# FINITE GROUP ACTIONS ON $C^*$-ALGEBRAS WITH THE ROHLIN PROPERTY. II

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ABSTRACT. We classify finite group actions on some class of $C^*$-algebras with the Rohlin property in terms of the $K$-groups. A group cohomological obstruction is obtained for the $K$-groups of a $C^*$-algebra allowing such an action, for example, the Tate cohomology groups with the $K$-groups as the coefficient modules must vanish. Model actions with the Rohlin property are constructed for given invariants satisfying some conditions.

## 1. INTRODUCTION

This is the second part of our analysis initiated in [21] about finite group actions on $C^*$-algebras with the Rohlin property, a property that already appeared in [25, 17, 18] with different names. For a detailed description of the history and significance of the subject, we refer the reader to the introduction of the first part [21]. (See also [26, 27, 33, 19] and references therein for related topics.) In our previous work [21], we established basic properties of finite group actions with the Rohlin property, and in particular we obtained strong $K$-theoretical constraints for such actions. As an application of these results, in this second part, we develop a classification theory of such actions on some "classifiable classes" of nuclear $C^*$-algebras in terms of the $K$-groups. More precisely, we deal with two classes of $C^*$-algebras, which are known to be completely classified by their $K$-groups [38, 24, 35], [14, 12, 11, 30]: simple nuclear separable purely infinite $C^*$-algebras satisfying the universal coefficient theorem [1, 41] (abbreviated as UCT), and simple nuclear separable $C^*$-algebras of tracial topological rank zero satisfying the UCT. We refer to recent monographs [31, 39] for the classification theory of these two classes of $C^*$-algebras.

Since its first appearance in operator algebras [7], the Rohlin property has always served as a means to establish cohomology vanishing type results (see, for example, [22, 34, 5]). Here, the coefficient "modules" of the relevant cohomology are usually the unitary groups of some operator algebras. When a $C^*$-algebra $A$ allows an action of a finite group $G$, the $K$-groups of $A$ have $G$-module structure induced by the action. One of the central issues of this paper is the group cohomology of $G$ with the $K$-groups as coefficient modules. When the action possesses the Rohlin property, we obtain a considerably strong vanishing theorem (Theorem 3.3), namely the $K$-groups should be completely cohomologically trivial (see Definition 3.8). This property has several important consequences. For example, in order to prove our classification theorems

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(Theorem 4.2, Theorem 4.3), where the complete invariants are the ordinary $K$-groups, one of the difficulties we must overcome is that we need to fill the gap between the ordinary $K$-groups and the $K$-groups with coefficient modules. Theorem 3.3 is the main tool for this procedure. In many cases, Theorem 3.3 also gives a constraint strong enough to characterize those $G$-modules coming from the $K$-groups of $C^*$-algebras allowing $G$-actions with the Rohlin property (Theorem 5.5).

Throughout this paper, we mainly work on the class of unital simple nuclear separable purely infinite $C^*$-algebras satisfying UCT. Whenever the proofs are valid without essential change, we state the corresponding results for the class of unital simple nuclear separable $C^*$-algebras of tracial topological rank zero satisfying UCT. Note that the contents of Section 5 work only for the former class so far.

In Section 6, we give some consequences of our main theorems in the case of finite cyclic group actions on the Cuntz algebras. Among others, we observe some subtle difference among the quasi-free actions about their $K$-theoretical behavior (Theorem 6.5), which is related to whether they fit into our classification scheme using the Rohlin property.

## 2. Preliminaries

In this section, we fix notations and collect basic facts frequently used in this paper.

2.1. **Notations.** Throughout this paper, $C^*$-algebras are always assumed to be unital unless particularly mentioned. For $C^*$-algebras $A$, $B$, $A \otimes B$ always stands for the minimal tensor product of $A$ and $B$. We denote by $U(A)$ the unitary group of $A$, and by $\mathrm{Aut}(A)$ the automorphism group of $A$. $\mathrm{Aut}(A)$ is equipped with the topology of pointwise norm convergence. We denote by $\mathrm{Inn}(A)$ the inner automorphism group of $A$, and by $\overline{\mathrm{Inn}(A)}$ the closure of $\mathrm{Inn}(A)$, whose elements are called approximately inner automorphisms. Two $*$-homomorphisms $\varphi$ and $\psi$ from $A$ into another $C^*$-algebra $B$ are said to be *approximately unitarily equivalent* if there exists a sequence of unitaries $\{u_n\}_{n=1}^{\infty}$ in $B$ such that for all $a \in A$, $\{\mathrm{Ad}(u_n) \cdot \varphi(a)\}_{n=1}^{\infty}$ converges to $\psi(a)$.

Let $G$ be a finite group. We denote by $|G|$ the order of $G$. A $G$-action on a $C^*$-algebra $A$ is a homomorphism from $G$ into $\mathrm{Aut}(A)$. For a $G$-action $\alpha$ on $A$, $A^{\alpha}$ denotes the fixed-point subalgebra of $A$ under $\alpha$. Two $G$-actions $\alpha$ on $A$ and $\beta$ on $B$ are said to be *conjugate*, and denoted by $(A, \alpha) \cong (B, \beta)$, if there exists an isomorphism $\theta$ from $A$ onto $B$ such that $\theta \cdot \alpha_g = \beta_g \cdot \theta$ holds for every $g \in G$.

We denote by $\mathbb{M}_n$, $\mathbb{K}$, $\mathbb{M}_{n^\infty}$ $(n \in \mathbb{N})$, and $\mathcal{O}_n$ $(n = 2, 3, \cdots, \infty)$ the $n$ by $n$ matrix algebra, the $C^*$-algebra of all compact operators on $\ell^2(\mathbb{N})$, the UHF algebra of type $n^\infty$, and the Cuntz algebra with the canonical generators $\{S_i\}_{i=1}^n$, [8] respectively.

We denote by $\mathcal{N}$ the UCT class of Robsenberg and Schochet [1, 41].

For abelian groups, we use the additive notation. Let $n$ be a natural number. We denote by $\mathbb{Z}_n$ the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of order $n$. For an abelian group $\Gamma$, we set

$$n\Gamma = \{na;\ a \in \Gamma\},$$

$${}_n\Gamma = \mathrm{Tor}(\Gamma, \mathbb{Z}_n) = \{a \in \Gamma;\ na = 0\},$$

$$\Gamma_n = \Gamma \otimes_{\mathbb{Z}} \mathbb{Z}_n = \Gamma/n\Gamma.$$

Likewise, for a homomorphism $\theta \in \mathrm{Hom}(\Gamma^1, \Gamma^2)$ between two abelian groups $\Gamma^1$ and $\Gamma^2$, we set ${}_n\theta = \theta|_{\mathrm{Tor}(\Gamma^1, \mathbb{Z}_n)}$ and $\theta_n = \theta \otimes \mathrm{id}_{\mathbb{Z}_n}$. We denote by $\Gamma_{\mathrm{tor}}$ the torsion part $\mathrm{Tor}(\Gamma, \mathbb{Q}/\mathbb{Z})$ of $\Gamma$. For a subset $S \subset \Gamma$, $\langle S \rangle$ denotes the subgroup of $\Gamma$ generated by $S$. $\Gamma$ is said to be *uniquely $n$-divisible* if multiplying by $n$ is a group automorphism of $\Gamma$.

2.2. **The entire $K$-group.** First, we give a brief survey on the mod $p$ $K$-theory of $C^*$-algebras introduced by Schochet [43]. For our practical use, we define the $K$-group of a $C^*$-algebra $A$ with the coefficient module $\mathbb{Z}_n$ ($n \in \mathbb{N}$) by

$$K_i(A; \mathbb{Z}_n) = K_i(A \otimes \mathcal{O}_{n+1}) \quad (i = 0, 1).$$

Although our definition looks different from the original one, it gives an equivalent theory to the conventional one [43, Theorem 6.4]. For notational convenience, we set $K_i(A; \mathbb{Z}_n) = K_i(A)$ for $n = 0$.

For a $*$-homomorphism $\varphi$ from a $C^*$-algebra $A$ to another $C^*$-algebra $B$, we denote by $K_i(\varphi; \mathbb{Z}_n)$, the homomorphism from $K_i(A; \mathbb{Z}_n)$ to $K_i(B; \mathbb{Z}_n)$ induced by $\varphi$. We set $K_i(\varphi) = K_i(\varphi; \mathbb{Z}_n)$ for $n = 0$.

There are several natural transformations among these $K$-groups. For example, we have the Künneth exact sequence

$$0 \to K_i(A)_n \xrightarrow{\rho_n^i} K_i(A; \mathbb{Z}_n) \xrightarrow{\beta_n^i} {}_nK_{1-i}(A) \to 0. \tag{2.1}$$

Also, there are so called the Bockstein operations

$$\kappa_{n,m}^i : K_i(A; \mathbb{Z}_m) \to K_i(A; \mathbb{Z}_n).$$

These transformations satisfy certain relations (see, for example, [43, 13, 31]). Among others, we have the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_i(A)_m \arrow[r, "\rho_m^i"] \arrow[d, "{\kappa_{n,m}^i}'"] & K_i(A; \mathbb{Z}_m) \arrow[r, "\beta_m^i"] \arrow[d, "\kappa_{n,m}^i"] & {}_mK_{1-i}(A) \arrow[r] \arrow[d, "{\kappa_{n,m}^i}''"] & 0 \\
0 \arrow[r] & K_i(A)_n \arrow[r, "\rho_n^i"] & K_i(A; \mathbb{Z}_n) \arrow[r, "\beta_n^i"] & {}_nK_{1-i}(A) \arrow[r] & 0,
\end{tikzcd}$$

where ${\kappa_{n,m}^i}'$ and ${\kappa_{n,m}^i}''$ are induced by multiplication by $n/(m,n)$ and $m/(m,n)$ respectively.

**Definition 2.1.** The entire $K$-group $\underline{K}(A)$ of $A$ is defined by

$$\underline{K}(A) = \bigoplus_{n=0}^{\infty} (K_0(A; \mathbb{Z}_n) \oplus K_0(A; \mathbb{Z}_n)).$$

Since the Künneth sequence splits, the group structure of $\underline{K}(A)$ is completely determined by $K_i(A)$ ($i = 0, 1$). However, the main point of introducing the entire $K$-groups is that the set of morphisms between them may be strictly larger than that for the ordinary $K$-groups. This is due to the fact that there is no natural splitting of the Künneth sequence (2.1) in general.

For $C^*$-algebras $A$, $B$, we denote by $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ the set of all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ preserving the direct sum decomposition and commuting with the natural transformations mentioned above (see [13], [31] for details). For $\theta \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$, we denote by $\theta^i$ and $\theta^{i,n}$ its components in $\mathrm{Hom}(K_i(A), K_i(B))$ and $\mathrm{Hom}(K_i(A;\mathbb{Z}_n), K_i(B;\mathbb{Z}_n))$ respectively. We denote by $\mathrm{Aut}_\Lambda(\underline{K}(A))$ the set of invertible elements in $\mathrm{End}_\Lambda(\underline{K}(A)) := \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(A))$. Let $\varphi : A \to B$ be a $*$-homomorphism. Then, $\{K_i(\varphi;\mathbb{Z}_n)\}_{i,n}$ induce an element of $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$, which will be denoted by $\underline{K}(\varphi)$. More generally, there exists a natural homomorphism $\Gamma : KK(A,B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ defined via the Kasparov product. Denoting by $KK(\varphi)$ the element of $KK(A,B)$ given by $\varphi$, we have the relation $\Gamma(KK(\varphi)) = \underline{K}(\varphi)$.

Dadarlat and Loring [13] proved the following universal multicoefficient theorem (UMCT):

**Theorem 2.2.** *Let $A \in \mathcal{N}$ and $B$ be a $\sigma$-unital $C^*$-algebra. Then, there is a short exact sequence*

$$0 \to \bigoplus_{i=0,1} \mathrm{Pext}(K_i(A), K_{1-i}(B)) \xrightarrow{\delta} KK(A,B) \xrightarrow{\Gamma} \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \to 0,$$

*where $\mathrm{Pext}(K_i(A), K_{1-i}(B))$ is the subgroup of $\mathrm{Ext}(K_i(A), K_{1-i}(B))$ consisting of the pure extensions. The sequence is natural in each variable.*

Following [13], we set

$$F_P\underline{K}(A) = K_0(A) \oplus K_1(A) \oplus \bigoplus_{p\in\mathcal{P}} \bigoplus_{m=1}^{\infty} (K_0(A;\mathbb{Z}_{p^m}) \oplus K_1(A;\mathbb{Z}_{p^m})),$$

where $\mathcal{P}$ is the set of all prime numbers. For our purpose, it is much more convenient to deal with $F_P\underline{K}(A)$ than $\underline{K}(A)$. We denote by $\mathrm{Hom}_\Lambda(F_P\underline{K}(A), F_P\underline{K}(B))$ (respectively $\mathrm{Aut}_\Lambda(F_P\underline{K}(A))$) the restriction of the elements in $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ (respectively $\mathrm{Aut}(\underline{K}(A))$) to $F_P\underline{K}(A)$. The following lemma [13, Proposition 2.10] shows that one does not loose any information by passing from $\underline{K}(A)$ to $F_P\underline{K}(A)$:

**Lemma 2.3.** *Let $A \in \mathcal{N}$ and $B$ be a $\sigma$-unital $C^*$-algebra. Then, the natural restriction maps*

$$\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \to \mathrm{Hom}_\Lambda(F_P\underline{K}(A), F_P\underline{K}(B))$$
$$\mathrm{Aut}_\Lambda(\underline{K}(A)) \to \mathrm{Aut}_\Lambda(F_P\underline{K}(A))$$

*are isomorphisms.*

It is a routine work to show that $\mathrm{Hom}_\Lambda(F_P\underline{K}(A), F_P\underline{K}(B))$ consists of those homomorphisms from $F_P\underline{K}(A)$ to $F_P\underline{K}(B)$ preserving the direct sum decomposition and commuting with $\rho^i_{p^m}$, $\beta^i_{p^m}$, $\kappa^i_{p^m,p^{m+1}}$, $\kappa^i_{p^{m+1},p^m}$ for all $i = 0, 1$, $p \in \mathcal{P}$, $m \in \mathbb{N}$. On the other hand, as partly pointed out in [43, Proposition 2.5] and essentially proven in [3, 4], there exists a family of splitting maps

$$s^i_{p^m} : {}_{p^m}K_{1-i}(A) \to K_i(A;\mathbb{Z}_{p^m})$$

of the Künneth sequence satisfying

$$\beta^i_{p^m} \cdot s^i_{p^m} = \mathrm{id}_{{}_{p^m}K_{1-i}(A)}, \tag{2.2}$$

$$\kappa^i_{p^m,p^{m+1}} \cdot s^i_{p^{m+1}} = s^i_{p^m} \cdot {\kappa^i_{p^m,p^{m+1}}}'', \tag{2.3}$$

$$\kappa^i_{p^{m+1},p^m} \cdot s^i_{p^m} = s^i_{p^{m+1}} \cdot {\kappa^i_{p^{m+1},p^m}}''. \tag{2.4}$$

Fixing such $s^i_{p^m}$, we may identify $K_i(A;\mathbb{Z}_{p^m})$ with $K_i(A)_{p^m} \oplus {}_{p^m}K_{1-i}(A)$ by the isomorphism

$$K_i(A;\mathbb{Z}_{p^m}) \ni x \mapsto (x - s^i_{p^m} \cdot \beta^i_{p^m}(x) \oplus \beta^i_{p^m}(x)) \in K_i(A)_{p^m} \oplus {}_{p^m}K_{1-i}(A),$$

where we regard $K_i(A)_{p^m}$ as a subgroup of $K_i(A,\mathbb{Z}_{p^m})$ through the injective homomorphism $\rho^i_{p^m}$. Let

$$\tilde{\kappa}^i_{p^{m+1},p^m} : K_i(A)_{p^m} \oplus {}_{p^m}K_{1-i}(A) \to K_i(A)_{p^{m+1}} \oplus {}_{p^{m+1}}K_{1-i}(A),$$
$$\tilde{\kappa}^i_{p^m,p^{m+1}} : K_i(A)_{p^{m+1}} \oplus {}_{p^{m+1}}K_{1-i}(A) \to K_i(A)_{p^m} \oplus {}_{p^m}K_{1-i}(A)$$

be the maps corresponding to $\kappa^i_{p^{m+1},p^m}$ and $\kappa^i_{p^m,p^{m+1}}$ under this identification. Then, they are given by

$$\tilde{\kappa}^i_{p^{m+1},p^m} = p' \oplus \iota,$$
$$\tilde{\kappa}^i_{p^m,p^{m+1}} = \pi \oplus p''.$$

In the above, $p'$ and $p''$ stand for multiplication by $p$ considered as a map from $K_i(A)_{p^m}$ to $K_i(A)_{p^{m+1}}$ and that from ${}_{p^{m+1}}K_{1-i}(A)$ to ${}_{p^m}K_{1-i}(A)$. $\iota$ is the inclusion map of ${}_{p^m}K_{1-i}(A)$ into ${}_{p^{m+1}}K_{1-i}(A)$, and $\pi$ is the quotient map of $K_i(A)_{p^{m+1}}$ onto $K_i(A)_{p^m}$.

For two $C^*$-algebras $A$, $B$, we fix $s^i_{p^m}$ as above for both $A$ and $B$, and identify $K_i(A;\mathbb{Z}_{p^m})$ and $K_i(B;\mathbb{Z}_{p^m})$ with $K_i(A)_{p^m} \oplus {}_{p^m}K_{1-i}(A)$ and $K_i(B)_{p^m} \oplus {}_{p^m}K_{1-i}(B)$ respectively. For $\theta \in \mathrm{Hom}_\Lambda(\underline{K}(A),\underline{K}(B))$, let $\tilde{\theta}^{i,p^m}$ be the element corresponding to $\theta^{i,p^m}$ in

$$\mathrm{Hom}(K_i(A)_{p^m} \oplus {}_{p^m}K_{1-i}(A), K_i(B)_{p^m} \oplus {}_{p^m}K_{1-i}(B)),$$

under the above identification. Then, the diagonal part of $\tilde{\theta}^{i,p^m}$ is nothing but $\theta^i_{p^m}$ and ${}_{p^m}\theta^{1-i}$. Since $\rho^i_{p^m}$ is natural, $\tilde{\theta}^{i,p^m}$ maps $K_i(A)_{p^m}$ into $K_i(B)_{p^m}$. Therefore, the only component of $\tilde{\theta}^{i,p^m}$ that is not determined by $\theta^0$, $\theta^1$ is the off diagonal one in $\mathrm{Hom}({}_{p^m}K_{1-i}(A), K_i(B)_{p^m})$, which will be denoted by $\tilde{\theta}^{i,p^m}_{\mathrm{off}}$. As we have

$$\tilde{\theta}^i_{p^{m+1}} \cdot \tilde{\kappa}^i_{p^{m+1},p^m} = \tilde{\kappa}^i_{p^{m+1},p^m} \cdot \tilde{\theta}^i_{p^m},$$
$$\tilde{\theta}^i_{p^m} \cdot \tilde{\kappa}^i_{p^m,p^{m+1}} = \tilde{\kappa}^i_{p^m,p^{m+1}} \cdot \tilde{\theta}^i_{p^{m+1}},$$

we get

$$\tilde{\theta}^{i,p^{m+1}}_{\mathrm{off}} \cdot \iota = p' \cdot \tilde{\theta}^{i,p^m}_{\mathrm{off}}, \tag{2.5}$$

$$\tilde{\theta}^{i,p^m}_{\mathrm{off}} \cdot p'' = \pi \cdot \tilde{\theta}^{i,p^{m+1}}_{\mathrm{off}}. \tag{2.6}$$

Let $\mathrm{Hom}^s_\Lambda(\underline{K}(A),\underline{K}(B))$ be the set of all $(\vartheta^i, \vartheta^{i,p^m}_{\mathrm{off}})$ with $\vartheta^i \in \mathrm{Hom}(K_i(A),K_i(B))$, $\vartheta^{i,p^m}_{\mathrm{off}} \in \mathrm{Hom}({}_{p^m}K_{1-i}(A), K_i(B)_{p^m})$ ($i=0,1$, $p \in \mathcal{P}$, $m \in \mathbb{N}$) satisfying (2.5) and (2.6). Then, $\mathrm{Hom}^s_\Lambda(\underline{K}(A),\underline{K}(B))$ is an additive group by the componentwise addition. Moreover, we can introduce a composition of $\vartheta \in \mathrm{Hom}^s_\Lambda(\underline{K}(A),\underline{K}(B))$ and $\varsigma \in \mathrm{Hom}^s_\Lambda(\underline{K}(B),\underline{K}(C))$ by

$$(\varsigma \cdot \vartheta)^i = \varsigma^i \cdot \vartheta^i, \tag{2.7}$$

$$(\varsigma \cdot \vartheta)^{i,p^m}_{\text{off}} = \varsigma^i_{p^m} \cdot \vartheta^{i,p^m}_{\text{off}} + \varsigma^{i,p^m}_{\text{off}} \cdot {}_{p^m}\vartheta^{1-i}. \tag{2.8}$$

Summing up the arguments so far, we get

**Lemma 2.4.** *There exists an isomorphism of abelian groups*

$$\Phi : \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \to \mathrm{Hom}^s_\Lambda(\underline{K}(A), \underline{K}(B))$$

*preserving composition such that it satisfies* $\Phi(\theta)^i = \theta^i$.

We denote by $\mathrm{Aut}^s_\Lambda(\underline{K}(A)) = \Phi(\mathrm{Aut}(\underline{K}(A)))$ the set of all invertible elements in $\mathrm{Hom}^s_\Lambda(\underline{K}(A), \underline{K}(A))$. $\mathrm{Aut}^s_\Lambda(\underline{K}(A))$ consists of those $\vartheta \in \mathrm{Hom}^s_\Lambda(\underline{K}(A), \underline{K}(A))$ with $\vartheta^i \in \mathrm{Aut}(K_i(A))$ $(i = 0, 1)$. $\vartheta^{-1}$ is given by

$$\vartheta^{-1} = ({\vartheta^i}^{-1}, -{\vartheta^i}^{-1}{}_{p^m} \cdot \vartheta^{i,p^m}_{\text{off}} \cdot {}_{p^m}{\vartheta^{1-i}}^{-1}). \tag{2.9}$$

2.3. **Kirchberg algebras.** A simple unital $C^*$-algebra $A$ is said to be *purely infinite* if for every non-zero element $x \in A$, there exist $a, b \in A$ such that $1 = axb$ holds. There are various remarkable properties fulfilled by purely infinite $C^*$-algebras, and we use them freely in the sequel as far as they are found in [39, Chapter 4]. A purely infinite simple unital $C^*$-algebra is said to be *in the Cuntz standard form* if $[1] = 0$ in $K_0(A)$. Pure infiniteness is a property preserved by passing to the crossed product and the fixed-point subalgebra for a finite group action [28], [23], or more generally, it is a property inherited through an inclusion with a finite index [20].

In this subsection, we summarize a few consequences of the classification theory of Kirchberg algebras mainly due to Kirchberg, Phillips, and Rørdam.

**Definition 2.5.** A *Kirchberg algebra* is a purely infinite simple nuclear separable $C^*$-algebra.

Kichberg algebras in the UCT class $\mathcal{N}$ are completely classified by their $K$-groups, and every element in the $KK$-group for them is given by a $*$-homomorphism [35]. As every Kirchberg algebra is approximately divisible [24], [39, Corollary 7.2.6], the following lemma holds thanks to [2]:

**Lemma 2.6.** *Let $A$ be a Kirchberg algebra and $B$ be a simple $C^*$-algebra. Then, the tensor product $A \otimes B$ is purely infinite simple.*

We will often use this fact for $A = \mathcal{O}_\infty$ or $A = \mathcal{P}_\infty$, where $\mathcal{P}_\infty$ is the unique (up to isomorphism) Kirchberg algebra in $\mathcal{N}$ satisfying $K_0(\mathcal{P}_\infty) = \{0\}$ and $K_1(\mathcal{P}_\infty) \cong \mathbb{Z}$. Taking tensor product with $\mathcal{P}_\infty$ plays the role of suspension within the class of simple purely infinite $C^*$-algebras.

Let $A \in \mathcal{N}$ and $B$ be a $\sigma$-unital $C^*$-algebra. Thanks to the UCT, there is an injective homomorphism from $\mathrm{Pext}(K_0(A), K_1(B)) \oplus \mathrm{Pext}(K_1(A), K_0(B))$ into $KK(A, B)$. Rørdam [38] introduced $KL(A, B)$ by setting

$$KL(A, B) = KK(A, B) / \bigoplus_{i=0,1} \mathrm{Pext}(K_i(A), K_{1-i}(B)).$$

If two $*$-homomorphisms $\varphi$ and $\psi$ from $A$ to $B$ are approximately unitarily equivalent, it is immediate to show that $\underline{K}(\varphi) = \underline{K}(\psi)$ holds. On the other hand, this and Theorem 2.2 imply $KL(\varphi) = KL(\psi)$. For the Kirchberg algebras in $\mathcal{N}$, the converse is also true [39, Example 8.4.14]. Summing up these results, we get

**Theorem 2.7.** *Let $A$ and $B$ be unital Kirchberg algebras in the UCT class $\mathcal{N}$. Then, the following holds:*

(1) *For any element $\phi \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ satisfying $\phi^0([1_A]) = [1_B]$, there exists a unital $*$-homomorphism $\varphi$ from $A$ to $B$ such that $\phi = \underline{K}(\varphi)$. $\varphi$ is unique up to approximate unitary equivalence.*

(2) *The natural sequence*
$$0 \to \overline{\mathrm{Inn}(A)} \to \mathrm{Aut}(A) \to \mathrm{Aut}_\Lambda(\underline{K}(A), [1_A]) \to 0$$
*is exact, where*
$$\mathrm{Aut}_\Lambda(\underline{K}(A), [1_A]) = \{\theta \in \mathrm{Aut}_\Lambda(\underline{K}(A));\ \theta^0([1_A]) = [1_A]\}.$$

2.4. **The Rohlin property.** For a $C^*$-algebra $A$, we set
$$c_0(A) = \{(a_n) \in \ell^\infty(\mathbb{N}, A);\ \lim_{n\to\infty} ||a_n|| = 0\},$$
$$A^\infty = \ell^\infty(\mathbb{N}, A)/c_0(A).$$
We identify $A$ with the $C^*$-subalgebra of $A^\infty$ consisting of the equivalence classes of constant sequences, and set
$$A_\infty = A^\infty \cap A'.$$
The usual perturbation argument shows that every unitary (respectively projection) in $A^\infty$ has a representing sequence in $\ell^\infty(\mathbb{N}, A)$ consisting of unitaries (respectively projections). The same statement is also true for a partition of unity consisting of projections. For an automorphism $\alpha \in \mathrm{Aut}(A)$, we denote by $\alpha^\infty$ and $\alpha_\infty$ the automorphisms of $A^\infty$ and $A_\infty$ induced by $\alpha$ respectively.

**Definition 2.8.** Let $\alpha$ be an action of a finite group $G$ on a unital $C^*$-algebra $A$. $\alpha$ is said to have the *Rohlin property* if there exists a partition of unity $\{e_g\}_{g\in G} \subset A_\infty$ consisting of projections satisfying
$$\alpha_{g\infty}(e_h) = e_{gh}, \quad g, h \in G.$$
We call $\{e_g\}_{g\in G}$ the *Rohlin projections.*

We recall two results from [21, Theorem 3.5, Theorem 3.13]. The next theorem follows from a finite group version of Evans-Kishimoto's intertwining argument [15].

**Theorem 2.9.** *Let $A$ be a separable unital $C^*$-algebra and $\alpha$ and $\beta$ be actions of a finite group $G$ on $A$ with the Rohlin property. Assume that $\alpha_g$ and $\beta_g$ are approximately unitarily equivalent for all $g \in G$. Then, there exists approximately inner automorphism $\theta \in \mathrm{Aut}(A)$ such that*
$$\theta \cdot \alpha_g \cdot \theta^{-1} = \beta_g, \quad g \in G.$$

The following statement is one of the most important $K$-theoretical consequences of the Rohlin property.

**Theorem 2.10.** *Let $A$ be a simple unital $C^*$-algebra, and $\alpha$ be an action of a finite group $G$ on $A$ with the Rohlin property. Let $\iota$ be the inclusion map of $A^\alpha$ into $A$. Then, $K_i(\iota)$ $(i = 0, 1)$ are injective. Moreover, the following equation holds in $K_i(A)$ $(i = 0, 1)$:*

$$\mathrm{Im}(K_i(\iota)) = \mathrm{Im}(\sum_{g\in G} K_i(\alpha_g)) = K_i(A)^G,$$

*where* $K_i(A)^G = \{x \in K_i(A);\ K_i(\alpha_g)(x) = x,\ \forall g \in G\}$.

Before ending this section, we give a simplest example of an action with the Rohlin property, which serves as a model for those actions trivially acting on the $K$-groups (see Theorem 3.4, 3.5 below). Let $G$ be a finite group, and $\lambda$ be the left regular representation of $G$. We identify $\mathbb{B}(\ell^2(G))$ with $\mathbb{M}_{|G|}$ and introduce an action of $G$ on $\mathbb{M}_{|G|^\infty}$ by

$$\mu_g^G = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(\lambda(g)), \quad g \in G.$$

Then, it is straightforward to show that $\mu^G$ has the Rohlin property.

## 3. A vanishing theorem for the Tate cohomology

The main purpose of this section is to deduce some cohomological constraint for the $K$-groups of a $C^*$-algebra admitting a finite group action with the Rohlin property.

First, we recall the definition of the Tate cohomology. Our exposition here heavily relies on [6, Chapter VI]. We fix a finite group $G$, and denote by $N$ its norm element in the integral group ring $\mathbb{Z}G$, namely

$$N = \sum_{g\in G} g \in \mathbb{Z}G.$$

For a left $G$-module $M$, we set

$$M^G = \{m \in M;\ gm = m\}.$$

$$M_G = M/\langle gm - m;\ m \in M,\ g \in G\rangle \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M,$$

where $\mathbb{Z}$ is regarded as a trivial $G$-module. As $gm - m$ is annihilated by $N$ for all $g \in G$ and $m \in M$, $N$ induces a map $\overline{N} : M_G \to M^G$.

**Definition 3.1.** Let $G$ be a finite group and $M$ a $G$-module.

(1) The *Tate cohomology* $\hat{H}^n(G, M)$ $(n \in \mathbb{Z})$ is defined by

$$\hat{H}^n(G, M) = \begin{cases} H^n(G, M) & n > 0 \\ \mathrm{Coker}\overline{N} & n = 0 \\ \mathrm{Ker}\overline{N} & n = -1 \\ H_{-n-1}(G, M) & n < -1 \end{cases}$$

(2) $M$ is said to be *cohomologically trivial* (abbreviated as CT in the sequel) if $\hat{H}^n(H, M) = \{0\}$ for all $n \in \mathbb{Z}$ and for all subgroups $H \subset G$.

Being a cohomology theory, the Tate cohomology converts a short exact sequence $0 \to M' \to M \to M'' \to 0$ of $G$-modules into a long exact sequence

$$\cdots \xrightarrow{\delta} \hat{H}^n(G, M') \to \hat{H}^n(G, M) \to \hat{H}^n(G, M'') \to \hat{H}^{n+1}(G, M') \xrightarrow{\delta} \cdots .$$

Therefore, if two of $M$, $M'$, and $M''$ are CT, so is the remaining one.

A $G$-module $M$ is said to be *induced* if it has the form $\mathbb{Z}G \otimes_{\mathbb{Z}} M'$ with a trivial $G$-module $M'$. Note that since $G$ is a finite group, there is no distinction between induced and co-induced modules. Thanks to Shapiro's lemma [6, Proposition III.6.2], every induced $G$-module is CT. A $G$-module $M$ is said to be *relatively projective* if it is a direct summand of an induced $G$-module. Since a direct summand of a CT $G$-module is again CT, every relatively projective $G$-module is CT. In particular, every $\mathbb{Z}G$-projective module is CT.

We collect a few useful results from [6, Chapter 3, Section 8] in the next lemma.

**Lemma 3.2.** *Let $G$ be a finite group and $M$ be a $G$-module. Then, the following hold:*

(1) *$M$ is CT and $\mathbb{Z}$-free if and only if $M$ is $\mathbb{Z}G$-projective.*
(2) *For every prime number $p$ dividing $|G|$, choose a $p$-Sylow subgroup $G(p)$ of $G$. Then, $M$ is CT if and only if its restriction to $G(p)$ is CT for each $p$.*
(3) *Assume that $G$ is a $p$-group for a prime number $p$. Then, $M$ is CT if and only if $\hat{H}^n(G, M) = \{0\}$ for two consecutive integers.*
(4) *Assume that $G$ is a $p$-group for a prime number $p$, and $M$ is a $kG$-module for a field $k$ of characteristic $p$. Then, $M$ is CT if and only if $M$ is $kG$-free.*
(5) *If $M$ is a torsion group such that the order of every element in $M$ is prime to $|G|$, then $M$ is CT.*

Now we state the main theorem in this section. In what follows, when an action $\alpha$ of $G$ on a $C^*$-algebra is fixed, the $G$-module structure of the $K$-groups is understood as that coming from $\alpha$, and we omit the symbol $K_i(\alpha_g)$ for simplicity.

**Theorem 3.3.** *Let $\alpha$ be an action of a finite group $G$ on a unital simple $C^*$-algebra $A$. If $\alpha$ has the Rohlin property, then $K_i(A)$, $K_i(A)_n$, and ${}_nK_i(A)$ are CT for all $i = 0, 1$, $n \in \mathbb{N}$.*

*Proof.* We may and do assume that $A$ is purely infinite because it suffices to prove the statement for $(A \otimes \mathcal{O}_\infty, \alpha \otimes \mathrm{id}_{\mathcal{O}_\infty})$. Since the Rohlin property is inherited to the restriction of $\alpha$ to any subgroup of $G$, we may and do assume that $G$ is a $p$-group for some prime number $p$ thanks to Lemma 3.2,(2). Therefore, to show that some $G$-module is CT, we may apply Lemma 3.2,(3).

First, we show that $K_1(A)$ is CT. Since Theorem 2.10 implies that $\hat{H}^0(G, K_1(A)) = \{0\}$, it suffices to show $H^1(G, K_1(A)) = \{0\}$ thanks to Lemma 3.2,(3). Let $c$ be a cocycle in $Z^1(G, K_1(A))$, which is a map $c : G \to K_1(A)$ satisfying

$$gc(h) - c(gh) + c(g) = 0, \quad \forall g, h \in G.$$

For each $c(g)$, we choose a unitary $u_g \in U(A)$ representing the class $c(g) \in K_1(A)$. Since the $K_1(A)$-classes of $u_g$ and $u_{gh}\alpha_g(u_h^*)$ are the same, we can choose a homotopy $\{v_{g,h}^{(t)}\}_{t\in[0,1]}$ in $U(A)$ such that $v_{g,h}^{(0)} = u_{gh}\alpha_g(u_h^*)$ and $v_{g,h}^{(1)} = u_g$ as $A$ is purely infinite. Let $\{e_g\}_{g\in G}\subset A_\infty$ be the Rohlin projections. We take a representing sequence $\{e_g(n)\}_{n=\in\mathbb{N}}$ of $e_g$ such that for each $n\in\mathbb{N}$, $\{e(n)_g\}_{g\in G}$ is a partition of unity consisting of projections. We set

$$u(n)=\sum_{h\in G}e_h(n)u_h^*.$$

$$w_g^{(t)}(n)=\sum_{h\in G}e_{gh}(n)v_{g,h}^{(t)}.$$

Note that $\{u(n)\}_{n\in\mathbb{N}}$ and $\{w_g^{(t)}(n)\}_{n\in\mathbb{N}}$ give rise to unitaries in $A^\infty$, say $u$ and $w_g^{(t)}$ respectively, such that $u^*\alpha_g^\infty(u)=w_g^{(0)}$ and $w_g^{(1)}=u_g$. Moreover, we have the following norm estimate

$$||w_g^{(s)}(n)-w_g^{(t)}(n)||\leq\sum_{h\in G}||v_{g,h}^{(s)}-v_{g,h}^{(t)}||.$$

We take $0=t_0<t_1<t_2<\cdots<t_m=1$ such that

$$||v_{g,h}^{(t_i)}-v_{g,h}^{(t_{i-1})}||<\frac{1}{2|G|},\quad\forall g,h\in G,\ 1\leq\forall i\leq m.$$

We choose $n\in\mathbb{N}$ such that the distance from $u(n)$ to $U(A)$ and that from $w_g^{(t_i)}(n)$ to $U(A)$ are less than $1/2$, and $||u(n)^*\alpha_g(u(n))-w_g^{(0)}(n)||<1/2$ for all $0\leq i\leq m$ and $g\in G$. Then, one can easily conclude that $u(n)$ and $w_g^{(t_i)}(n)$ are invertible and

$$g[u(n)]-[u(n)]=[w_g^{(0)}(n)]=[w_g^{(t_1)}(n)]=[w_g^{(t_2)}(n)]=\cdots=[w_g^{(1)}(n)]=c(g),$$

which shows that $c$ is a coboundary. Therefore, $H^1(G,K_1(A))$ is trivial and $K_1(A)$ is CT. Applying the same argument to $(A\otimes\mathcal{P}_\infty,\alpha\otimes\mathrm{id}_{\mathcal{P}_\infty})$, we also conclude that $K_0(A)$ is CT.

The same argument applied to $A\otimes\mathcal{O}_{n+1}$ shows that $K_i(A;\mathbb{Z}_n)$ is CT for all $i=0,1$, $n\in\mathbb{Z}$. Thus, the long exact sequence for the Tate cohomology groups derived by the Künneth sequence (2.1) implies

$$\hat{H}^m(G,{}_nK_{1-i}(A))\cong\hat{H}^{m+1}(G,K_i(A)_n),\quad i=0,1,\ \forall m\in\mathbb{Z},\ \forall n\in\mathbb{N}.$$

Therefore, to prove the statement it suffices to show

$$\hat{H}^0(G,K_i(A)_n)=\hat{H}^1(G,K_i(A)_n)=\{0\},\quad i=0,1,\ \forall n\in\mathbb{N}$$

thanks to Lemma 3.2,(3).

Since $H^0(G,M)=M^G$ for every $G$-module $M$, the long exact sequence of the cohomology groups derived by the Künneth sequence (2.1) implies that the following sequence is exact:

$$0\to(K_i(A)_n)^G\to K_i(A;\mathbb{Z}_n)^G\to({}_nK_{1-i}(A))^G\to H^1(G,K_i(A)_n)\to 0,$$

where we use the fact $H^1(G,K_i(A;\mathbb{Z}_n))=\{0\}$. Let $\iota$ be the inclusion map $\iota:A^\alpha\to A$. Then, Theorem 2.10 implies that the two maps

$$K_i(\iota;\mathbb{Z}_n):K_i(A^\alpha,\mathbb{Z}_n)\to K_i(A;\mathbb{Z}_n)^G,$$

$${}_nK_{1-i}(\iota):{}_nK_{1-i}(A^\alpha)\to({}_nK_{1-i}(A))^G$$

are surjective isomorphisms. On the other hand, the Künneth sequence (2.1) applied to $A^\alpha$ implies the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_i(A^\alpha)_n & \longrightarrow & K_i(A^\alpha;\mathbb{Z}_n) & \longrightarrow & {}_nK_{1-i}(A^\alpha) & \longrightarrow & 0 \\ & & \big\downarrow {\scriptstyle K_i(\iota)_n} & & \big\downarrow {\scriptstyle K_i(\iota;\mathbb{Z}_n)} & & \big\downarrow {\scriptstyle {}_nK_{1-i}(\iota)} & & \\ 0 & \longrightarrow & (K_i(A)_n)^G & \longrightarrow & K_i(A;\mathbb{Z}_n)^G & \longrightarrow & ({}_nK_{1-i}(A))^G & & \end{array}.$$

Therefore, we conclude that

$$\hat{H}^1(G, K_i(A)_n) = H^1(G, K_i(A)_n) = \{0\},$$

and $K_i(\iota)_n$ is an isomorphism of $K_i(A^\alpha)_n$ onto $(K_i(A)_n)^G$. Using this fact and Theorem 2.10 again, one can show that for any $x \in (K_i(A)_n)^G$, there exists $y \in K_i(A)_n$ such that $x = Ny$, where $N \in \mathbb{Z}G$ is the norm element. Thus, $\hat{H}^0(G, K_i(A)_n) = \{0\}$, which finishes the proof. □

We now consider the case where $G$ is $\mathbb{Z}_n$, the cyclic group of order $n$. We denote by $\sigma$ the canonical generator of $G$. Let $M$ be a $G$-module, and we regard $N$ and $\sigma$ as elements of $\mathrm{End}(M)$. It is well-known that every finite cyclic group has the periodic Tate cohomology of period 2 [6, pp. 58], and we get

$$\hat{H}^{2n+1}(G, M) = M^G/NM,$$

$$\hat{H}^{2n}(G, M) = \mathrm{Ker}(N)/(1-\sigma)M.$$

Therefore, when $M$ is CT, we get the following exact sequence.

$$0 \to M^G \to M \xrightarrow{1-\sigma} M \xrightarrow{N} M^G \to 0. \tag{3.1}$$

Recall that an abelian group $\Gamma$ is said to be *uniquely $n$-divisible* if multiplying by $n$ is a group automorphism of $\Gamma$. If $M$ is a trivial $\mathbb{Z}_n$-module that is CT, the above exact sequence implies that $M$ is uniquely $n$-divisible.

As an application of Theorem 3.3, we can obtain, without appealing to Theorem 4.2, the following structure theorem of finite group actions with the Rohlin property acting on the $K$-groups trivially (cf. [25, 17, 18]).

**Theorem 3.4.** *Let $\alpha$ be an action of a finite group $G$ on a unital Kirchberg $A$ in the UCT class $\mathcal{N}$. Assume that $\alpha$ has the Rohlin property and acts on $K_i(A)$ $(i = 0, 1)$ trivially. Then, $(A, \alpha)$ is conjugate to $(A \otimes \mathbb{M}_{|G|^\infty}, \mathrm{id}_A \otimes \mu^G)$.*

*Proof.* First, we treat the case where $G$ is the cyclic group of order $n$, and denote by $\sigma$ the canonical generator of $G$. Then, the exact sequence (3.1) implies that $K_i(A)$ is uniquely $n$-divisible for $i = 0, 1$. Thus, Kichberg-Phillips's classification theorem implies $A \cong A \otimes \mathbb{M}_{n^\infty}$. We show that $\alpha_g$ is approximately inner for all $g \in G$, which will show the statement thanks to Theorem 2.9. Indeed, let $\mathrm{Aut}(A)_{\mathrm{tr}}$ be the subgroup of $\mathrm{Aut}(A)$ consisting of those automorphisms acting on $K_i(A)$ $(i = 0, 1)$ trivially. As in [29], we define a homomorphism $\eta$ by

$$\eta : \mathrm{Aut}(A)_{\mathrm{tr}} \ni \beta \mapsto 1 - KK(\beta) \in \mathrm{Ext}(K_0(A), K_1(A)) \oplus \mathrm{Ext}(K_1(A), K_0(A)).$$

Then, we get $n\eta(\alpha_\sigma) = \eta(\alpha_\sigma^n) = 0$. On the other hand, as $K_0(A)$ and $K_1(A)$ are uniquely $n$-divisible, so are $\mathrm{Ext}(K_0(A), K_1(A))$ and $\mathrm{Ext}(K_1(A), K_0(A))$, which implies $\eta(\alpha_\sigma) = 0$. This shows $KK(\alpha_\sigma) = KK(\mathrm{id}_A)$ and so, $\alpha_\sigma$ is approximately inner thanks to Kichberg-Phillips's classification theorem again.

Now, we consider the general case. Applying the above argument to the cyclic subgroup generated by $g$ for each $g \in G \setminus \{e\}$, one can show that $K_0(A)$ and $K_1(A)$ are uniquely $|G|$-divisible and $\alpha_g$ is approximately inner for every $g \in G$. Thus, we get the statement in the same way as above. □

If Kichberg-Phillips's classification theorem is replaced with Lin's classification theorem [30, 31], essentially the same argument works for a unital simple nuclear separable $C^*$-algebra of tracial topological rank zero in the UCT class (see [30, 31] for the definition).

**Theorem 3.5.** *Let $\alpha$ be an action of a finite group $G$ on a unital simple nuclear separable $C^*$-algebra $A$ of tracial topological rank zero in the UCT class $\mathcal{N}$. Assume that $\alpha$ has the Rohlin property and acts on $K_i(A)$ $(i = 0, 1)$ trivially. Then, $(A, \alpha)$ is conjugate to $(A \otimes \mathbb{M}_{|G|^\infty}, \mathrm{id}_A \otimes \mu^G)$.*

Theorem 3.3 also implies some no-go theorem.

**Theorem 3.6.** *Let $A$ be a simple unital $C^*$-algebra such that either $K_0(A)$ or $K_1(A)$ is isomorphic to $\mathbb{Z}$. Then, there is no non-trivial finite group action with the Rohlin property on $A$. In particular, there is no non-trivial finite group action with the Rohlin property on $\mathcal{O}_\infty$.*

*Proof.* Note that it suffices to show the statement for a cyclic group action by passing to a non-zero cyclic subgroup from a general finite group. Indeed, setting $M = \mathbb{Z}$ in (3.1), one can easily deduce a contradiction. □

We devote the rest of this section to structure analysis of $G$-modules satisfying the property in the conclusion of Theorem 3.3.

**Lemma 3.7.** *Let $G$ be a finite group and $M$ be a CT $G$-module. Then, the following conditions are equivalent.*

(1) *$nM$ is CT for all $n \in \mathbb{N}$.*
(2) *${}_nM$ is CT for all $n \in \mathbb{N}$.*
(3) *$M_n$ is CT for all $n \in \mathbb{N}$.*

*Proof.* The statement follows from the two exact sequences below:

$$0 \to nM \to M \to M_n \to 0,$$

$$0 \to {}_nM \to M \xrightarrow{n} nM \to 0.$$

□

Since the author could not find an appropriate term for $G$-modules satisfying the above conditions in the literature, we introduce the following definition:

**Definition 3.8.** Let $G$ be a finite group. A $G$-module $M$ is said to be *completely cohomologically trivial* (abbreviated as CCT in the sequel) if $M$ is CT and for all $n \in \mathbb{N}$ one of (actually all) $nM$, ${}_nM$, and $M_n$ is CT.

**Lemma 3.9.** *Let $G$ be a finite group.*

(1) *If $M$ is a CCT $G$-module and $\Gamma$ is an abelian group, then $M \otimes_{\mathbb{Z}} \Gamma$ and $\mathrm{Tor}(M, \Gamma)$ are CT, where $\Gamma$ is regarded as a trivial $G$-module.*
(2) *The class of CCT $G$-modules is closed under forming an inductive limit and a direct sum.*
(3) *Every relatively projective $G$-module is CCT. In particular, $\mathbb{Z}G$-free modules, $\mathbb{Z}G$-projective modules, and induced $G$-modules are CCT.*

*Proof.* (1) and (2) follows from Lemma 3.2 and the fact that the functor $H_*(G, \cdot) = \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, \cdot)$ commutes with inductive limits. (3) follows from the fact that every induced $G$-module is CCT and a direct summand of a CCT module is CCT again. $\square$

**Lemma 3.10.** *Let $G$ be a finite group and $M$ be a $G$-module.*

(1) *When $M$ is torsion-free, $M$ is CCT if and only if $M$ is CT.*
(2) *If $M$ is CCT, so are $M_{\mathrm{tor}}$ and $M/M_{\mathrm{tor}}$.*
(3) *If $M$ is CCT, $nM$, ${}_nM$, and $M_n$ are CCT for all $n \in \mathbb{N}$.*

*Proof.* (1) This is trivial because $M$ is isomorphic to $nM$ for all $n \in \mathbb{N}$.

(2) Assume that $M$ is CCT. Lemma 3.9,(1) implies that $M_{\mathrm{tor}} = \mathrm{Tor}(M, \mathbb{Q}/\mathbb{Z})$ is CT, and so $M/M_{\mathrm{tor}}$ is CT as well. Therefore, $M/M_{\mathrm{tor}}$ is CCT thanks to (1). For a natural number $n$, ${}_n(M_{\mathrm{tor}}) = {}_nM$ is CT, which shows that $M_{\mathrm{tor}}$ is CCT.

(3) This follows from $m(nM) = mnM$, $(M_n)_m = M_{(m,n)}$, and ${}_m({}_nM) = {}_{(m,n)}M$. $\square$

Let $M$ and $M'$ be $G$-modules. Then, $\mathrm{Hom}(M, M')$ has $G$-module structure with the diagonal action, that is

$$(g \cdot f)(m) = gf(g^{-1}m), \quad f \in \mathrm{Hom}(M, M'),\ g \in G,\ m \in M.$$

**Lemma 3.11.** *Let $G$ be a finite group.*

(1) *Let $M$, $M'$ be $G$-modules. If either $M$ or $M'$ is relatively projective, so is $\mathrm{Hom}(M, M')$. In particular, $\mathrm{Hom}(M, M')$ is CCT.*
(2) *Let*
$$0 \to M' \to M \to M'' \to 0$$
*be an exact sequence of $G$-modules that splits as an exact sequence of abelian groups. If $M'$ or $M''$ is relatively projective, the sequence splits as an exact sequence of $G$-modules.*

*Proof.* (1) This follows from the fact that if either $M$ or $M'$ is an induced $G$-module, so is $\mathrm{Hom}(M, M')$, which can be shown easily.

(2) Let $\varphi : M'' \to M$ be a splitting map. For $g \in G$, we set

$$c(g) = g \cdot \varphi - \varphi \in \mathrm{Hom}(M'', M'),$$

which gives a 1-cocycle in $Z^1(G, \mathrm{Hom}(M'', M'))$. Thanks to (1), $\mathrm{Hom}(M'', M')$ is relatively projective, and so $H^1(G, \mathrm{Hom}(M'', M')) = \{0\}$. Therefore, there exists

$\psi \in \mathrm{Hom}(M'', M')$ such that $g \cdot \varphi - \varphi = g \cdot \psi - \psi$ holds for all $g \in G$. Thus, $\varphi - \psi$ is a splitting map commuting with the $G$-action. □

Recall that an abelian torsion group $\Gamma$ is said to be *bounded* if $n\Gamma = \{0\}$ holds for some natural number $n$. It is well-known that every bounded group is a direct sum of cyclic groups (see, for example, [16, Theorem 12]).

**Lemma 3.12.** *Let $G$ be a finite group and $M$ be a CCT $G$-module.*

(1) *If $G$ is a $p$-group for a prime number $p$ and $M$ is a bounded $p$-group, then $M$ is induced.*
(2) *If $M$ is a bounded $p$-group for a prime number $p$, then $M$ is relatively projective.*
(3) *If $M$ is a torsion group, $M$ is an ascending union of relatively projective modules.*

*Proof.* (1) Let $m$ be the smallest integer such that $p^m M = \{0\}$ holds. Note that $p^{m-1}M$ is a $\mathbb{F}_p G$-module, where $\mathbb{F}_p$ is the finite field $\mathbb{Z}/p\mathbb{Z}$. Thus, if $m = 1$, we are done thanks to Lemma 3.2,(4), and so we assume $m > 1$. Since $p^{m-1}M$ is a CT $\mathbb{F}_p G$-module, Lemma 3.2,(4) implies that there exists a basis $\{x_{g,i}\}_{g\in G,\, i\in I}$ of $p^{m-1}M$ over $\mathbb{F}_p$ such that $gx_{h,i} = x_{gh,i}$ for all $g, h \in G$, $i \in I$. For each $i \in I$, we choose $y_{e,i} \in M$ satisfying $x_{e,i} = p^{m-1}y_{e,i}$ and set $y_{g,i} = gy_{e,i}$. Let $M'$ be the submodule of $M$ generated by $\{y_{g,i}\}_{g\in G,\, i\in I}$.

We claim that $M'$ is induced, more precisely,

$$M' = \bigoplus_{i\in I}\bigoplus_{g\in G}\langle y_{g,i}\rangle.$$

Indeed, assume that

$$\sum_{g\in G,\, i\in I_o} a_{g,i}y_{g,i} = 0 \tag{3.2}$$

holds, where $I_0$ is a finite subset of $I$ and $0 \le a_{g,i} < p^m$ is an integer. Multiplying the both sides of (3.2) by $p^{m-1}$, we get $a_{g,i} \equiv 0 \mod p$. Let $b_{g,i} = a_{g,i}/p$. Then, multiplying the both sides of (3.2) by $p^{m-2}$, we get $b_{g,i} \equiv 0 \mod p$. Repeating this procedure, we finally get $a_{g,i} = 0$, which shows the claim.

Next we claim that $M'$ is a pure subgroup of $M$ (see [16, Chapter II] for the definition). To prove this claim, it suffices to show $M' \cap p^k M = p^k M'$ for $0 < k < m$. Assume that

$$\sum_{g\in G,\, i\in I_0} c_{g,i}y_{g,i} = p^k z \tag{3.3}$$

holds for $z \in M$, where $I_0$ and $c_{g,i}$ are as before. Multiplying the both sides of (3.3) by $p^{m-k-1}$, we get

$$\sum_{g\in G,\, i\in I_0} c_{g,i}p^{m-1-k}y_{g,i} \in p^{m-1}M,$$

which shows that $c_{g,i}$ is a multiple of $p^k$. Therefore, there exists $z' \in M'$ such that $p^k z' = p^k z$ and the claim holds. Thanks to [16, Lemma 7], $M'$ is a direct summand of $M$. Moreover, Lemma 3.11,(2) implies that there exists a $G$-submodule $M'' \subset M$ such that $M = M' \oplus M''$. By construction, we get $p^{m-1}M'' = \{0\}$. Being a direct summand of a CCT $G$-module, $M''$ is CCT as well. Therefore, we get the statement by induction of $m$.

(2) We choose a $p$-Sylow subgroup $H$ of $G$. Then, (1) implies that $\mathrm{Res}^G_H M$ is an induced $H$-module, where $\mathrm{Res}^G_H M$ stands for the restriction of $M$ to $H$. Thus, $\mathrm{Ind}^G_H \mathrm{Res}^G_H M$ is an induced $G$-module, where $\mathrm{Res}^G_H M'$ stands for the $G$-module induced from an $H$-module $M'$. On the other hand, the following $G$-module isomorphism is well-known [6, pp.67]:

$$\mathrm{Ind}^G_H \mathrm{Res}^G_H M \cong \mathbb{Z}G/H \otimes_{\mathbb{Z}} M,$$

where $G$ acts on $\mathbb{Z}G/H \otimes_{\mathbb{Z}} M$ diagonally. Let $N$ be the norm element in $\mathbb{Z}G$. We claim that

$$P = \frac{1}{|G|}(N \otimes \mathrm{id}_M) \in \mathrm{End}_G(\mathbb{Z}G/H \otimes_{\mathbb{Z}} M)$$

is well defined (whose precise sense will be given soon). Indeed, we choose a transversal $\{g_i\}_{i=1}^{[G:H]}$ of $G/H$. Then,

$$(N \otimes \mathrm{id}_M)(gH \otimes x) = |H| \sum_{i=1}^{[G:H]} g_i H \otimes x, \quad g \in G,\ x \in M.$$

Note that since $[G:H]$ is prime to $p$ and $M$ is a direct sum of cyclic $p$-groups, $M$ is uniquely $[G:H]$-divisible. Thus, $P$ given by the following formula makes sense:

$$P(gH \otimes x) = \frac{1}{[G:H]} \sum_{i=1}^{[G:H]} g_i H \otimes x.$$

$P$ is clearly the projection onto a submodule isomorphic to $M$, and so $M$ is relatively projective.

(3) For each prime $p$, we denote by $M(p)$ the $p$-primary component of $M$. Then, $M$ is the direct sum of $M(p)$'s and $M(p)$ is CCT. Since $M(p)$ is the ascending union of $\{{}_{p^n}M\}_{n\in\mathbb{N}}$, the statement follows from (2). ∎

**Lemma 3.13.** *Let $G$ be a finite group.*

1. *If $M$ is a finitely generated CCT $G$-module, then $M$ is relatively projective.*
2. *If $M$ is a CCT $G$-module and $M/M_{\mathrm{tor}}$ is $\mathbb{Z}$-free, then $M$ is an ascending union of relatively projective $G$-modules.*

*Proof.* (1) Thanks to Lemma 3.11,(2), if $M$ is finitely generated CCT $G$-module, $M$ is isomorphic to the direct sum of $M_{\mathrm{tor}}$ and $M/M_{\mathrm{tor}}$ as $G$-modules. Lemma 3.12,(2) implies that $M_{\mathrm{tor}}$ is relatively projective, and Lemma 3.10,(2) with Lemma 3.2,(1) implies that $M/M_{\mathrm{tor}}$ is $\mathbb{Z}G$-projective. Therefore, $M$ is relatively projective.

(2) Thanks to Lemma 3.11,(2) and Lemma 3.2,(2), if $M$ satisfies the assumption, $M$ is isomorphic to the direct sum of $M_{\mathrm{tor}}$ and $M/M_{\mathrm{tor}}$ as $G$-modules. Thus, Lemma 3.12,(3) shows that $M$ is an ascending union of relatively projective modules. ∎

**Example 3.14.** Let $p$ be a prime number, $G$ be the cyclic group $\mathbb{Z}_p$ of order $p$, and $M$ be a finitely generated CCT $G$-module. Then, Theorem 3.2, Lemma 3.12,(1), and

Lemma 3.13,(1) imply that $M$ has the following decomposition (each component is allowed to be zero):

$$M = M_f \oplus M(p) \oplus M(q_1) \oplus M(q_2) \oplus \cdots \oplus M(q_k),$$

where $M_f$ is $\mathbb{Z}G$-projective, $p, q_1, q_2, \cdots, q_k$ are mutually distinct prime numbers, and $M(p)$ (respectively $M(q_i)$) is the $p$-component (respectively $q_i$-component) of $M_{\mathrm{tor}}$. $M(p)$ must be an induced module though there is no restriction for $M(q_i)$ (Theorem 3.2,(5)). If $p$ is less than 23, every projective $\mathbb{Z}G$-module is $\mathbb{Z}G$-free; however, it is not the case in general. The structure of projective $\mathbb{Z}G$-modules is determined by the class group of $\mathbb{Z}[e^{2\pi i/p}]$ [40, pp.157]. When $p = 2$, $M$ has the following form (each component is allowed to be zero):

$$M = (\mathbb{Z}^n \oplus \mathbb{Z}^n) \oplus (M_0(2) \oplus M_0(2)) \oplus M_+ \oplus M_-,$$

where $M_0(2)$ is a 2-group, and $M_+$ and $M_-$ are odd torsion groups. $G$ acts on $\mathbb{Z}^n \oplus \mathbb{Z}^n$ and $M_0(p) \oplus M_0(p)$ by flip of the components, and $G$ acts on $M_+$ and $M_-$ as multiplying by 1 and $-1$ respectively.

## 4. The classification theorems

In this section, we show that finite group actions with the Rohlin property are completely classified by their actions on the $K$-groups for two "classifiable classes" of $C^*$-algebras.

**Lemma 4.1.** *Let $\alpha$ be an action of a finite group $G$ on a unital simple $C^*$-algebra $A$. If $\alpha$ has the Rohlin property, for every prime number $p$, $m \in \mathbb{N}$, and $i = 0, 1$, there exists a splitting map*

$$s^i_{p^m} : {}_{p^m}K_{1-i}(A) \to K_i(A; \mathbb{Z}_{p^m})$$

*for the Künneth sequence (2.1) commuting with the $G$-action and satisfying (2.2), (2.3), and (2.4).*

*Proof.* We choose a system of splitting maps $t^i_{p^m} : {}_{p^m}K_{1-i}(A) \to K_i(A; \mathbb{Z}_{p^m})$ satisfying (2.2), (2.3), and (2.4). For $g \in G$, we set

$$c^i_{p^m}(g) = g \cdot t^i_{p^m} - t^i_{p^m} \in \mathrm{Hom}({}_{p^m}K_{1-i}(A), K_i(A)_{p^m}).$$

where $g \cdot t^i_{p^m}$ means $K_i(\alpha_g : \mathbb{Z}_{p^m}) \cdot t^i_{p^m} \cdot {}_{p^m}K_{1-i}(\alpha_g)^{-1}$ as in the previous section. Then, $c^i_{p^m}$ is a 1-cocycle in $Z^1(G, \mathrm{Hom}({}_{p^m}K_{1-i}(A), K_i(A)_{p^m}))$ satisfying

$$c^i_{p^m}(g) \cdot \iota = p' \cdot c^i_{p^{m-1}}(g), \quad m > 1,$$

$$c^i_{p^{m-1}}(g) \cdot p'' = \pi \cdot c^i_{p^m}(g), \quad m > 1,$$

where we use the notation in Subsection 2.2. To prove the statement, it suffices to show that there exists a family of maps $r^i_{p^m} \in \mathrm{Hom}({}_{p^m}K_{1-i}(A), K_i(A)_{p^m})$ satisfying the following equations for all $g \in G$.

$$c^i_{p^m}(g) = g \cdot r^i_{p^m} - r^i_{p^m}, \quad m \geq 1 \tag{4.1}$$

$$r^i_{p^m} \cdot \iota = p' \cdot r^i_{p^{m-1}}, \quad m > 1, \tag{4.2}$$

$$r^i_{p^{m-1}} \cdot p'' = \pi \cdot r^i_{p^m}, \quad m > 1. \tag{4.3}$$

Indeed, if there exists such a system, $s^i_{p^m} = t^i_{p^m} - r^i_{p^m}$ gives the desired map in the statement.

Thanks to Theorem 3.3, Lemma 3.10,(3), Lemma 3.12,(2), and Lemma 3.11,(2), the cohomology group $H^1(G, \mathrm{Hom}({}_{p^m}K_{1-i}(A), K_i(A)_{p^m}))$ vanishes. Therefore, we can choose $r^i_p \in \mathrm{Hom}({}_pK_{1-i}(A), K_i(A)_p)$ satisfying (4.1) for $m = 1$. Assume that we have maps $r^i_{p^m} \in \mathrm{Hom}({}_{p^m}K_{1-i}(A), K_i(A)_{p^m})$ for $m = 1, 2, \cdots, l$ satisfying (4.1), (4.2), and (4.3) for $m = 1, 2, \cdots, l$.

We claim that there exists a homomorphism $r : {}_{p^{l+1}}K_{1-i}(A) \to K_i(A)_{p^{l+1}}$, such that the following two equations hold:

$$r \cdot \iota = p' \cdot r^i_{p^l}, \tag{4.4}$$

$$r^i_{p^l} \cdot p'' = \pi \cdot r. \tag{4.5}$$

Indeed, being a bounded torsion group, ${}_{p^{l+1}}K_{1-i}(A)$ is a direct sum of cyclic groups [16, Theorem 12]. Moreover, the image of $p' \cdot r^i_{p^l}$ is $p$-divisible in $K_i(A)_{p^{l+1}}$. Therefore, the homomorphism $p' \cdot r^i_{p^l} : {}_{p^l}K_{1-i}(A) \to K_i(A)_{p^{l+1}}$ extends to a homomorphism from ${}_{p^{l+1}}K_{1-i}(A)$ to $K_i(A)_{p^{l+1}}$, say $r'$. Let $\Gamma = {}_{p^{l+1}}K_{1-i}(A)/{}_{p^l}K_{1-i}(A)$ and $q : {}_{p^{l+1}}K_{1-i}(A) \to \Gamma$ be the quotient map. Then, $\Gamma$ is a direct sum of cyclic groups isomorphic to $\mathbb{Z}_p$. As we have

$$(\pi \cdot r' - r^i_{p^l} \cdot p'') \cdot \iota = \pi \cdot p' \cdot r^i_{p^l} - r^i_{p^l} \cdot p'' \cdot \iota = 0,$$

there exists a homomorphism $r'' : \Gamma \to K_i(A)_{p^l}$ such that $\pi \cdot r' - r^i_{p^l} \cdot p'' = r'' \cdot q$. On the other hand,

$$p' \cdot (\pi \cdot r' - r^i_{p^l} \cdot p'') = pr' - p' \cdot r^i_{p^l} \cdot p'' = r' \cdot \iota \cdot p'' - p' \cdot r^i_{p^l} \cdot p'' = 0,$$

which implies $p' \cdot r'' = 0$. Therefore, there exists a homomorphism $r''' : \Gamma \to K_i(A)_{p^{l+1}}$ such that $r'' = \pi \cdot r'''$. Thus, $r = r' - r''' \cdot q$ satisfies the condition of the claim.

We set

$$c(g) = c^i_{p^{l+1}}(g) - (g \cdot r - r).$$

Then, it is a routine work to show that $c(g) \cdot \iota = 0$ and $\pi \cdot c(g) = 0$. Therefore, there exists a 1-cocycle $c' \in Z^1(G, \mathrm{Hom}(\Gamma, p^l(K_i(A)_{p^{l+1}})))$ such that $c(g) = c'(g) \cdot q$. Thanks to Lemma 3.11,(1) again, $c'$ is a coboundary, and so there exists a homomorphism $f : \Gamma \to p^l(K_i(A)_{p^{l+1}})$ such that $c'(g) = g \cdot f - f$. (4.4) and (4.5) show that $r^i_{p^{l+1}} = r + f \cdot q$ satisfies (4.1),(4.2), and (4.3) for $m = l + 1$. Therefore, we get the statement by induction. □

**Theorem 4.2.** *Let $\alpha$, $\beta$ be actions of a finite group $G$ on a unital Kirchberg algebra $A$ in the UCT class $\mathcal{N}$. If $\alpha$ and $\beta$ have the Rohlin property and $K_i(\alpha_g) = K_i(\beta_g)$ for all $i = 0, 1$, $g \in G$, then there exists $\theta \in \mathrm{Aut}(A)$ such that $\theta$ acts on $K_i(A)$ $(i = 0, 1)$ trivially and $\alpha_g = \theta \cdot \beta_g \cdot \theta^{-1}$ holds for all $g \in G$.*

*Proof.* We show that there exists an automorphism $\varphi \in \mathrm{Aut}(A)$ acting on $K_i(A)$ $(i = 0, 1)$ trivially such that $\varphi^{-1} \cdot \alpha_g \cdot \varphi$ and $\beta_g$ are approximately unitarily equivalent for all $g \in G$. Then, the statement will follow from Theorem 2.9.

For $G$-module structure of $K_i(A)$ and $K_i(A;\mathbb{Z}_{p^m})$, we adopt that coming from $\alpha$ in the sequel, and $K_i(\beta_g;\mathbb{Z}_{p^m})$ will be always explicitly written. Using Lemma 4.1, we choose a family of splitting maps

$$s^i_{p^m} : {}_{p^m}K_{1-i}(A) \to K_i(A;\mathbb{Z}_{p^m})$$

for the Künneth sequence (2.1) commuting with the $G$-action (coming from $\alpha$) and satisfying (2.2), (2.3), and (2.4). For $g \in G$, we set

$$c^i_{p^m}(g) = K_i(\beta_g;\mathbb{Z}_{p^m}) \cdot s^i_{p^m} \cdot {}_{p^m}K_{1-i}(\beta_g)^{-1} - s^i_{p^m}.$$

The same argument as in the proof of Lemma 4.1 implies that there exists a family of homomorphisms $r^i_{p^m} \in \mathrm{Hom}({}_{p^m}K_{1-i}(A), K_i(A)_{p^m})$ satisfying (4.1), (4.2), and (4.3). Let $\Phi$ be the homomorphism given in Lemma 2.4 for the case $B = A$. Then we get

$$\Phi(\underline{K}(\alpha_g)) = (K_0(\alpha_g), K_1(\alpha_g), 0),$$

$$\begin{aligned}
\Phi(\underline{K}(\beta_g)) &= (K_0(\alpha_g), K_1(\alpha_g), c^i_{p^m}(g) \cdot {}_{p^m}K_{1-i}(\beta_g)) \\
&= (K_0(\alpha_g), K_1(\alpha_g), (g \cdot r^i_{p^m} - r^i_{p^m}) \cdot {}_{p^m}K_{1-i}(\alpha_g)) \\
&= (K_0(\alpha_g), K_1(\alpha_g), (K_i(\alpha_g)_{p^m} \cdot r^i_{p^m} - r^i_{p^m} \cdot {}_{p^m}K_{1-i}(\alpha_g)).
\end{aligned}$$

Let $\phi = (\mathrm{id}_{K_0(A)}, \mathrm{id}_{K_1(A)}, r^i_{p^m}) \in \mathrm{Aut}^s_\Lambda(\underline{K}(A))$. Then, $\phi^{-1} = (\mathrm{id}_{K_0(A)}, \mathrm{id}_{K_1(A)}, -r^i_{p^m})$ and so

$$\Phi(\underline{K}(\beta_g)) = \phi^{-1} \cdot \Phi(\underline{K}(\alpha_g)) \cdot \phi.$$

Thanks to Theorem 2.7, there exists an automorphism $\varphi \in \mathrm{Aut}(A)$ satisfying $\phi = \Phi(\underline{K}(\varphi))$, and so $\beta_g$ and $\varphi^{-1} \cdot \alpha_g \cdot \varphi$ are approximately unitarily equivalent for all $g \in G$. Thus, we get the statement. $\square$

Since $\phi$ in the above proof acts on $K_0(A)$ trivially, essentially the same argument works for a unital simple nuclear separable $C^*$-algebra of tracial topological rank zero in the UCT class if Lin's classification theorem [30, 31] is used.

**Theorem 4.3.** *Let $\alpha$, $\beta$ be actions of a finite group $G$ on a unital simple nuclear separable $C^*$-algebra $A$ of tracial topological rank zero in the UCT class $\mathcal{N}$. If $\alpha$ and $\beta$ have the Rohlin property and $K_i(\alpha_g) = K_i(\beta_g)$ for all $i = 0, 1$, $g \in G$, then there exists $\theta \in \mathrm{Aut}(A)$ such that $\theta$ acts on $K_i(A)$ $(i = 0, 1)$ trivially and $\alpha_g = \theta \cdot \beta_g \cdot \theta^{-1}$ holds for all $g \in G$.*

## 5. Model actions

The purpose of this section is to construct models for actions with the Rohlin property. For a given pair of countable CCT $G$-modules in the class $\mathcal{RM}_c(G)$ introduced below, we show that there exists a $G$-action with the Rohlin property on a Kirchberg algebra in the Cuntz standard form in $\mathcal{N}$, whose $K$-groups are isomorphic to the prescribed ones as $G$-modules. Though the class $\mathcal{RM}_c(G)$ could be considerably vast, it is still an open question, at least to the author, whether $\mathcal{RM}_c(G)$ exhausts all the countable CCT $G$-modules (see Remark 5.9 below).

**Definition 5.1.** Let $G$ be a finite group. We denote by $\mathcal{RM}_c(G)$ the smallest category of countable $G$-modules satisfying

(1) $\mathcal{RM}_c(G)$ contains every countable induced $G$-module.
(2) The direct sum of any two $G$-modules in $\mathcal{RM}_c(G)$ is again in $\mathcal{RM}_c(G)$.
(3) The inductive limit of any countably many $G$-modules in $\mathcal{RM}_c(G)$ is again in $\mathcal{RM}_c(G)$.

*Remark* 5.2. One can immediately see from the definition that every element of $\mathcal{RM}_c(G)$ is CCT and that $\mathcal{RM}_c(G)$ is closed under countable direct sums. Moreover, $\mathcal{RM}_c(G)$ contains every countable relatively projective $G$-module. Indeed, if $M$ is a countable relatively projective $G$-module, one can show that there exists a countable induced module $M'$ containing $M$ such that there exists a projection $\pi : M' \to M$. For $n \in \mathbb{N}$, set $M_n = M'$ and $\varphi_n = \pi$ that is considered to be a homomorphism from $M_n$ to $M_{n+1}$. Then, $M$ is isomorphic to the inductive limit of the system $\{M_n, \varphi_n\}_{n\in\mathbb{N}}$. In particular, $\mathcal{RM}_c(G)$ contains every finitely generated CCT $G$-module thanks to Lemma 3.13,(1).

**Lemma 5.3.** *Let $A$ and $B$ be unital Kirchberg algebras in the Cuntz standard form in the UCT class $\mathcal{N}$. Let $\alpha$ and $\beta$ be actions of a finite group $G$ on $A$ and $B$ respectively with the Rohlin property. If $\phi^i$ $(i = 0, 1)$ are homomorphisms from $K_i(A)$ to $K_i(B)$ satisfying $\phi^i \cdot K_i(\alpha_g) = K_i(\beta_g) \cdot \phi^i$ for all $g \in G$, $i = 0, 1$, then there exists a unital $*$-homomorphism $\varphi$ from $A$ to $B$ such that $\varphi \cdot \alpha_g = \beta_g \cdot \varphi$ and $K_i(\varphi) = \phi^i$ hold for all $g \in G$, $i = 0, 1$.*

*Proof.* Applying Lemma 4.1 to $(A, \alpha)$ and $(B, \beta)$, we fix systems of the splitting maps for the Künneth sequence (2.1) commuting with the $G$-actions and satisfying (2.2), (2.3), and (2.4). Let $\phi = (\phi^0, \phi^1, 0) \in \mathrm{Hom}^s_\Lambda(\underline{K}(A), \underline{K}(B))$. Then, thanks to Theorem 2.7, there exists a unital $*$-homomorphism $\varphi_0 : A \to B$ such that $\Phi(\underline{K}(\varphi_0)) = \phi$. As $\phi \cdot \Phi(\underline{K}(\alpha_g)) = \Phi(\underline{K}(\beta_g)) \cdot \phi$ holds for all $g \in G$, $\varphi_0 \cdot \alpha_g$ and $\beta_g \cdot \varphi_0$ are approximately unitarily equivalent thanks to Theorem 2.7.

The rest of the proof is essentially the same as that of [21, Theorem 3.5]. Using the Rohlin property of $\beta$, one can show as in [21, Lemma 3.3] that if $\psi : A \to B$ is a unital $*$-homomorphism such that $\beta_g \cdot \psi$ and $\psi \cdot \alpha_g$ are approximately unitarily equivalent, for any $\epsilon > 0$ and any finite set $F \subset A$, there exists a unitary $u \in B$ such that

$$||\mathrm{Ad}(u^*) \cdot \beta_g \cdot \mathrm{Ad}(u) \cdot \psi(x) - \psi \cdot \alpha_g(x)|| < \epsilon, \quad \forall g \in G,\ \forall x \in F,$$
$$||[u, \psi(x)]|| < \epsilon + \sup_{g\in G} ||\beta_g \cdot \psi \cdot \alpha_{g^{-1}}(x) - \psi(x)||, \quad \forall x \in F.$$

We choose an increase sequence of finite subsets $\{F_n\}_{n=1}^\infty$ of $A$ whose union is dense in $A$. We further assume that $\alpha_g(F_n) = F_n$ holds for all $g \in G$ and all $n \in \mathbb{N}$. Then, repeated use of the above statement implies that there exist a sequence of unital $*$-homomorphisms $\{\varphi_n\}_{n=1}^\infty$ from $A$ to $B$ and a sequence of unitaries $\{u_n\}_{n=1}^\infty$ in $B$ such that

$$\varphi_n = \mathrm{Ad}(u_n) \cdot \varphi_{n-1}, \quad n \in \mathbb{N},$$
$$||\beta_g \cdot \varphi_n(x) - \varphi_n \cdot \alpha_g(x)|| < \frac{1}{2^n}, \quad g \in G,\ x \in F_n,\ n \in \mathbb{N},$$

$$||[u_n, \varphi_{n-1}(x)]|| < \frac{3}{2^n}, \quad x \in F_{n-1},\ n > 1.$$

For $n < m$, we get

$$||\varphi_{m+1}(x) - \varphi_m(x)|| = ||[u_{m+1}, \varphi_m(x)]|| < \frac{3}{2^{m+1}}, \quad x \in F_n.$$

Since $\cup_{n\in\mathbb{N}} F_n$ is dense in $A$, $\{\varphi_n\}_{n\in\mathbb{N}}$ converges to a unital $*$-homomorphism, which satisfies the desired property. $\square$

**Lemma 5.4.** *For every finite group $G$, there exists a $G$-action $\nu^G$ with the Rohlin property on a unital Kirchberg algebra $D$ in the UCT class $\mathcal{N}$ such that $K_0(D) \cong \mathbb{Z}G$ and $K_1(D) = \{0\}$ as $G$-modules.*

*Proof.* We choose a partition of unity $\{p_g\}_{g\in G}$ of $\mathcal{O}_\infty$ consisting of non-zero projections such that $[p_g] = 0$ for $g \neq e$ and $[p_g] = [1]$ for $g = e$ in $K_0(\mathcal{O}_\infty)$. Note that $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$, $K_0(\mathcal{O}_\infty) = \{0\}$, and that $[1]$ is the generator of $K_0(\mathcal{O}_\infty)$. We set

$$D_n = C(G, \bigotimes_{k=1}^{n} \mathcal{O}_\infty)$$

and introduce a $G$-action $\nu_n^G$ on $D_n$ by the left translation

$$\nu_{n,g}^G(f)(h) = f(g^{-1}h), \quad g, h \in G.$$

Then, $K_1(D_n)$ is trivial and $K_0(D_n)$ is identified with $\mathbb{Z}G$ as a $G$-module. We introduce a unital connecting map $\varphi_n : D_n \to D_{n+1}$ by

$$\varphi_n(f)(g) = \sum_{h\in G} f(gh^{-1}) \otimes p_h,$$

which are compatible with the above actions $\nu_n^G$. Note that on the level of the $K_0$-groups, $\varphi_n$ induces an identity map. Let $(D, \nu^G)$ be the inductive limit of the inductive system $\{(D_n, \nu_n^G), \varphi_n\}_{n\in\mathbb{N}}$. Then, the pair $(D, \nu^G)$ satisfies the desired property. $\square$

**Theorem 5.5.** *Let $G$ be a finite group and $M_i$ $(i = 0, 1)$ be a pair of $G$-modules in $\mathcal{RM}_c(G)$. Then, there exists a unique (up to conjugacy) $G$-action $\alpha$ with the Rohlin property on a Kirchberg algebra $A$ in the Cuntz standard form in the UCT class $\mathcal{N}$ such that $K_i(A)$ and $M_i$ $(i = 0, 1)$ are isomorphic as $G$-modules.*

*Proof.* Uniqueness up to conjugacy follows from Theorem 4.3.

Let $\mathcal{RM}_K(G)$ be the category of countable $G$-modules $M$ such that there exists a $G$-action $\alpha$ on a $C^*$-algebra $A$ with the properties described in the statement of the theorem such that $K_0(A) \cong M$ and $K_1(A) = \{0\}$ as $G$-modules. We call $(A, \alpha)$ a realization of $M \in \mathcal{RM}_K(G)$. We prove $\mathcal{RM}_c(G) \subset \mathcal{RM}_K(G)$ by showing that $\mathcal{RM}_K(G)$ satisfies the three properties in the definition of $\mathcal{RM}_c(G)$.

Let $M = \mathbb{Z}G \otimes_{\mathbb{Z}} M'$ be a countable induce module, where $M'$ is a trivial $G$-module. We choose a Kirchberg algebra $A$ in the Cuntz standard form in the UCT class $\mathcal{N}$ such that $K_0(A) \cong M'$ and $K_1(A) = \{0\}$. Let $D$ and $\nu^G$ be as in Lemma 5.4. Then, $(A \otimes D, \mathrm{id}_A \otimes \nu^G)$ gives a desired realization, which shows $M \in \mathcal{RM}_K(G)$.

Let $M, M' \in \mathcal{RM}_K(G)$ and $(A, \alpha)$ and $(B, \beta)$ be their realizations. We choose projections $p \in A^\alpha$ and $q \in B^\beta$ such that $p \neq 0, 1$, $q \neq 0, 1$, $[p] = [1-p] = 0$ in $K_0(A)$ and $[q] = [1-q] = 0$ in $K_0(B)$. Using Lemma 5.3, we can construct four unital $*$-homomorphisms

$$\varphi_{11} : A \to pAp, \quad \varphi_{21} : A \to (1-q)B(1-q),$$
$$\varphi_{12} : B \to (1-p)A(1-p), \quad \varphi_{22} : B \to qBq,$$

such that they intertwine the $G$-actions, $K_0(\varphi_{11})$ and $K_0(\varphi_{22})$ are isomorphisms, and $K_0(\varphi_{12}) = 0$, $K_0(\varphi_{21}) = 0$. We denote by $\varphi$ the unital endomorphism of $A \oplus B$ obtained by assembling these 4 homomorphisms. We set $C_n = A \oplus B$, $\varphi_n = \varphi$, and $\gamma_{n,g} = \alpha_g \oplus \beta_g$ for $g \in G$. Then, the inductive limit of the system $\{(C_n, \gamma_n), \varphi_n\}_{n \in \mathbb{N}}$ gives a realization of $M \oplus M'$, which shows $M \oplus M' \in \mathcal{RM}_K(G)$.

Let $M$ be the inductive limit of a system of $G$-modules $\{M_n, \phi_n\}_{n \in \mathbb{N}}$ in $\mathcal{RM}_K(G)$. We choose a realization $(A_n, \alpha_n)$ of $M_n$ for each $n \in \mathbb{N}$. Lemma 5.3 implies that there exists a unital $*$ homomorphism $\varphi_n : A_n \to A_{n+1}$ that intertwines the $G$-actions and satisfies $K_0(\varphi_n) = \phi_n$. Then, the inductive limit of the system $\{(A_n, \alpha_n), \varphi_n\}_{n \in \mathbb{N}}$ gives a realization of $M$, which shows $M \in \mathcal{RM}_K(G)$. Therefore, we conclude $\mathcal{RM}_c(G) \subset \mathcal{RM}_K(G)$.

Now, we finish the proof. Let $M_i \in \mathcal{RM}_K(G)$ $(i = 0, 1)$. We choose a realization $(A_i, \alpha_i)$ of $M_i$. Applying the same argument as above to

$$(A_0 \oplus (A_1 \otimes \mathcal{P}_\infty), \alpha_0 \oplus (\alpha_1 \otimes \mathrm{id}_{\mathcal{P}_\infty})),$$

we can construct an appropriate inductive system whose inductive limit gives a desired action in the statement of the theorem. □

**Corollary 5.6.** *Let $G$ be a finite group, $M_i$ $(i = 0, 1)$ be a pair of $G$-modules in $\mathcal{RM}_c(G)$, and $a \in M_0^G$. Then, there exists a unique (up to conjugacy) $G$-action $\alpha$ with the Rohlin property on a unital Kirchberg algebra $A$ in the UCT class $\mathcal{N}$ such that there exist $G$-module isomorphisms from $K_i(A)$ onto $M_i$ $(i = 0, 1)$ that take $[1_A] \in K_0(A)$ to $a$.*

*Proof.* Let $A, \alpha$ be as in Theorem 5.5. Thanks to Theorem 2.10, there exists a non-zero projection $e \in A^\alpha$ such that $[e]$ in $K_0(A)$ corresponds to $a \in M_0^G$. Let $\alpha^e$ be the restriction of $\alpha$ to $eAe$. Then, $\alpha^e$ has the desired property. □

**Corollary 5.7.** *Let $\alpha$ be an action of a finite group $G$ with the Rohlin property on a unital Kirchberg algebra $A$ in the UCT class $\mathcal{N}$. If $K_i(A) \in \mathcal{RM}_c(G)$ $(i = 0, 1)$, the crossed product $A \rtimes_\alpha G$ is a Kirchberg algebra in the UCT class $\mathcal{N}$.*

*Proof.* $A \rtimes_\alpha G$ being a Kirchberg algebra follows from [23, 28]. Let $D$, $\nu^G$ be as in Lemma 5.4. Then, it is clear by construction that $D \rtimes_{\nu^G} G \in \mathcal{N}$. Thus, the statement follows from Theorem 4.2 and the proof of Theorem 5.5. □

Before closing this section, we note the following criterion, which is an easy consequence of Lemma 3.13.

**Lemma 5.8.** *Let $G$ be a finite group. If $M$ is a countable CCT $G$-module such that $M/M_{\mathrm{tor}}$ is a countable inductive limit of countable $\mathbb{Z}G$-projective modules (this is the case if $M/M_{\mathrm{tor}}$ is $\mathbb{Z}$-free thanks to Lemma 3.2,(1)), then $M$ is in $\mathcal{RM}_c(G)$.*

*Remark* 5.9. It is tempting to conjecture that for every finite group $G$, very countable torsion-free CT $G$-module is an inductive limit of a system of countably many countable $\mathbb{Z}G$-projective modules. The affirmative solution of this conjecture would imply that every countable CCT $G$-module is in $\mathcal{RM}_c(G)$, meaning that the only $K$-theoretical constraint for the Rohlin property is the $K$-groups being countable CCT.

## 6. Cyclic group actions on the Cuntz algebras

We give a few consequences of the general analysis so far concerning finite cyclic group actions on the Cuntz algebras. In the sequel, for a cyclic group action $\alpha$, we abuse the notation and $\alpha_1$ is simply denote by $\alpha$. $\hat{\alpha}$ always means the dual action.

Before getting into specific arguments, we recall the notion of approximately representable actions introduced in [21].

**Definition 6.1.** Let $A$ be a unital $C^*$-algebra, and $\alpha$ be an action of a finite abelian group $G$ on $A$. $\alpha$ is said to be *approximately representable* if there exists a unitary representation $\{u(g)\}_{g\in G}$ of $G$ in $(A^\alpha)^\infty$ such that

$$\alpha_g(x) = u(g)xu(g)^*, \quad x \in A,\ g \in G.$$

As one can see from the next lemma [21, Lemma 3.8], approximate representability is the dual notion of the Rohlin property.

**Lemma 6.2.** *Let $A$ be a separable $C^*$-algebra, and $\alpha$ be an action of a finite abelian group $G$ on $A$. We denote by $\widehat{\alpha}$ the dual action of $\alpha$ on $A \rtimes_\alpha G$. Then,*

1. *$\alpha$ has the Rohlin property if and only if $\widehat{\alpha}$ is approximately representable.*
2. *$\alpha$ is approximately representable if and only if $\widehat{\alpha}$ has the Rohlin property.*

### 6.1. $\mathcal{O}_2$ case.

Let $p$ be a prime number and $q = p^m$, $m \in \mathbb{N}$. If $\alpha$ is an action of $\mathbb{Z}_q$ on $\mathcal{O}_2$, the $K$-groups of the crossed product $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_q$ are uniquely $p$-divisible and the following equation holds [21, Lemma 4.4, Remark 4.9]:

$$\sum_{i=0}^{q-1} KK(\hat{\alpha}^i) = 0.$$

In [21, Lemma 4.7], for any given two uniquely 2-divisible countable abelian groups $M_0$ and $M_1$, we constructed an outer approximately representable action $\alpha$ of $\mathbb{Z}_2$ on $\mathcal{O}_2$ such that $K_i(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2) \cong M_i$ $(i = 0, 1)$. Now, we generalize this statement.

**Lemma 6.3.** *Let $p$ be a prime number and $q = p^m$, $m \in \mathbb{N}$. Then, every uniquely $p$-divisible $\mathbb{Z}_q$-module is relatively projective.*

*Proof.* Let $M$ be a uniquely $p$-divisible $\mathbb{Z}_q$-module. We set $M' = \mathbb{Z}\mathbb{Z}_q \otimes_{\mathbb{Z}} M$, which is a $\mathbb{Z}_q$-module by the diagonal action. Note that $M'$ is an induce module no matter what kind of an action on $M$ we start with. As $M'$ is uniquely $p$-divisible, $P \in \mathrm{End}_{\mathbb{Z}_q}(M')$ defined by the following formula makes sense:

$$P = \frac{1}{q}(N \otimes \mathrm{id}_M),$$

where $N$ is the norm element. Clearly, $P$ is a projection onto a submodule isomorphic to $M$, which shows that $M$ is relatively projective. $\square$

**Theorem 6.4.** *Let $p$ be a prime number and $q = p^m$, $m \in \mathbb{N}$. For any pair $M_i$ $(i = 0, 1)$ of uniquely $p$-divisible countable $\hat{\mathbb{Z}}_q$-modules such that $M_i^{\hat{\mathbb{Z}}_q} = \{0\}$, there exists an outer approximately representable $\mathbb{Z}_q$-action $\alpha$ on $\mathcal{O}_2$ such that $K_i(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_q) \cong M_i$ $(i = 0, 1)$ as $\hat{\mathbb{Z}}_q$-modules, where the $\hat{\mathbb{Z}}_q$-module structure of $K_i(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_q)$ is given by the dual action $\hat{\alpha}$.*

*Proof.* Since $\hat{\mathbb{Z}}_q$ is isomorphic to $\mathbb{Z}_q$, Lemma 6.3 implies that $M_i \in \mathcal{RM}_c(G)$. Thus, we can apply Theorem 5.5 to $M_i$, and there exists a $\hat{\mathbb{Z}}_q$-action $\beta$ with the Rohlin property on a unital Kirchberg algebra $A$ in the UCT class such that $K_i(A) \cong M_i$ $(i = 0, 1)$ as $\hat{\mathbb{Z}}_q$-modules. Moreover, Corollary 5.7 shows that $A \rtimes_\beta \mathbb{Z}_q$ is in the UCT class $\mathcal{N}$. Thanks to Theorem 2.10, $K_i(A \rtimes_\beta \mathbb{Z}_q)$ is isomorphic to $M_i^{\hat{\mathbb{Z}}_q} = \{0\}$ because $A \rtimes_\beta \mathbb{Z}_q$ is stably isomorphic to the fixed point algebra $A^\alpha$. Thus, Kichberg-Phillips' classification theorem implies that $A \rtimes_\beta \mathbb{Z}_q$ is isomorphic to $\mathcal{O}_2$. It is a routine work to show that $\alpha = \hat{\beta}$ has the desired property via the duality theorem of crossed products [44]. $\square$

6.2. **Quasi-free actions.** A quasi-free action of a group on the Cuntz algebra $\mathcal{O}_n$ is an action induced by a unitary transformation of the canonical generators of $\mathcal{O}_n$. Let $p$ be a prime number and $q = p^m$, $m \in \mathbb{N}$. In [21, Theorem 5.9], we showed that if $n \not\equiv 1 \mod p$, every quasi-free $\mathbb{Z}_q$-action on $\mathcal{O}_n$ is approximately representable. Here, we give a result complementing this statement.

We set $\zeta_q = e^{2\pi i/q}$. Up to conjugacy, every quasi-free $\mathbb{Z}_q$-action has the following form: there exists a partition

$$\{1, 2, \cdots, n-1, n\} = \bigcup_{j=0}^{q-1} I_j$$

such that

$$\alpha(S_i) = \zeta_q^j S_i, \quad i \in I_j.$$

We denote by $|I_j|$ the cardinality of $I_j$.

**Theorem 6.5.** *Let $p$ be a prime number and $q = p^m$, $m \in \mathbb{N}$. We assume that $\alpha$ is a quasi-free action of $\mathbb{Z}_q$ on the Cuntz algebra $\mathcal{O}_n$ of the above form, and that $n = p^k l + 1$ with $l$ prime to $p$ and $k > 0$. Then, unless*

$$|I_0| - 1 \equiv |I_1| \equiv |I_2| \equiv \cdots \equiv |I_{q-1}| \equiv 0 \mod p^k,$$

*$\alpha$ is not approximately representable.*

*Proof.* We may and do assume that $\alpha$ is not trivial. Let $\lambda$ be the permutation matrix of order $q$, i.e.

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Then, it is known that the crossed product $\mathcal{O}_n \rtimes_\alpha \mathbb{Z}_q$ is stably isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_M$ [10, 32], where the matrix $M$ is given by

$$M = \sum_{i=0}^{q-1} |I_i| \lambda^i.$$

Let $A = \mathcal{O}_n \otimes \mathcal{O}_{p^k+1}$ and $\beta = \alpha \otimes \mathrm{id}_{\mathcal{O}_{p^k+1}}$. Then, the Künneth formula [1, 42] implies

$$K_0(A) \cong K_1(A) \cong \mathbb{Z}_{p^k}.$$

Using the crossed product decomposition of $\mathbb{K} \otimes \mathcal{O}_M \otimes \mathcal{O}_{p^k+1}$ and the Pimsner-Voiculescu exact sequence [1, 36], we get the following:

$$K_0(A \rtimes_\beta \mathbb{Z}_q) \cong \mathrm{Ker}(I - M : \mathbb{Z}_{p^k}^q \to \mathbb{Z}_{p^k}^q),$$

$$K_1(A \rtimes_\beta \mathbb{Z}_q) \cong \mathrm{Coker}(I - M : \mathbb{Z}_{p^k}^q \to \mathbb{Z}_{p^k}^q).$$

Now, assume that $\alpha$ is approximately representable. Then, so is $\beta$, and the dual action $\hat{\beta}$ has the Rohlin property due to Lemma 6.2. Therefore, Theorem 3.3 implies that $K_i(A \rtimes_\beta \mathbb{Z}_q)$ $(i = 0, 1)$ are CCT $\hat{\mathbb{Z}}_q$-modules. On the other hand, as $K_i(A \rtimes_\beta \mathbb{Z}_q)$ and $\hat{\mathbb{Z}}_q$ are $p$-groups, Lemma 3.12,(1) implies that $K_i(A \rtimes_\beta \mathbb{Z}_q)$ $(i = 0, 1)$ are induced $\hat{\mathbb{Z}}_q$-modules. Moreover, Theorem 2.10 shows that the fixed point modules $K_i(A \rtimes_\beta \mathbb{Z}_q)^{\hat{\mathbb{Z}}_q}$ $(i = 0, 1)$ are isomorphic to $K_i(A)$ $(i = 0, 1)$. This means that $K_i(A \rtimes_\beta \mathbb{Z}_q)$ $(i = 0, 1)$ are isomorphic to $\mathbb{Z}_{p^k}^q$, and so $I - M$ acts on $\mathbb{Z}_{p^k}^q$ trivially. Thus, we get

$$|I_0| - 1 \equiv |I_1| \equiv |I_2| \equiv \cdots \equiv |I_{q-1}| \equiv 0 \mod p^k,$$

which finishes the proof. □

It is an interesting question to ask what really happens in the exceptional case in Theorem 6.5. The first test case is the $\mathbb{Z}_2$-action $\alpha$ on $\mathcal{O}_3$ given by

$$\alpha(S_1) = S_1, \ \alpha(S_2) = -S_2, \ \alpha(S_3) = -S_3.$$

There is no $K$-theoretical obstruction for this action to be approximately representable.

6.3. **A splitting problem.** For a $C^*$-algebra $A$, one may ask the following question: Does the exact sequence

$$0 \to \overline{\mathrm{Inn}(A)} \to \mathrm{Aut}(A) \to \mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)} \to 0 \tag{6.1}$$

split? Note that if $A$ is a "classifiable $C^*$-algebra", the structure of $\mathrm{Aut}(A)/\overline{\mathrm{Inn}(A)}$ is well-understood in terms of the $K$-groups in a generalized sense (see e.g. Theorem 2.7). Quite surprisingly, there are only few cases where the answer to this fundamental question is known except for some trivial cases. When $A$ is $\mathcal{O}_{n+1}^{\mathrm{st}} := \mathbb{M}_n(\mathcal{O}_{n+1})$, the sequence (6.1) takes the following form (Theorem 2.7, [37, Section 8]):

$$0 \to \overline{\mathrm{Inn}(\mathcal{O}_{n+1}^{\mathrm{st}})} \to \mathrm{Aut}(\mathcal{O}_{n+1}^{\mathrm{st}}) \to \mathbb{Z}_n^\times \to 0. \tag{6.2}$$

**Theorem 6.6.** *Let $A$ be a Kirchberg algebra in the Cuntz standard form in the UCT class such that $K_i(A)$ is a finite group and $K_{1-i}(A) = \{0\}$ for either $i = 0$ or $i = 1$. If $K_i(A)$ is a CCT $\mathrm{Aut}(K_i(A))$-module with respect to the natural action, then the sequence (6.1) splits. In particular, if $n$ is a prime number, (6.2) splits.*

*Proof.* The first half follows from Theorem 2.7, Lemma 3.13,(1), and Theorem 5.5. When $n$ is a prime number, $\mathbb{Z}_n^\times$ is isomorphic to $\mathbb{Z}_{n-1}$. Thus, Lemma 3.2,(5) implies that $\mathbb{Z}_n$ is a CCT $\mathbb{Z}_n^\times$-module. □

Note that thanks to Lemma 3.2, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is a CCT $\mathrm{Aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ ($\cong \mathfrak{S}_3$)-module, which gives another example where Theorem 6.6 is applicable.

*Remark* 6.7. One can push Theorem 6.6 a little further. Let $p$ be an odd prime and $n = p^2$. Then, the exact sequence (6.2) splits, though $n = 4$ case is still open. We give a sketch of the construction of the splitting map below.

Let $\zeta_p = e^{2\pi i/p}$. We define an action $\alpha$ of $\mathbb{Z}_p$ on $\mathcal{O}_{p+1}$ by

$$\alpha(S_j) = \zeta_p^j S_j, \quad 1 \le j \le p+1.$$

Then, the $K$-groups of the crossed product can be computed by using [32], and we get

$$K_0(\mathcal{O}_{p+1} \rtimes_\alpha \mathbb{Z}_p) \cong \mathbb{Z}_{p^2},$$
$$K_1(\mathcal{O}_{p+1} \rtimes_\alpha \mathbb{Z}_p) = \{0\}.$$

The dual action $\hat{\alpha}$ acts on $K_0(\mathcal{O}_{p+1} \rtimes_\alpha \mathbb{Z}_p)$ as multiplication by $1-p$.

Next we set

$$B = \bigotimes_{k=1}^{\infty} \mathbb{M}_{p^2}(\mathcal{O}_\infty),$$

and introduce a $\mathbb{Z}_{p-1}$-action $\beta$ on $B$. We take $a \in \mathbb{Z}$ whose order in $\mathbb{Z}_{p^2}^\times$ is $p-1$. It is well-known that $\mathbb{Z}_{p^2}^\times$ is a cyclic group of order $p(p-1)$, and such $a$ always exists. Note that the following equation holds:

$$1 + a + a^2 + \cdots + a^{p-2} \equiv 0 \mod p^2.$$

We choose a partition of unity $\{e_i\}_{i=0}^{p-2}$ in $\mathbb{M}_{p^2}(\mathcal{O}_\infty)$ consisting of non-zero projections such that if we identify $K_0(\mathbb{M}_{p^2}(\mathcal{O}_\infty))$ with $\mathbb{Z}$, the following holds:

$$[e_0] = p^2 - (a + a^2 + \cdots + a^{p-2}),$$

$$[e_i] = a^i, \quad i = 1, 2, \cdots, p-2.$$

We define a unitary $u \in \mathbb{M}_{p^2}(\mathcal{O}_\infty)$ by

$$u = \sum_{l=0}^{p-2} \zeta_{p-1}^l e_l,$$

and set

$$\beta = \bigotimes_{k=1}^{\infty} \mathrm{Ad}(u).$$

Let $A = \mathcal{O}_{p+1} \otimes B$. We introduce a $\mathbb{Z}_p \times \mathbb{Z}_{p-1}$-action $\gamma$ on $A$ by

$$\gamma_{(s,t)} = \alpha_s \otimes \beta_t, \quad (s,t) \in \mathbb{Z}_p \times \mathbb{Z}_{p-1}.$$

The $K$-groups of the crossed product $A \rtimes_\gamma (\mathbb{Z}_p \times \mathbb{Z}_{p-1})$ can be computed from the inductive limit of the system

$$\{K_i((\mathcal{O}_{p+1} \rtimes_\alpha \mathbb{Z}_p) \otimes ((\bigotimes_{k=1}^{n} \mathbb{M}_{2}(\mathcal{O}_\infty)) \rtimes_\beta \mathbb{Z}_{p-1}))\}_{n\in\mathbb{N}},$$

and we get

$$K_0(A \rtimes_\gamma (\mathbb{Z}_p \times \mathbb{Z}_{p-1})) \cong \mathbb{Z}_{p^2}, \quad [1] = 0,$$
$$K_1(A \rtimes_\gamma (\mathbb{Z}_p \times \mathbb{Z}_{p-1})) = \{0\}.$$

Therefore, Kirchberg-Phillips' classification theorem implies

$$A \rtimes_\gamma (\mathbb{Z}_p \times \mathbb{Z}_{p-1}) \cong \mathcal{O}_{p^2+1}^{\mathrm{st}}.$$

The dual action $\hat{\gamma}$ gives the desired splitting map.

## References


[1] Blackadar, B. *K-Theory for Operator Algebras.* Second Edition. Mathematical Sciences Research Institute Publications, 5. Springer-Verlag, New York, 1998.

[2] Blackadar, B.; Kumjian, A.; Rørdam, M. *Approximately central matrix units and the structure of noncommutative tori.* $K$-Theory **6** (1992), 267–284.

[3] Bödigheimer, C.-F. *Splitting the Kunneth sequence in K-theory.* Math. Ann. **242** (1979), 159–171.

[4] Bödigheimer, C.-F. *Splitting the Kunneth sequence in K-theory. II.* Math. Ann. **251** (1980), 249–252.

[5] Bratteli, O.; Kishimoto, A.; Rørdam, M.; Størmer, E. *The crossed product of a UHF algebra by a shift.* Ergodic Theory Dynam. Systems **13** (1993), 615–626.

[6] Brown, K. S. *Cohomology of Groups.* Graduate Texts in Mathematics, 87. Springer-Verlag, New York-Berlin, 1982.

[7] Connes, A. *Outer conjugacy classes of automorphisms of factors.* Ann. Sci. École Norm. Sup. (4) **8** (1975), 383–419.

[8] Cuntz, J. *Simple $C^*$-algebras generated by isometries.* Comm. Math. Phys. **57** (1977), 173–185.

[9] Cuntz, J. *K-theory for certain $C^*$-algebras.* Ann. of Math. (2) **113** (1981), 181–197.

[10] Cuntz, J.; Krieger, W. *A class of $C^*$-algebras and topological Markov chains.* Invent. Math. **56** (1980), 251–268.

[11] Dadarlat, M.; Eilers, S. *On the classification of nuclear $C^*$-algebras.* Proc. London Math. Soc. (3) **85** (2002), 168–210.

[12] Dadarlat, M.; Gong, Guihua *A classification result for approximately homogeneous $C^*$-algebras of real rank zero.* Geom. Funct. Anal. **7** (1997), 646–711.
[13] Dadarlat, M.; Loring, T. A. *A universal multicoefficient theorem for the Kasparov groups.* Duke Math. J. **84** (1996), 355–377.
[14] Elliott, G. A.; Gong, Guihua *On the classification of $C^*$-algebras of real rank zero. II.* Ann. of Math. (2) **144** (1996), 497–610.
[15] Evans, D. E.; Kishimoto, A. *Trace scaling automorphisms of certain stable AF algebras.* Hokkaido Math. J. **26** (1997), 211–224.
[16] Griffith, P. A. *Infinite Abelian Group Theory.* The University of Chicago Press, Chicago and London 1970.
[17] Herman, R. H.; Jones, V. F. R. *Period two automorphisms of* UHF *$C^*$-algebras.* J. Funct. Anal. **45** (1982), 169–176.
[18] Herman, R. H.; Jones, V. F. R. *Models of finite group actions.* Math. Scand. **52** (1983), 312–320.
[19] Izumi, M. *The Rohlin property for automorphisms of $C^*$-algebras.* Mathematical Physics in Mathematics and Physics (Siena, 2000), 191–206, Fields Inst. Commun., 30, Amer. Math. Soc., Providence, RI, 2001.
[20] Izumi, M. *Inclusions of simple $C^*$-algebras.* J. Reine Angew. Math. **547** (2002), 97-138.
[21] Izumi, M. *Finite group actions on $C^*$-algebras with the Rohlin property. I.* Preprint, Kyoto-Math 2002-04.
[22] Jones, V. F. R. *Actions of finite groups on the hyperfinite type* $\mathrm{II}_1$ *factor.* Mem. Amer. Math. Soc. **28** (1980), no. 237.
[23] Jeong, J. A.; Kodaka, K. ; Osaka, H. *Purely infinite simple $C^*$-crossed products. II.* Canad. Math. Bull. **39** (1996), 203–210.
[24] Kirchberg, E.; Phillips, N. C. *Embedding of exact $C^*$-algebras in the Cuntz algebra $\mathcal{O}_2$.* J. Reine Angew. Math. **525** (2000), 17–53.
[25] Kishimoto, A. *On the fixed point algebra of a UHF algebra under a periodic automorphism of product type.* Publ. Res. Inst. Math. Sci. **13** (1977/78), 777–791.
[26] Kishimoto, A. *Unbounded derivations in AT algebras.* J. Funct. Anal. **160** (1998), 270–311.
[27] Kishimoto, A. *Non-commutative shifts and crossed products.* preprint, 2000
[28] Kishimoto, A.; Kumjian, A. *Crossed products of Cuntz algebras by quasi-free automorphisms.* Operator algebras and their applications (Waterloo, ON, 1994/1995), pp. 173–192, Fields Inst. Commun., 13, Amer. Math. Soc., Providence, RI, 1997.
[29] Kishimoto, A.; Kumjian, A. *The Ext class of an approximately inner automorphism.* Trans. Amer. Math. Soc. **350** (1998), 4127–4148.
[30] Lin, Huaxin. *Classification of simple $C^*$-algebras of tracial topological rank zero.* preprint, 2000.
[31] Lin, Huaxin. *An Introduction to the Classification of Amenable $C^*$-Algebras.* World Scientific, 2001.
[32] Mann, M. H.; Raeburn, I.; Sutherland, C. E. *Representations of finite groups and Cuntz-Krieger algebras.* Bull. Austral. Math. Soc. **46** (1992), 225–243.
[33] Nakamura, H. *Aperiodic automorphisms of nuclear purely infinite simple $C^*$-algebras.* Ergodic Theory Dynam. Systems **20** (2000), 1749 –1765.
[34] Ocneanu, A. *Actions of discrete amenable groups on von Neumann algebras.* Lecture Notes in Mathematics, 1138. Springer-Verlag, Berlin-New York, 1985.
[35] Phillips, N. C. *A classification theorem for nuclear purely infinite simple $C^*$-algebras.* Doc. Math. **5** (2000), 49–114.
[36] Pimsner, M. V.; Voiculescu, D. *Exact sequences for $K$-groups and Ext-groups of certain cross-product $C^*$-algebras.* J. Operator Theory **4** (1980), 93–118.
[37] Rørdam, M. *Classification of inductive limits of Cuntz algebras.* J. Reine Angew. Math. **440** (1993), 175–200.
[38] Rørdam, M. *Classification of certain infinite simple $C^*$-algebras.* J. Funct. Anal. **131** (1995), 415–458.

[39] Rørdam, M.; Størmer, E. *Classification of Nuclear $C^*$-algebras. Entropy in Operator Algebras.* Operator Algebras and Non-commutative Geometry VII. Encyclopedia of Mathematical Sciences, Springer, 2001.

[40] Rosenberg, J. *Algebraic $K$-theory and its Applications.* Graduate Texts in Mathematics, 147. Springer-Verlag, New York, 1994.

[41] Rosenberg, J.; Schochet, C. *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized $K$-functor.* Duke Math. J. **55** (1987), 431–474.

[42] Schochet, C. *Topological methods for $C^*$-algebras. II. Geometry resolutions and the Künneth formula.* Pacific J. Math. **98** (1982), 443–458.

[43] Schochet, C. *Topological methods for $C^*$-algebras. IV. Mod $p$ homology.* Pacific J. Math. **114** (1984), 447–468.

[44] Takai, H. *On a duality for crossed products of $C^*$-algebras.* J. Funct. Anal. **19** (1975), 25–39.

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