}\right)\\ &\qquad\to R^1p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*(\kappa^{-1}\otimes K_C)\right)^+.\end{aligned}$$

$\kappa^{-1}\otimes K_C$ and κ are isomorphic as line bundles on C, but their i-actions differ by (-1)-multiplication. Hence

$$R^1p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*(\kappa\otimes K_C)\right)^+ \simeq R^1p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*\kappa\right)^-.$$

This isomorphism and the equality

$$\begin{aligned}&\mathrm{Div}\left(R^1p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*\kappa\right)^+\right)\\ &=\mathrm{Div}\left(\mathcal{E}xt^1\left(R^1p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*\kappa\right)^+,\mathcal{O}_{\mathcal{M}^{invo}_{C/\mathbb{P}^1}}\right)\right)\end{aligned}$$

implies the claim. □

This claim implies that

$$\mathrm{Div}\left(R^1p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*\kappa\right)\right)=2\mathrm{Div}\left(R^1p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*\kappa\right)^+\right).$$

This proves (i).

Suppose that $\mathrm{H}^0(C,\kappa)\neq 0$. We shall show that $\mathrm{H}^0(C,\mathcal{E}nd^\circ(E)\otimes\kappa)\neq 0$ for any $E\in\mathcal{M}^{invo}_{C/\mathbb{P}^1}$. Since $\chi(\mathcal{E}nd^\circ(E)\otimes\mathcal{O}_C(V+(g-1-|V|)c_{2g+2}))^+=(-g-1+|V|)/2\neq 0$, either $\mathrm{H}^0(C,\mathcal{E}nd^\circ(E)\otimes\mathcal{O}_C(V+(g-1-|V|)c_{2g+2}))^+$ or $\mathrm{H}^1(C,\mathcal{E}nd^\circ(E)\otimes\mathcal{O}_C(V+(g-1-|V|)c_{2g+2}))^+$ is nonzero. A similar argument as in the proof of the above claim implies that $\dim\mathrm{H}^0(C,\mathcal{E}nd^\circ(E)\otimes\mathcal{O}_C(V+(g-1-|V|)c_{2g+2}))=\dim\mathrm{H}^0(C,\mathcal{E}nd^\circ(E)\otimes\mathcal{O}_C(V+(g-1-|V|)c_{2g+2}))^++\mathrm{H}^1(C,\mathcal{E}nd^\circ(E)\otimes\mathcal{O}_C(V+(g-1-|V|)c_{2g+2}))^+$. Hence $\mathrm{H}^0(C,\mathcal{E}nd^\circ(E)\otimes\mathcal{O}_C(V+(g-1-|V|)c_{2g+2}))\neq 0$. $\square$

5. Reducedness of the divisors $D_{(\mathfrak{A},\lambda)}$

In this section, we shall show that the divisors $D_{(\mathfrak{A},\lambda)}$ are reduced.

For a non-negative integer l, let Z_l be the closed subset of $\mathcal{M}^{invo}_{C/\mathbb{P}^1}$ consisting of involutionally stable involutional vector bundle $E\in\mathcal{M}^{invo}_{C/\mathbb{P}^1}$ that satisfies $\dim\mathrm{H}^0(\mathcal{E}nd^\circ(E)\otimes(\mathfrak{A}+\lambda c_{2g+2}))^+\geq l$. We give Z_l a scheme structure as follows. For some open neighborhood U of $E\in Z_l$, we can take a resolution of involutional vector bundles $0\to\mathcal{G}\to\mathcal{O}(-Nc_{2g+2})\otimes V\to\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})\to 0$ on $U\times C$, where $\mathcal{E}$ is the universal involutional vector bundle (strictly speaking, only $\mathcal{E}nd^\circ(\mathcal{E})$ exists on $\mathcal{M}^{invo}_{C/\mathbb{P}^1}\times C$), N is a sufficiently large positive integer and V is a vector space with an i-action. This resolution gives rise to a short exact sequence

$$0\to R^1p_*\mathcal{G}^+\xrightarrow{g}R^1p_*\left(\mathcal{O}(-Nc_{2g+2})\otimes V\right)^+\to R^1p_*\left(\mathcal{E}nd^\circ(\mathcal{E})\otimes p_C^*\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})\right)^+\to 0,$$

where p is the projection $p:U\times C\to U$. By trivializing $R^1p_*\mathcal{G}^+$ and $R^1p_*\left(\mathcal{O}(-Nc_{2g+2})\otimes V\right)^+$ over an open neighborhood W of $E\in\mathcal{M}^{invo}_{C/\mathbb{P}^1}$, g can be expressed by an $r\times r$ matrix $\{g_{ij}\}$ with $g_{ij}\in\mathcal{O}_{\mathcal{M}^{invo}_{C/\mathbb{P}^1}}(W)$, where $r=\mathrm{rank}R^1p_*\mathcal{G}^+=\mathrm{rank}R^1p_*\left(\mathcal{O}(-Nc_{2g+2})\otimes V\right)^+$. The ideal sheaf of Z_l is generated, over W, by all the $l\times l$ minors of this matrix. Note that, with this scheme structure, Z_1 is nothing but $D^\circ_{(\mathfrak{A},\lambda)}$.

For $E\in\mathcal{M}^{invo}_{C/\mathbb{P}^1}$, let ν_E be the map

$$\begin{aligned}\mathrm{H}^0\left(\mathcal{E}nd^\circ(E)\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})\right)^+\otimes\mathrm{H}^0\left(\mathcal{E}nd^\circ(E)\otimes K_C\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2})^{-1}\right)^+\\ \to\mathrm{H}^0\left(\mathcal{E}nd^\circ(E)\otimes K_C\right)^+,\end{aligned}$$

taking $\phi\otimes\psi$ to $[\phi,\psi]$ $(:=\phi\circ\psi-\psi\circ\phi)$.

Proposition 5.1. *Assume that $|I|$ is even [resp. odd]. There exists a closed subset $\mathcal{Y}$ of $\mathcal{M}^{invo}_{C/\mathbb{P}^1}$ of dimension at most $\lfloor\frac{4g-5}{3}\rfloor$ [resp. $\lfloor\frac{4g-7}{3}\rfloor$] such that if E is not in $\mathcal{Y}$ then $\dim\mathrm{Ker}\nu_E\leq l^2-2l+1$, where $l=\dim\mathrm{H}^0(C,\mathcal{E}nd^\circ(E)\otimes\mathcal{O}(\mathfrak{A}+\lambda c_{2g+2}))^+$.*

Proof. We give a proof for the case when $|I|$ is even. The case when $|I|$ is odd is very similar.

Suppose that $\dim \mathrm{Ker}\nu_E > l^2 - 2l + 1$. Then we can find a non-zero element $\phi \otimes \psi$ in $\mathrm{Ker}\nu_E$.

Claim 5.1.1. Either ϕ or ψ is of rank one.

Proof of Claim5.1.1. For c_j, $1 \leq j \leq 2g+1$, fix an isomorphism $\alpha : \mathbb{C}^{\oplus 2} \xrightarrow{\sim} E \otimes \mathbb{C}_{c_j}$ such that $\alpha({}^t(1,0))$ is, for the i-action, an eigen vector with the eigen value 1 and $\alpha({}^t(0,1))$ is an eigen vector with eigen value -1. With this trivialization, at c_j, ϕ and ψ are expressed, respectively, by $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right)$ with $b, c \in \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1}_{c_j}$ and $a \in K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1}_{c_j}$ if $c_j \in \mathfrak{A}$ because ϕ and ψ are traceless and compatible with the i-action. The relation $\phi \circ \psi = \psi \circ \phi$ implies that $\left(\begin{smallmatrix} 0 & -ab \\ ac & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & ab \\ -ac & 0 \end{smallmatrix}\right)$. Hence either ϕ or ψ is a zero map on the fiber over c_j. If $c_j \notin \mathfrak{A}$, switching the role of $\mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})$ and $K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1}$, we know that ϕ or ψ is zero map on the fiber over c_j. Therefore $\phi \circ \psi$ is a zero map on the fiber over c_j for $1 \leq \forall j \leq 2g+1$. If $\phi \circ \psi$ is injective, then $\deg E + 2(2g-2) = \deg E \otimes K_C \geq \deg E + 2(2g+1)$, which is a contradiction. Hence $\phi \circ \psi$ is not injective, which implies that either ϕ or ψ is of rank one. □

Without loss of generality we may assume that ϕ is of rank one. Put $L := \mathrm{Ker}\phi$. Since L is i-invariant, L is isomorphic to $\mathcal{O}(\mathfrak{B} + \mu c_{2g+2})$, for some $\mathfrak{B} \subset \{c_1, \dots, c_{2g+1}\}$ and $\mu \in \mathbb{Z}$, as involutional bundles. E/L is isomorphic to $\mathcal{O}(\mathfrak{B}' - (2g+1+\mu)c_{2g+2})$, where $\mathfrak{B}' := \{c_1, \dots, c_{2g+1}\} \setminus \mathfrak{B}$.

Claim 5.1.2. We have $|\mathfrak{A}| + |\mathfrak{B}| + \mu + \frac{\lambda}{2} - g - 1 \geq 0$ if $\mathrm{H}^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ = 0$. We have $|\mathfrak{B}| - |\mathfrak{A}| + \mu - \frac{\lambda}{2} + g - 1 \geq 0$ if $\mathrm{H}^0(C, K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1})^+ = 0$.

Proof of Claim5.1.2. First note that either $\mathrm{H}^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+$ or $\mathrm{H}^0(C, K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1})^+$ is zero because $\mathrm{H}^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ = \max\{\frac{\lambda}{2} + 1, 0\}$ and $\mathrm{H}^0(C, K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1})^+ = \max\{-\frac{\lambda}{2} - 1, 0\}$. If $\mathrm{H}^0(C, \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ = 0$, then ϕ induces a non-zero i-equivariant morphism $E/L \to L \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})$ because the composite $E/L \to E \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}) \to E/L \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})$ is zero. This nonzero i-equivariant morphism corresponds to a non-zero element of $\mathrm{H}^0(C, L \otimes (E/L)^{-1} \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+$. Since $\mathrm{H}^0(C, L \otimes (E/L)^{-1} \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ \simeq \mathrm{H}^0(C, \mathcal{O}(\mathfrak{B} + \mathfrak{A} - \mathfrak{B}' + (2g+1+2\mu+\lambda)c_{2g+2})^+ \simeq \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(|\mathfrak{A}| + |\mathfrak{B}| + \mu + \frac{\lambda}{2} - g - 1))$, we have $|\mathfrak{A}| + |\mathfrak{B}| + \mu + \frac{\lambda}{2} - g - 1 \geq 0$. Next let us consider the case $\mathrm{H}^0(C, K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1})^+ = 0$. Since $\phi \circ \psi = \psi \circ \phi$ and $L = \mathrm{Ker}\phi$, the restriction of ψ to L factors through $L \otimes K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1}$, hence is a zero map because $\mathrm{H}^0(C, K_C \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2})^{-1})^+ = 0$. This means that $L = \mathrm{Ker}\psi$. Then a similar argument as above implies that $|\mathfrak{B}| - |\mathfrak{A}| + \mu - \frac{\lambda}{2} + g - 1 \geq 0$. □

In order to complete the proof, it suffices to prove that $\dim \mathrm{H}^1(C, \mathcal{H}om(E/L, L))^+ - 1 \leq \lfloor \frac{4g-5}{3} \rfloor$ because $\mathrm{H}^1(C, \mathcal{H}om(E/L, L))^+$ parameterizes extensions of E/L by L that are compatible with the i-actions. Since $\mathrm{H}^1(C, \mathcal{H}om(E/L, L))^+ \simeq \mathrm{H}^1(C, \mathcal{O}(\mathfrak{B} - \mathfrak{B}' + (2g+1+2\mu)c_{2g+2}))^+ \simeq \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}(|\mathfrak{B}| + \mu - g - 1))$, $\dim \mathrm{H}^1(C, \mathcal{H}om(E/L, L))^+ = \max\{g - \mu - |\mathfrak{B}|, 0\}$. By the above claim, $\dim \mathrm{H}^1(C, \mathcal{H}om(E/L, L))^+ \leq \max\{|\mathfrak{A}| + \frac{\lambda}{2} - 1, 0\} \leq \lfloor \frac{4g-2}{3} \rfloor$. □

Before stating and proving the main theorem of this section, we recall an important property of ν_E.

Proposition 5.2. *For $E \in Z_l \backslash Z_{l+1}$, the codimension of the irreducible component of Z_l containing E is greater than or equal to* $\dim \mathrm{Im}\nu_E$. *If ν_E is injective, Z_l is smooth at E and of codimension l^2.*

This proposition is essentially proved in [12] and we omit the proof.

Theorem 5.3. *All the divisors $D_{(\mathfrak{A},\lambda)}$ are reduced.*

Proof. Note that the closed subset $\mathcal{Y}$ of Proposition5.1 is of codimension greater than one. Therefore we have only to prove that $D_{(\mathfrak{A},\lambda)}|_{\mathcal{M}^{invo}_{C/\mathbb{P}^1} \backslash \mathcal{Y}}$ is reduced. For $E \in D_{(\mathfrak{A},\lambda)} \backslash \mathcal{Y}$, if $\dim \mathrm{H}^0(C, \mathcal{E}nd^\circ(E) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+ = 1$, then by Proposition5.1 ν_E is injective, hence $D_{(\mathfrak{A},\lambda)}$ is smooth at E by Proposition5.2. Therefore if suffices to prove that $\mathrm{codim}(Z_2 \backslash \mathcal{Y}, \mathcal{M}^{invo}_{C/\mathbb{P}^1} \backslash \mathcal{Y}) \geq 2$. For $E \in Z_2 \setminus \mathcal{Y}$, $\dim \mathrm{Im}\nu_E \geq 2l - 1$ by Proposition5.1, where $l = \dim \mathrm{H}^0(C, \mathcal{E}nd^\circ(E) \otimes \mathcal{O}(\mathfrak{A} + \lambda c_{2g+2}))^+$. By Proposition5.2, the codimension of Z_l at E is greater than or equal to $2l - 1 > 2$. □

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