ON A SUBFACTOR ANALOGUE OF THE SECOND COHOMOLOGY

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Dedicated to Professor Huzihiro Araki on his seventieth birthday

Abstract. The set of equivalence classes of Longo's Q-systems is shown to serve as a right subfactor analogue of the second cohomology. This "cohomology" is computed for several classes of subfactors, which tells us if a subfactor is uniquely determined up to inner conjugacy by the associated canonical endomorphism. As a byproduct of our analysis, two different groups of order 64 are shown to possess the same representation category as an abstract tensor category. Thus, this category has two permutation symmetries with non-isomorphic groups via the Doplicher-Roberts duality.

1. Introduction

The starting point of the present work is the simple question for an inclusion $M \supset N$ of factors: Does Longo's canonical endomorphism γ_N (as a sector) uniquely determine the subfactor N up to inner conjugacy ? In terms of bimodules, this can be rephrased as follows: Does the basic extension M_1 (as an M-M bimodule) uniquely determine the subfactor N up to inner conjugacy ? The answer is negative in general, and the obstruction for uniqueness is analyzed in the present article.

The notion of a Q-system due to R. Longo ([21]) was initially introduced to present an abstract characterization of the canonical endomorphism of a subfactor of finite index. In [22] it was employed as a tool for dealing with nets of observable algebras, and R. Longo and K.-H. Rehren introduced a new construction for a subfactor (now referred to as the Longo-Rehren inclusion) from a given finite system of endomorphisms of a type III factor. This notion was further translated into type II_1 language by T. Masuda to show that A. Ocneanu's asymptotic inclusion, or S. Popa's symmetric enveloping algebra ([26]), is essentially the same object as the Longo-Rehren inclusion (see [24]). A Q-system also played a crucial role in the first-named author's construction [15] of the Haagerup subfactor ([1]) from a Cuntz algebra endomorphism.

For fixed-point algebra subfactors the obstruction for the above-mentioned uniqueness question is closely related to the second cohomology obstruction [17, 28] for cocycle conjugacy of group actions. Therefore, one expects that systematic study for uniqueness for more general subfactors might reveal a new interplay between operator algebras and cohomological objects. We will see that this is indeed the case. In §2 the "second cohomology" $H^2(M \supset N)$ is defined as the set of equivalence classes of the Q-systems with a common endomorphism γ_N. This cohomology parameterizes

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the inner conjugacy classes of subfactors with the canonical endomorphism γ_N, and it will be determined for depth 2 subfactors (in §3), group-subgroup subfactors (in §5), and subfactors of index at most 4 (in §6).

Depth 2 subfactors give us Kac algebras, and for such a subfactor $H^2(M \supset N)$ is identified as the second cohomology of the corresponding Kac algebra. Hence, (as pointed out in [31, 34]) Hopf algebras cannot be recovered from the abstract tensor categories of representations without specifying Hilbert space realization. This phenomenon occurs because cocycle perturbation of the coproduct might change the Hopf algebra structure ([16, 34]), however, to the best of our knowledge, it has been unknown if it occurs within cocommutative Hopf algebras, i.e., group algebras. As a byproduct of our analysis the answer is given to this question in §4 based on A. Wassermann's observation on the correspondence between the second cohomology of a group dual and ergodic actions of a compact group ([32]). We indeed show that two different groups of order 64 possess the same representation category as an abstract tensor category. Consequently, via the Doplicher-Roberts duality ([6]) this category has at least two permutation symmetries providing non-isomorphic groups.

Our basic reference for subfactors is [8] (see [18] for the type III case), and we follow the notations in [14, §2] for sectors. The first-named author would like to thank S. Yamagami for stimulating discussions on the topics covered in §4.

2. Q-systems and second cohomology

Let $M \supset N$ be an inclusion of infinite factors of finite index with the canonical endomorphism $\gamma_N = \mathrm{Ad}(J_N J_M) : M \to N$, where J_M, J_N are modular conjugations of M, N respectively. Although γ_N depends on the choice of natural cones for M and N, the sector $[\gamma_N] \in \mathrm{Sect}(N, M)$ is uniquely determined by N. Any representative of $[\gamma_N]$ is also called the canonical endomorphism in the sequel. It is known that one can find isometries $S \in (\mathrm{Id}, \gamma_N)$ and $T \in (\mathrm{Id}|_N, \gamma_N|_N)$ satisfying

$$T^*S = \gamma(S)^*T = [M : N]_0^{-1/2}.$$

In order to get an abstract characterization of the canonical endomorphisms, R. Longo introduced the notion of Q-systems in [21].

Definition 2.1. A triple (γ, S, T) consisting of an endomorphism γ of M and isometries $S \in (\mathrm{Id}, \gamma), T \in (\gamma, \gamma^2)$ is called a *Q-system* if the following relations hold:

$$T^*S = \gamma(S^*)T = \frac{1}{d} \in \mathbb{R}_+, \tag{2.1}$$

$$TT^* = \gamma(T^*)T, \tag{2.2}$$

$$T^2 = \gamma(T)T. \tag{2.3}$$

Note that (2.2) and (2.3) are equivalent as long as (2.1) is assumed. Indeed, the implication (2.3) $\Rightarrow$ (2.2) was shown in [23]. On the other hand, (2.2) shows $T^*\gamma(T)\gamma(T^*) = T^*\gamma^2(T^*)\gamma(T)$. Another use of (2.2) shows that the left side here is equal to $TT^*\gamma(T^*)$. But, the right side is

$$T^*\gamma^2(T^*)\gamma(T) = \gamma(T^*)T^*\gamma(T) = \gamma(T^*)TT^* = TT^*T^*$$

due to $T \in (\gamma, \gamma^2)$ and (2.2). Therefore, we have $TT^*\gamma(T^*) = TT^*T^*$, showing (2.3).

Longo proved that every Q-system comes from an inclusion of factors.

Theorem 2.2 (Longo [21]). *For a given Q-system $\Xi = (\gamma, S, T)$ one can construct a subfactor N in such a way that γ is the canonical endomorphism for the inclusion $M \supset N$ and $T \in (\mathrm{Id}|_N, \gamma|_N)$. Moreover, N is given by*

$$N = \{x \in N;\ Tx = \gamma(x)T,\ Tx^* = \gamma(x^*)T\},$$

and $E_\Xi(\cdot) = T^*\gamma(\cdot)T$ gives a conditional expectation onto N.

We say that two Q-systems $\Xi_1 = (\gamma_1, S_1, T_1)$ and $\Xi_2 = (\gamma_2, S_2, T_2)$ are *equivalent* if there exists a unitary $u \in (\gamma_1, \gamma_2)$ satisfying

$$S_2 = uS_1, \quad T_2 = u\gamma_1(u)T_1u^*.$$

This is clearly an equivalence relation (note $\gamma_2(\cdot) = u\gamma_1(\cdot)u^*$). The two relevant conditional expectations are related by

$$E_{\Xi_2}(x) = T_2^*\gamma_2(x)T_2 = uT_1^*\gamma_1(u^*)\gamma_1(x)\gamma_1(u)T_1u^* = uE_{\Xi_1}(u^*xu)u^*$$

so that two equivalent systems give inner conjugate subfactors. On the other hand, it is easy to show that two inner conjugate subfactors give rise to equivalent Q-systems.

Definition 2.3. For an endomorphism γ, we denote by $Q(\gamma)$ the set of Q-systems of the form (γ, S, T) with $S^*T = 1/\sqrt{d(\gamma)}$. For an inclusion $M \supset N$, we define $H^2(M \supset N)$ to be the set of equivalence classes of the elements in $Q(\gamma_N)$.

When a subfactor P $(\subset M)$ satisfies $[\gamma_P] = [\gamma_N]$, i.e., $\gamma_P(\cdot) = u\gamma_N(\cdot)u^*$ for some unitary $u \in M$, we have $J_{u^*Pu} = u^*J_Pu$ and the commutativity $[u, J_MxJ_M] = 0$ $(x \in M)$ shows

$$\gamma_{u^*Pu}(x) = u^*J_PuJ_MxJ_Mu^*J_Pu = u^*J_PJ_MxJ_MJ_Pu = u^*\gamma_P(x)u = \gamma_N(x).$$

Consequently, there is a natural one-to-one correspondence between $H^2(M \supset N)$ and the inner conjugacy classes of subfactors P satisfying $[\gamma_N] = [\gamma_P]$ in $\mathrm{Sect}(M)$. The following elementary, but instructive, example justifies our notation $H^2(M \supset N)$: Let α be an outer action of a finite group G on M with the fixed-point algebra $N = M^\alpha$. Then, we have

$$[\gamma_N] = \bigoplus_{g \in G} [\alpha_g],$$

and hence $[\gamma_N] = [\gamma_P]$ holds if and only if P is the fixed-point algebra under inner perturbation of α (see [12]). Recall ([3]) that every α-cocycle is a coboundary since G is a finite group. Therefore, there is a one-to-one correspondence between $H^2(M \supset N)$ and the second cohomology group $H^2(G, \mathbb{T})$ ([17, 28]). This means that $[\gamma_N]$ fails to determine the inner conjugacy class of N when $H^2(G, \mathbb{T})$ is non-trivial.

The next theorem generalizes the well-known fact $\#H^2(G, \mathbb{T}) < \infty$ for a finite group G [29].

Theorem 2.4. *We have $\#H^2(M \supset N) < \infty$ for an inclusion $M \supset N$ of factors of finite index.*

Proof. Let us regard $Q(\gamma_N)$ as a compact subset in the finite-dimensional space $(\mathrm{Id},\gamma)\times(\gamma,\gamma^2)$, and it suffices to show the existence of a positive constant ε with the following property: two Q-systems Ξ_1,Ξ_2 as above are equivalent as long as $||S_1-S_2||<\varepsilon$ and $||T_1-T_2||<\varepsilon$. Note that $||T_1-T_2||<\varepsilon$ implies $||E_{\Xi_1}-E_{\Xi_2}||<2\varepsilon$ (the last part in Theorem 2.2). Thus, in the type II_1 setting ε small can be chosen to guarantee inner conjugacy, i.e., equivalence (see Proposition after Theorem 2 in [25]). Note $H^2(M\supset N)$ depends only on the paragroup of $M\supset N$, hence, to show $\#H^2(M\supset N)<\infty$ in the current case, we may replace $M\supset N$ by an inclusion of II_1 factors with the same paragroup due to T. Masuda's formulation [24] of Q-systems, and we are done. □

Remark 2.5. A "II_1 factor-free" proof is possible for irreducible inclusions $M\supset N$. Here is such a proof inspired by A. Ocneanu's argument for finiteness of the equivalence classes of $6j$-symbols for a given finite-dimensional fusion algebra.

Since we have $\dim(\mathrm{Id},\gamma_N)=1$, for given two Q-systems as above we may assume $S_1=S_2=S$ replacing Ξ_2 with an equivalent Q-system if necessary. We set $\delta T=T_1-T_2$, and $\approx$ will mean an equality up to the order of $||\delta T||^2$ in the sequel. The defining conditions for Q-systems imply

$$\delta T^*T_2+T_2^*\delta T\approx 0, \tag{2.4}$$

$$S^*\delta T=\gamma_N(S^*)\delta T=0, \tag{2.5}$$

$$\delta TT_2^*+T_2\delta T^*\approx\gamma_N(\delta T^*)T_2+\gamma_N(T_2^*)\delta T. \tag{2.6}$$

Let $a=dS^*E_{\Xi_2}(\delta T)\in(\gamma_N,\gamma_N)$ with $d=[M:N]_0^{1/2}$. Then, we get

$$\begin{aligned} a &= dS^*T_2^*\gamma_N(\delta T)T_2\approx dS^*(\delta TT_2^*+T_2\delta T^*-\delta T^*\gamma_N(T_2))T_2\\ &= \delta T^*T_2-dS^*\delta T^*T_2T_2=\delta T^*T_2-\langle\delta T^*T_2S,S\rangle\end{aligned}$$

thanks to (2.6), (2.5) together with the fact that $\delta T^*T_2\in(\gamma_N,\gamma_N)$ and $S^*\delta T^*T_2$ is proportional to S^*. Thus, we have

$$aS\approx\delta T^*T_2S-\langle\delta T^*T_2S,S\rangle S=0.$$

Thanks to (2.4), we have $a\approx-a^*$. With $u=e^{(a-a^*)/2}$, a unitary in (γ_N,γ_N), we have $uS\approx S+aS\approx S$ and

$$\begin{aligned} u\gamma_N(u)T_2u^* &\approx T_2+aT_2-T_2a+\gamma_N(a)T_2\\ &\approx T_2+\delta T^*T_2^2-T_2\delta T^*T_2+d\gamma_N(S^*E_{\Xi_2}(\delta T))T_2\\ &= T_2+\delta T^*T_2^2-T_2\delta T^*T_2+T_2^*\gamma_N(\delta T)T_2\\ &\approx T_2+\delta T^*T_2^2+(\delta TT_2^*-\delta T^*\gamma_N(T))T_2\\ &= T_2+\delta T=T_1,\end{aligned}$$

where (2.6) was used. Let $c\in\mathbb{T}$ satisfying $uS=cS$. Then, by setting $v=\overline{c}u$, we get $vS=S$ and $v\gamma_N(v)T_2v^*\approx T_1$. A little more careful examination of the above computation actually shows

$$||1-v||<C_1||T_1-T_2||\quad\text{and}\quad||T_1-v\gamma_N(v)T_2v^*||<C_2||T_1-T_2||^2$$

with constants C_1, C_2 (depending only on d). Therefore, by starting from T_2 satisfying $C_2||T_1 - T_2|| < 1$, we can choose converging unitaries $\{v_n\}_{n=1}^{\infty} \subset (\gamma_N, \gamma_N)$ satisfying

$$v_n S = S \quad \text{and} \quad ||v_n \gamma_N(v_n) T_2 v_n^* - T_1|| < C_2^{2^n - 1} ||T_1 - T_2||^{2^n},$$

showing the equivalence between Ξ_1 and Ξ_2.

3. The depth 2 case

When an irreducible inclusion $M \supset N$ of infinite factors (of index n) is of depth 2, N is realized as the fixed-point algebra under a Kac algebra action on M ([5], [30]). In this section, $H^2(M \supset N)$ is identified with the second cohomology of the corresponding Kac algebra (in the depth 2 case).

Definition 3.1. Let $\mathcal{A}$ be a Kac algebra with a coproduct δ. A unitary $\omega \in \mathcal{A} \otimes \mathcal{A}$ is called a *2-cocycle* (or simply a cocycle) if it satisfies the cocycle relation

$$(\delta \otimes \mathrm{Id})(\omega)(\omega \otimes 1) = (\mathrm{Id} \otimes \delta)(\omega)(1 \otimes \omega).$$

We say that $\omega_1, \omega_2 \in Z^2(\mathcal{A})$, the set of 2-cocycles for $\mathcal{A}$, are *equivalent* if

$$\omega_2 = \delta(\xi^*)\omega_1(\xi \otimes \xi) \text{ for some unitary } \xi \in \mathcal{A}.$$

The set of equivalence classes of 2-cocycles for $\mathcal{A}$ is denote by $H^2(\mathcal{A})$.

Let ρ be the inclusion map from N into M and $\overline{\rho}$ be the conjugate morphism from M into N. We fix two isometries $S_0 \in (\mathrm{Id}, \rho\overline{\rho})$, $T_0 \in (\mathrm{Id}, \overline{\rho}\rho)$ forming a Q-system $(\rho\overline{\rho}, S_0, T_0)$. We identify the C^*-algebra generated by $\mathcal{H} = (\rho, \rho\overline{\rho}\rho)$, an n-dimensional Hilbert space, with the Cuntz algebra $\mathcal{O}_n$. A unitary $u \in \mathcal{O}_n$ determines the endomorphism λ_u of $\mathcal{O}_n$ by $\lambda_u(S) = uS$, $S \in \mathcal{H}$. A Hilbert space $\mathcal{K} \subset \mathcal{O}_n$ with left support 1 (with an orthonormal basis $\{T_i\}_i$) gives us the endomorphism $\sigma_{\mathcal{K}}(\cdot) = \sum T_i \cdot T_i^*$ of $\mathcal{O}_n$. We identify $\mathcal{H}^m \mathcal{H}^{*n}$ with $\mathbb{B}(\mathcal{H}^{\otimes n}, \mathcal{H}^{\otimes m})$ and often use the tensor notation inside $\mathcal{O}_n$. For example, $x \otimes y$ means $x\sigma_{\mathcal{H}}(y)$ for $x, y \in \mathbb{B}(\mathcal{H})$. With this notation, we have $\lambda_F = \sigma_{\mathcal{H}}$ with the flip F (of the tensor components of $\mathcal{H} \otimes \mathcal{H}$).

We employ the Kac algebra action on $\mathcal{O}_n$ investigated in [4], [14], [21] as our model action. As pointed out in [21], $\rho\overline{\rho}$ globally preserves $\mathcal{O}_n$ and $\rho\overline{\rho}|_{\mathcal{O}_n}$ is given by λ_{VF}, where V is an irreducible multiplicative unitary ([2]). For the objects associated with V, we use the same notation as in [14, §4] in what follows. Here, the most important observation for us is $((\rho\overline{\rho})^m, (\rho\overline{\rho})^n) = (\lambda_{VF}^m, \lambda_{VF}^n)$ for each n, m, which enables us to work with $\mathcal{O}_n$ for determination of $H^2(M \supset N)$.

Note that S_0, T_0 were denoted by $\overline{R}, R$ instead in [14, p.439], and so we have

$$S_0 = \Lambda_\varphi(1) = \sqrt{n}\Lambda_{\hat{\varphi}}(\hat{e}), \quad T_0 = \sqrt{n}\Lambda_\varphi(e) = \Lambda_{\hat{\varphi}}(1).$$

$$\overline{\rho}_V(S_0) = S_0, \quad \rho_V(T_0) = T_0.$$

We introduce Kac algebra structure on $\widehat{\mathcal{A}}'$ (the commutant here is taken in $\mathbb{B}(\mathcal{H})$) from $\widehat{\mathcal{A}}$ through the conjugate linear isomorphism $\mathrm{Ad}\hat{J}$. For a $*$-subalgebra $\mathcal{B}$ of $\mathbb{B}(\mathcal{H})$ we denote by $E_{\mathcal{B}}$ the trace preserving conditional expectation from $\mathbb{B}(\mathcal{H})$ onto $\mathcal{B}$. Note that the Haar state φ of $\mathcal{A}$ (respectively $\hat{\varphi}$ of $\widehat{\mathcal{A}}$) is the restriction of the normalized trace of $\mathbb{B}(\mathcal{H})$. We denote by E_{ρ_V} and $E_{\overline{\rho}_V}$ the conditional expectations from $\mathcal{O}_n$ onto $\rho_V(\mathcal{O}_n)$ and $\overline{\rho}_V(\mathcal{O}_n)$ respectively given by

$$E_{\rho_V}(x) = T_0^*\lambda_{VF}(x)T_0 \quad \text{and} \quad E_{\overline{\rho}_V}(x) = S_0^*\lambda_{\hat{V}F}(x)S_0.$$

Lemma 3.2. *With the notations so far the following assertions hold:*

(1) $T_0VxV^*T_0 = \hat{\varphi}(x)1$ *for each* $x \in \widehat{\mathcal{A}}$,

(2) $E_{\widehat{\mathcal{A}}'}$ *is given by*

$$E_{\widehat{\mathcal{A}}'}(x) = \sum_{\pi\in\Pi}\sum_{a,b=1}^{d(\pi)} \frac{d(\pi)}{n}\,\hat{u}(\pi)_{ab}x\hat{u}(\pi)^*_{ab}.$$

(3) $(E_{\widehat{\mathcal{A}}'}\otimes \mathrm{Id})(\hat{\delta}(\widehat{\mathcal{A}})') = \widehat{\mathcal{A}}'\otimes\widehat{\mathcal{A}}'$.

(4) $V^*T_0 = UVT_0$.

(5) *The coproduct* $\hat{\delta}'$ *of* $\widehat{\mathcal{A}}'$ *satisfies*

$$V(1\otimes x)V^* = (U\otimes 1)\hat{\delta}'(x)(U\otimes 1) \in \widehat{\mathcal{A}}\otimes\widehat{\mathcal{A}}', \quad x\in\widehat{\mathcal{A}}',$$
$$\hat{\hat{V}}^*(1\otimes x)\hat{\hat{V}} = F\hat{\delta}'(x)F, \quad x\in\widehat{\mathcal{A}}'.$$

Proof. (1). This follows from [14, (4.1.6)] and the fact that $\hat{\varphi}$ is a trace.
(2). Thanks to [14, Corollary 4.4], we have $\mathbb{B}(\mathcal{H})\cap\overline{\rho}(\mathcal{O}_n) = \widehat{\mathcal{A}}'$, and so $E_{\widehat{\mathcal{A}}'}$ is given by the restriction of $E_{\overline{\rho}_V}$. Thus, using [14, Lemma 4.1, (4.1.6)], we get

$$\begin{aligned}
E_{\widehat{\mathcal{A}}'}(x) &= S_0^*\lambda_{\hat{V}F}(x)S_0 = \sigma_{\mathcal{H}}(S_0^*)VxV^*\sigma_{\mathcal{H}}(S_0)\\
&= \sum_{\pi,a,b,c}\hat{u}(\pi)_{ac}x\hat{u}(\pi)^*_{bc}\sigma_{\mathcal{H}}(S_0^*e(\pi)_{ab}S_0) = \sum_{\pi,a,b,c}\frac{d(\pi)}{n}\,\hat{u}(\pi)_{ac}x\hat{u}(\pi)^*_{ac},
\end{aligned}$$

where the last equality follows from [14, (4.1.11)].
(3). It suffices to check the commuting square condition for

$$\begin{array}{ccc}
\widehat{\mathcal{A}}'\otimes\mathbb{B}(\mathcal{H}) & \subset & \mathbb{B}(\mathcal{H})\otimes\mathbb{B}(\mathcal{H})\\
\cup & & \cup\\
\widehat{\mathcal{A}}'\otimes\widehat{\mathcal{A}}' & \subset & \hat{\delta}(\widehat{\mathcal{A}})',
\end{array}$$

and hence $E_{\hat{\delta}(\widehat{\mathcal{A}})'}(\widehat{\mathcal{A}}'\otimes\mathbb{B}(\mathcal{H})) = \widehat{\mathcal{A}}'\otimes\widehat{\mathcal{A}}'$ has to be seen. Since $\hat{\delta}(\widehat{\mathcal{A}}) = V^*(\mathbb{C}\otimes\widehat{\mathcal{A}})V$, we have $E_{\hat{\delta}(\widehat{\mathcal{A}})'} = \mathrm{Ad}(V^*)\cdot(\mathrm{Id}\otimes E_{\widehat{\mathcal{A}}'})\cdot\mathrm{Ad}(V)$, and for $x\in\widehat{\mathcal{A}}'\otimes\mathbb{B}(\mathcal{H})$ we compute

$$\begin{aligned}
E_{\hat{\delta}(\widehat{\mathcal{A}})'}(x) &= \sum_{\pi\in\Pi,a,b}\frac{d(\pi)}{n}\,V^*(1\otimes\hat{u}(\pi)_{ab})VxV^*(1\otimes\hat{u}(\pi)^*_{ab})V\\
&= \sum_{\pi\in\Pi,a,b}\frac{d(\pi)}{n}\,\hat{\delta}(\hat{u}(\pi)_{ab})x\hat{\delta}(\hat{u}(\pi)^*_{ab})\\
&= \sum_{\pi\in\Pi,a,b,c,d}\frac{d(\pi)}{n}\,(\hat{u}(\pi)_{ac}\hat{u}(\pi)^*_{ad}\otimes\hat{u}(\pi)_{cb})x(1\otimes\hat{u}(\pi)^*_{db})\\
&= \sum_{\pi\in\Pi,a,b,c}\frac{d(\pi)}{n}\,(1\otimes\hat{u}(\pi)_{cb})x(1\otimes\hat{u}(\pi)^*_{cb}) = (\mathrm{Id}\otimes E_{\widehat{\mathcal{A}}'})(x)\in\widehat{\mathcal{A}}'\otimes\widehat{\mathcal{A}}'.
\end{aligned}$$

(4). This follows from [14, (4.1.6)] and $U\Lambda_{\hat{\varphi}}(x) = \Lambda_{\hat{\varphi}}(\hat{\kappa}(x))$, $x \in \widehat{\mathcal{A}}$.
(5). Using $V^* = (J \otimes \hat{J})V(J \otimes \hat{J})$ [9], we get the following for $x \in \widehat{\mathcal{A}}'$:

$$\begin{aligned} V(1 \otimes x)V^* &= (J \otimes \hat{J})V^*(1 \otimes \hat{J}x\hat{J})V(J \otimes \hat{J}) = (J \otimes \hat{J})\hat{\delta}(\hat{J}x\hat{J})(J \otimes \hat{J}) \\ &= (U \otimes 1)\hat{\delta}'(x)(U \otimes 1). \end{aligned}$$

Since $JxJ = \hat{\kappa}(x^*)$ for $x \in \widehat{\mathcal{A}}$, we have

$$\begin{aligned} \hat{\hat{V}}^*(1 \otimes x)\hat{\hat{V}} &= (U \otimes U)V^*(1 \otimes \hat{\kappa}(\hat{J}x^*\hat{J}))V(U \otimes U) = (U \otimes U)\hat{\delta}(\hat{\kappa}(\hat{J}x^*\hat{J}))(U \otimes U) \\ &= (\hat{J} \otimes \hat{J})(\hat{\kappa} \otimes \hat{\kappa}) \cdot \hat{\delta}(\hat{\kappa}(\hat{J}x\hat{J}))(\hat{J} \otimes \hat{J}) = F\hat{\delta}'(x)F. \end{aligned}$$

□

We are now ready to prove the main result in the section.

Theorem 3.3. *For a given Q-system (λ_{VF}, S, T), there exists a unique 2-cocycle $\omega' \in Z^2(\widehat{\mathcal{A}}')$ satisfying $S = \omega'^* S_0$, $T = V\omega'^* V^* T_0$. On the other hand, every 2-cocycle arises in this way, and furthermore two Q-systems are equivalent if and only if the corresponding 2-cocycles are equivalent. In consequence, we can identify $H^2(M \supset N)$ with $H^2(\widehat{\mathcal{A}}')$ (and also with $H^2(\widehat{\mathcal{A}})$).*

Proof. For a Q-system (λ_{VF}, S, T) we set $\tilde{T}_0 = \overline{\rho}_V(T_0) = V^*T_0$ and $\tilde{T} = \overline{\rho}_V(T)$. Due to $V \in \widehat{\mathcal{A}} \otimes \mathcal{A}$ and $T \in (\lambda_{VF}, \lambda_{VF}^2) = (\lambda_{VF}, \sigma_{\mathcal{H}}\lambda_{VF}) = \mathcal{H}\widehat{\mathcal{A}}'$, we have $\tilde{T} = \lambda_{V^*}(T) = V^*T \in \mathcal{H}^2\mathcal{H}^*$. Note that $T_0T^* \in \mathbb{B}(\mathcal{H}) \otimes \widehat{\mathcal{A}}'$ implies $\tilde{T}_0\tilde{T}^* \in \hat{\delta}(\widehat{\mathcal{A}})'$ because

$$V^*T_0T^*V\hat{\delta}(x) = V^*T_0T^*(1 \otimes x)V = V^*(1 \otimes x)T_0T^*V, \quad x \in \widehat{\mathcal{A}}.$$

We set $\omega' = n(E_{\widehat{\mathcal{A}}'} \otimes \mathrm{Id})(\tilde{T}_0\tilde{T}^*)$, which belongs to $\widehat{\mathcal{A}}' \otimes \widehat{\mathcal{A}}'$ thanks to Lemma 3.2, (3). Lemma 3.2, (1) and (2) imply

$$\begin{aligned} V\omega'^*V^*T_0 &= \sum_{\pi,a,b} d(\pi)V\hat{u}(\pi)_{ab}\tilde{T}T_0^*V\hat{u}(\pi)_{ab}^*V^*T_0 = \sum_{\pi,a,b} d(\pi)\hat{\varphi}(\hat{u}(\pi)_{ab}^*)V\hat{u}(\pi)_{ab}\tilde{T} \\ &= V\tilde{T} = T. \end{aligned}$$

We show that ω' is a unitary. Since $(\rho_V\overline{\rho}_V, S, T)$ is a Q-system, we have

$$\tilde{T}\tilde{T}^* = \overline{\rho}_V(\rho_V\overline{\rho}_V(T^*)T) = \overline{\rho}_V\rho_V\overline{\rho}_V(T^*)\overline{\rho}_V(T) = \sigma_{\mathcal{H}}\overline{\rho}_V(T^*)\overline{\rho}_V(T) = \sigma_{\mathcal{H}}(\tilde{T}^*)\tilde{T}.$$

Using [14, (4.1.10),(4.1.12)], Lemma 3.2, (1), (2), and $T \in \mathcal{H}\widehat{\mathcal{A}}'$, we get

$$\begin{aligned} \omega'^*\omega' &= \sum_{\pi,\sigma,a,b,c,d} d(\pi)d(\sigma)\hat{u}(\pi)_{ab}\tilde{T}T_0^*V\hat{u}(\pi)_{ab}^*\hat{u}(\sigma)_{cd}V^*T_0\tilde{T}^*\hat{u}(\sigma)_{cd}^* \\ &= \sum_{\pi,a,b} d(\pi)\hat{u}(\pi)_{ab}\tilde{T}\tilde{T}^*\hat{u}(\pi)_{ab}^* = \sum_{\pi,a,b} d(\pi)\sigma_{\mathcal{H}}(\tilde{T}^*)\hat{u}(\pi)_{ab}V^*T\hat{u}(\pi)_{ab}^* \\ &= \sum_{\pi,a,b} d(\pi)\sigma_{\mathcal{H}}(\tilde{T}^*)\hat{u}(\pi)_{ab}V^*\sigma_{\mathcal{H}}(\hat{u}(\pi)_{ab}^*)T \\ &= \sum_{\pi,a,b} d(\pi)\sigma_{\mathcal{H}}(\tilde{T}^*)\hat{u}(\pi)_{ab}\hat{\delta}(\hat{u}(\pi)_{ab}^*)\tilde{T} = \sum_{\pi,a,b} d(\pi)\sigma_{\mathcal{H}}(\tilde{T}^*)\sigma_{\mathcal{H}}(\hat{u}(\pi)_{bb}^*)\tilde{T} \\ &= n\sigma_{\mathcal{H}}(\tilde{T}^*)\sigma_{\mathcal{H}}(\hat{e})\tilde{T} = n\sigma_{\mathcal{H}}(\tilde{T}^*)\sigma_{\mathcal{H}}(\overline{\rho}_V(\hat{e}))\tilde{T} = n\overline{\rho}(\rho\overline{\rho}(T^*SS^*)T) = 1 \end{aligned}$$

as desired. Note that the above argument also shows uniqueness of ω'.

Now we show that ω' is a 2-cocycle. $T_0^2 = \lambda_{VF}(T_0)T_0$ and $T^2 = \lambda_{VF}(T)T$ imply $\sigma_{\mathcal{H}}(\tilde{T}_0)\tilde{T}_0 = \tilde{T}_0^2$, $\sigma_{\mathcal{H}}(\tilde{T})\tilde{T} = \tilde{T}^2$. Thanks to Lemma 3.2, (4) and (5), the left side of the second equation is equal to

$$\begin{aligned}\sigma_{\mathcal{H}}(\omega' UVT_0)\omega'^*\tilde{T}_0 &= \omega'^*_{23}(1\otimes U\otimes 1)V_{23}\omega'^*_{13}\sigma_{\mathcal{H}}(T_0)\tilde{T}_0\\ &= \omega'^*_{23}(1\otimes U\otimes 1)V_{23}\omega'^*_{13}V^*_{23}(1\otimes U\otimes 1)\sigma_{\mathcal{H}}(\tilde{T}_0)\tilde{T}_0\\ &= \omega'^*_{23}(\mathrm{Id}\otimes\hat{\delta}')(\omega'^*)\tilde{T}_0^2.\end{aligned}$$

On the other hand, the right side is

$$\omega'^*UVT_0\omega'^*\tilde{T}_0 = \omega'^*_{12}(U\otimes 1\otimes 1)V_{12}\omega'^*_{23}V^*_{12}(U\otimes 1\otimes 1)\tilde{T}_0^2 = \omega'^*_{12}(\hat{\delta}'\otimes\mathrm{Id})(\omega'^*)\tilde{T}_0^2.$$

We claim $n^2(Id\otimes E_{\widehat{\mathcal{A}}'}\otimes E_{\widehat{\mathcal{A}}'})(\tilde{T}_0^2\tilde{T}_0^{*2}) = 1$, which will show the cocycle relation of ω'. Indeed, since we have

$$\tilde{T}_0^2\tilde{T}_0^{*2} = V^*T_0V^*T_0T_0^*VT_0V = V^*_{12}V^*_{23}(e\otimes e\otimes 1)V_{23}V_{12},$$

it suffices to show $n(\mathrm{Id}\otimes E_{\widehat{\mathcal{A}}'})(V^*(e\otimes 1)V) = 1$. Thanks to the commuting square condition shown in [14, p.442], the restriction of $E_{\widehat{\mathcal{A}}'}$ to $\mathcal{A}$ is the Haar state φ. Thus, Lemma 3.2, (2) implies

$$\begin{aligned}(\mathrm{Id}\otimes E_{\widehat{\mathcal{A}}'})(V^*(e\otimes 1)V) &= \sum_{\pi,a,b,c}\hat{u}(\pi)^*_{ca}e\hat{u}(\pi)_{cb}\otimes E_{\widehat{\mathcal{A}}'}(e(\pi)_{ab})\\ &= \sum_{\pi,a,c}\frac{d(\pi)}{n}\hat{u}(\pi)^*_{ca}e\hat{u}(\pi)_{ca}\otimes 1 = E_{\widehat{\mathcal{A}}'}(e)\otimes 1 = \frac{1}{n},\end{aligned}$$

where we use the fact that $\{\hat{u}(\pi)^*_{ab}\}_{ab}$ is also an irreducible unitary corepresentation as $\widehat{\mathcal{A}}$ is a finite-dimensional Kac algebra. Therefore, ω' is a 2-cocycle.

Let $\hat{\epsilon}'$ be the counit of $\widehat{\mathcal{A}}'$. Then, the cocycle relation implies

$$(\hat{\epsilon}'\otimes\mathrm{Id})(\omega') = (\mathrm{Id}\otimes\hat{\epsilon}')(\omega')\in\mathbb{T}.$$

Note that $xS_0 = \hat{\epsilon}'(x)S_0$ holds for any $x\in\widehat{\mathcal{A}}'$. Since $(\mathrm{Id},\rho\overline{\rho}) = \mathbb{C}S_0$, we have $S = cS_0$ for some $c\in\mathbb{T}$ and we compute

$$\frac{1}{\sqrt{n}} = S^*T = \overline{c}S_0^*V\omega'V^*T_0 = \overline{c}(\hat{\epsilon}\otimes\mathrm{Id})(\omega'^*)S_0^*T_0 = \overline{c}(\hat{\epsilon}\otimes\mathrm{Id})(\omega'^*)\frac{1}{\sqrt{n}},$$

showing $S = (\hat{\epsilon}'\otimes\mathrm{Id})(\omega'^*)S_0 = \omega'^*S_0$.

Let $\xi\in(\lambda_{VF},\lambda_{VF}) = \widehat{\mathcal{A}}'$ be a unitary. We claim that the cocycle corresponding to the Q-system $(\lambda_{VF},\xi S,\xi\lambda_{VF}(\xi)T\xi^*)$ is $\hat{\delta}'(\xi)\omega'(\xi^*\otimes\xi^*)$. Direct computation shows

$$V^*\xi\lambda_{VF}(\xi)T\xi^* = V^*\xi V\sigma_{\mathcal{H}}(\xi)V^*T\xi = \xi\sigma_{\mathcal{H}}(\xi)\omega'\tilde{T}_0\xi^*.$$

Therefore, to prove the claim it suffices to show $n(E_{\widehat{\mathcal{A}}'}\otimes \mathrm{Id})(\tilde{T}_0\xi\tilde{T}_0^*)=\hat{\delta}'(\xi)$. However, by Lemma 3.2,(4) and (5) we compute

$$\begin{aligned}(E_{\widehat{\mathcal{A}}'}\otimes \mathrm{Id})(\tilde{T}_0\xi\tilde{T}_0^*) &= (E_{\widehat{\mathcal{A}}'}\otimes \mathrm{Id})((U\otimes 1)V(U\otimes 1)(e\otimes\xi)(U\otimes 1)V^*(U\otimes 1))\\ &= (U\otimes 1)V(U\otimes 1)(E_{\widehat{\mathcal{A}}'}\otimes \mathrm{Id})((e\otimes\xi))(U\otimes 1)V^*(U\otimes 1)\\ &= \frac{1}{n}(U\otimes 1)V(1\otimes\xi)V^*(U\otimes 1)=\frac{1}{n}\hat{\delta}'(\xi).\end{aligned}$$

Finally, we show that every cocycle arises in this way. We set

$$S=\omega'^*S_0,\quad T=V\omega'^*V^*T_0$$

for a given cocycle ω', and show that (λ_{VF},S,T) is a Q-system. It is easy to show $S^*T=\lambda_{VF}(S^*)T=1/\sqrt{n}$. We claim $\overline{\rho}_V(T)=\omega'^*\tilde{T}_0$. Indeed, direct computation based on the pentagon equation yields

$$\begin{aligned}\overline{\rho}_V(T) &= V^*\sigma_{\mathcal{H}}(V^*)V\omega'^*V^*T_0V=V_{12}^*V_{23}^*V_{12}\omega'^*_{12}V_{12}^*V_{23}T_0\\ &= V_{23}^*V_{13}^*\omega'^*_{12}V_{12}^*V_{23}T_0=\omega'^*_{12}V_{23}^*V_{13}^*V_{12}^*V_{23}T_0\\ &= \omega'^*_{12}V_{12}^*V_{23}^*V_{12}V_{12}^*V_{23}T_0=\omega'^*\tilde{T}_0,\end{aligned}$$

which shows the claim. It is easy to show $T\in(\lambda_{VF},\lambda_{VF}^2)$ from the claim, as $\omega'^*\tilde{T}_0$ belongs to $(\sigma_{\mathcal{H}}\overline{\rho},\sigma_{\mathcal{H}}^2\overline{\rho}_V)=(\overline{\rho}_V\lambda_{VF},\overline{\rho}_V\lambda_{VF}^2)$. To prove $T^2=\lambda_{VF}(T)T$, it suffices to show $\overline{\rho}_V(T)^2=\sigma_{\mathcal{H}}\overline{\rho}_V(T)\overline{\rho}_V(T)$, which follows from direct computation using Lemma 3.2, (4), (5) and the claim. □

In particular, when $M=N\rtimes G$, a crossed product by G, $H^2(M\supset N)$ can be identified with the second cohomology $H^2(\widehat{G})$ of the group dual $\widehat{G}$ (see the first part in §4). Also, from Theorem 2.4 and Theorem 3.3 we conclude

Corollary 3.4. *The second cohomology $H^2(\mathcal{A})$ is a finite set for a finite-dimensional Kac algebra $\mathcal{A}$.*

Let ω be a 2-cocycle of $\widehat{\mathcal{A}}$ and set $\omega'=(\hat{J}\otimes\hat{J})\omega(\hat{J}\otimes\hat{J})$, a 2-cocycle of $\widehat{\mathcal{A}}'$. We keep the notations S, T, $\tilde{T}_0$, and $\tilde{T}$ in the above proof. We would like to determine the Kac algebra corresponding to the subfactor associated with the Q-system $(\rho\overline{\rho},S,T)$. We denote by E_ω the conditional expectation defined by $E_\omega(x)=T^*\rho\overline{\rho}(x)T$. Let P be the image of E_ω, and we set

$$\mathcal{K}=\{X\in M;\ Xy=\rho\overline{\rho}(y)X \text{ for each } y\in P\}.$$

Then, the multiplicative unitary $V_P\in\mathcal{K}^2\mathcal{K}^{2*}$ describing the new inclusion $M\supset P$ is characterized by $\lambda_{VF}(X)=V_PF_{\mathcal{K}}X$, $X\in\mathcal{K}$ with the flip $F_{\mathcal{K}}$ of $\mathcal{K}\otimes\mathcal{K}$ ([21]).

Lemma 3.5. *With the above notations we have*
(1) $\lambda_{VF}(T)=U\omega'\sigma_{\mathcal{H}}(T)\omega'^*U$,
(2) $\mathcal{K}=U\omega'\mathcal{H}$.

Proof. (1). We note $V\sigma_{\mathcal{H}}(\tilde{T}_0)V^*=\sigma_{\mathcal{H}}(\tilde{T}_0)$ because the pentagon equation shows

$$\begin{aligned}V\sigma_{\mathcal{H}}(\tilde{T}_0)V^* &= V\sigma_{\mathcal{H}}(V^*T_0)V^*=V_{12}V_{23}^*V_{13}^*\sigma_{\mathcal{H}}(T_0)=V_{23}^*V_{12}\sigma_{\mathcal{H}}(T_0)\\ &= \sigma_{\mathcal{H}}(V^*T_0)=\sigma_{\mathcal{H}}(\tilde{T}_0).\end{aligned}$$

Thus, by the pentagon equation and the above equality we compute

$$\begin{aligned}
\lambda_{VF}(T) &= VF\sigma_{\mathcal{H}}(VF)V\omega'^*V^*T_0FV^* = V_{12}F_{12}V_{23}F_{23}V_{12}\omega'^*_{12}V^*_{12}F_{23}T_0V^*_{12} \\
&= V_{12}V_{13}V_{23}\omega'^*_{23}\sigma_{\mathcal{H}}(\tilde{T}_0)V^*_{12} = V_{23}V_{12}\omega'^*_{23}V^*_{12}\sigma_{\mathcal{H}}(\tilde{T}_0) \\
&= V_{23}(U\otimes 1\otimes 1)(\hat{\delta}'\otimes \mathrm{Id})(\omega'^*)(U\otimes 1\otimes 1)\sigma_{\mathcal{H}}(\tilde{T}_0) \\
&= (U\otimes 1\otimes 1)\omega'_{12}V_{23}\omega'^*_{23}(\mathrm{Id}\otimes\hat{\delta}')(\omega'^*)(1\otimes U\otimes 1)V_{23}\sigma_{\mathcal{H}}(T_0)(U\otimes 1\otimes 1) \\
&= (U\otimes 1\otimes 1)\omega'_{12}V_{23}\omega'^*_{23}(1\otimes U\otimes 1)V_{23}\omega'^*_{13}\sigma_{\mathcal{H}}(T_0)(U\otimes 1\otimes 1) \\
&= (U\otimes 1\otimes 1)\omega'_{12}\sigma_{\mathcal{H}}(T)\omega'^*_{12}(U\otimes 1\otimes 1),
\end{aligned}$$

showing (1).
(2). Using (1), we compute

$$\begin{aligned}
\rho\overline{\rho}(E_\omega(x)) &= \lambda_{VF}(T^*)(\rho\overline{\rho})^2(x)\lambda_{VF}(T) \\
&= U\omega'\sigma_{\mathcal{H}}(T^*)\omega'^*U\sigma_{\mathcal{H}}\rho\overline{\rho}(x)U\omega'\sigma_{\mathcal{H}}(T)\omega'^*U \\
&= U\omega'\sigma_{\mathcal{H}}(T^*\rho\overline{\rho}_{VF}(x)T)\omega'^*U = U\omega'\sigma_{\mathcal{H}}(E_\omega(x))\omega'^*U,
\end{aligned}$$

where we used the fact $(\rho\overline{\rho},\rho\overline{\rho}) = (\lambda_{VF},\lambda_{VF}) = \widehat{\mathcal{A}}'$. This shows $\mathcal{K} = U\omega'\mathcal{H}$. □

We set $W = (U\otimes 1)\omega'(U\otimes 1)\in\widehat{\mathcal{A}}\otimes\widehat{\mathcal{A}}'$, which satisfies $\mathcal{K} = W\mathcal{H}$. Then, we get

$$\begin{aligned}
V_PF_{\mathcal{K}} &= \sum_{i=1}^n \lambda_{VF}(WS_i)S_i^*W^* = \lambda_{VF}(W)VFW^* \\
&= VF\sigma_{\mathcal{H}}(VF)W\sigma_{\mathcal{H}}(FV^*)FV^*VFW^* = VF\sigma_{\mathcal{H}}(VF)W\sigma_{\mathcal{H}}(FV^*)W^*,
\end{aligned}$$

where $\{S_i\}_{i=1}^n$ is an orthonormal basis of $\mathcal{H}$.

Let $V_\omega\in\mathcal{H}^2\mathcal{H}^{*2}$ be the multiplicative unitary that is unitary equivalent to V_P through the unitary transformation $W:\mathcal{H}\longrightarrow\mathcal{K}$. Then, using the cocycle relation and Lemma 3.2, (5), we compute

$$\begin{aligned}
V_\omega &= \sigma_{\mathcal{H}}(W^*)W^*V_PF_{\mathcal{K}}W\sigma_{\mathcal{H}}(W)F = W^*_{23}W^*_{12}V_{12}F_{12}V_{23}F_{23}W_{12}F_{23}V^*_{23}W_{23}F_{12} \\
&= W^*_{12}W^*_{23}V_{12}(1\otimes U\otimes 1)F_{12}V_{23}\omega'_{13}V^*_{23}(U\otimes 1\otimes 1)W_{23}F_{12} \\
&= W^*_{12}W^*_{23}V_{12}(U\otimes U\otimes 1)F_{12}(\mathrm{Id}\otimes\hat{\delta}')(\omega')\omega'_{23}(U\otimes U\otimes 1)F_{12} \\
&= W^*_{12}W^*_{23}V_{12}(U\otimes U\otimes 1)F_{12}(\hat{\delta}'\otimes\mathrm{Id})(\omega')\omega'_{12}(U\otimes U\otimes 1)F_{12} \\
&= W^*_{12}W^*_{23}V_{12}(U\otimes U\otimes 1)\hat{\hat{V}}^*_{12}\omega'_{23}\hat{\hat{V}}_{12}((F\omega'(U\otimes U)F)\otimes 1) \\
&= W^*_{12}W^*_{23}(1\otimes((U\otimes 1)\omega'(U\otimes 1))((VF(U\otimes U)\omega'(U\otimes U)F)\otimes 1) \\
&= (((U\otimes 1)\omega'^*(U\otimes 1)VF(U\otimes U)\omega'(U\otimes U)F)\otimes 1).
\end{aligned}$$

Moreover, when ω is a normalized cocycle (such a choice is always possible up to equivalence as shown in [16, Chapter 8]), we have

$$F(U\otimes U)\omega'(U\otimes U)F = F(J\otimes J)\omega(J\otimes J)F = F(\hat{\kappa}\otimes\hat{\kappa})(\omega^*)F = \omega,$$

and so $V_\omega = (1\otimes U)F\omega^*F(1\otimes U)V\omega$.

Let $\widehat{\mathcal{A}}^\omega$ be the Kac algebra whose underlying algebra is $\widehat{\mathcal{A}}$ with the coproduct $\hat{\delta}^\omega$ coming from V_ω, i.e., $\hat{\delta}_\omega(x) = V^*_\omega(1\otimes x)V_\omega$ (and the other Kac algebra operations unchanged). Let $\Gamma : M \to M\otimes\widehat{\mathcal{A}}$, $\Gamma^\omega : M \to M\otimes\widehat{\mathcal{A}}^\omega$ be actions of $\widehat{\mathcal{A}}, \widehat{\mathcal{A}}^\omega$ respectively defined by

$$\begin{aligned}\Gamma(x) &= \sum_{i,j=1}^{n} S_i^*\rho\overline{\rho}(x)S_j \otimes S_iS_j^*,\\ \Gamma^\omega(x) &= \sum_{i,j=1}^{n} S_i^*W^*\rho\overline{\rho}(x)WS_j \otimes S_iS_j^*.\end{aligned}$$

Then, N and P are the fixed-point algebras of the actions Γ and Γ^ω respectively. From the arguments so far we have

Corollary 3.6. *With the above notations we have*

(1) $\lambda_{VF}\lambda_W = \lambda_W\lambda_{V_\omega F}$.
(2) $\hat{\delta}^\omega(x) = \omega^*\hat{\delta}(x)\omega$, $x \in \widehat{\mathcal{A}}$.
(3) $\Gamma^\omega = Ad(Z)\cdot\Gamma$ *with*

$$Z = \sum_{i,j=1}^{n} S_i^*W^*S_j \otimes S_iS_j^*,$$

and Z satisfies $(\mathrm{Id}\otimes\hat{\delta})(Z) = (1\otimes\omega)(Z\otimes 1)(\Gamma\otimes\mathrm{Id})(Z)$.
(4) *The representation categories of $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{A}}^\omega$ are isomorphic as an abstract tensor category.*

Proof. The first three facts follow from easy computations, and (4) follows from the fact that the two categories have the same realization in $\mathrm{End}(M)$ generated by $\rho\overline{\rho}$. □

Corollary 3.7. *Let the notations be as above.*

(1) *Let P be a subfactor of M satisfying $[\gamma_P] = [\gamma_N]$ in $\mathrm{Sect}(M)$. Then, there exists a normalized cocycle $\omega \in Z^2(\widehat{\mathcal{A}})$ and a unitary $Z_P \in M\otimes\widehat{\mathcal{A}}$ satisfying $(\mathrm{Id}\otimes\hat{\delta})(Z_P) = (1\otimes\omega_P)(Z_P\otimes 1)(\Gamma\otimes\mathrm{Id})(Z_P)$ such that $\Gamma_P = \mathrm{Ad}(Z_P)\cdot\Gamma$ is an action of $\widehat{\mathcal{A}}^\omega$ on M whose fixed-point algebra is P.*
(2) *Assume that P and Q are subfactors satisfying the above condition. Then, P and Q are inner conjugate if and only if $[\omega_P] = [\omega_Q]$ in $H^2(\widehat{\mathcal{A}})$.*

Proof. We must see inner conjugacy of P and Q for $[\omega_P] = [\omega_Q] \in H^2(\widehat{\mathcal{A}})$. We may and do assume $\omega_P = \omega_Q$ as we may replace Z_P with $(1\otimes v^*)Z_P$ if $v \in \widehat{\mathcal{A}}$ is a unitary satisfying $\omega_P = \delta_G(v)\omega_Q(v^*\otimes v^*)$. Then, direct computation shows that $X = Z_QZ_P^*$ is a Γ_P-cocycle, i.e., $(\mathrm{Id}\otimes\delta_G^{\omega_P})(X) = (X\otimes 1)(\Gamma_P\otimes\mathrm{Id})(X)$. With $\Gamma_Q = Ad(X)\cdot\Gamma_P$ we can show, as in the case of group actions, that the identity of M extends to an isomorphism from the crossed product by Γ_P onto that by Γ_Q. Since the crossed product by Γ_P is the basic extension of $P = M^{\Gamma_P} \subset M$, the desired inner conjugacy comes from uniqueness (up to inner perturbation) of downward basic construction. □

4. Non-uniqueness of permutation symmetries

Let G be a finite group, and $\mathbb{C}G$ be its group algebra with the usual coproduct δ_G. We identify G with its image in the regular representation and keep the notations in the previous section. We freely use the terminology in [32] for cocycles of the dual of G, and the readers are recommended to consult [32] or references there for the facts stated below without proofs. We use the notations $Z^2(\widehat{G})$ and $H^2(\widehat{G})$ for $Z^2(\mathbb{C}G)$ and $H^2(\mathbb{C}G)$ respectively. Note that when G is abelian $Z^2(\mathbb{C}G)$ (resp. $H^2(\mathbb{C}G)$) is the usual 2-cocycle group $Z^2(\widehat{G},\mathbb{T})$ (resp. the second cohomology $H^2(\widehat{G},\mathbb{T})$) of the dual group $\widehat{G}$ via the duality $\mathbb{C}G \cong \ell^\infty(\widehat{G})$.

As was seen in the previous section, for a normalized cocycle $\omega \in Z^2(\widehat{G})$ the Kac algebra $(\mathbb{C}G)^\omega$ has the same representation category as G. However, in general δ_G^ω is not cocommutative. Even if δ_G^ω is cocommutative, i.e., $(\mathbb{C}G)^\omega$ is isomorphic to the group algebra $\mathbb{C}G_\omega$ with some group G_ω thanks to the structure theorem of cocommutative Kac algebras, there is no reason to expect that the new group G_ω is isomorphic to G. Indeed, the purpose of this section is to construct an example of a group G and a cocycle $\omega \in Z^2(\widehat{G})$ such that G_ω is not isomorphic to G. Such an example shows that it is inevitable to fix a permutation symmetry in the Doplicher-Roberts duality theorem in order to uniquely recover the group from its representation category (see [6]).

For a cocycle $\omega \in Z^2(K,\mathbb{T})$ of a finite abelian group K, $\beta_\omega(k,l) = \omega(k,l)\overline{\omega(l,k)}$ $(k,l \in K)$ gives us a skew-symmetric bicharacter. It is well-known that the map $\omega \mapsto \beta_\omega$ induces an isomorphism from $H^2(K,\mathbb{T})$ into the set of skew-symmetric bicharacters.

Lemma 4.1. *Let G be a finite group, and we assume that $\omega \in Z^2(\mathbb{C}G)$ is a normalized cocycle. Then the following two conditions are equivalent*:

1. *δ_G^ω is cocommutative.*
2. *There exist an abelian normal subgroup $N \subset G$ and a non-degenerate cocycle $\omega_1 \in Z^2(\widehat{N})$ satisfying $[\omega] = [\omega_1]$ in $H^2(\widehat{G})$ and $[{}^g\omega_1] = [\omega_1]$ in $H^2(\widehat{N})$ for each $g \in G$, where ${}^g\omega_1 = (g\otimes g)\omega_1(g^{-1}\otimes g^{-1})$. (Note that $Z^2(\widehat{N})$ is regarded as a subset of $Z^2(\widehat{G})$ since $\mathbb{C}N$ is a Hopf subalgebra of $\mathbb{C}G$).*

Proof. Thanks to [32, Theorem 2], the class of ω uniquely corresponds to a conjugacy class of a full multiplicity ergodic action θ of G on a finite-dimensional von Neumann algebra B. Then, θ is induced by a full multiplicity ergodic action θ^1 of a subgroup N on a full matrix algebra B_1. Let $\omega_1 \in Z^2(\widehat{N})$ be the cocycle whose class corresponds to the class of θ^1. Then, since B_1 is a matrix algebra, ω_1 is non-degenerate ([32, Theorem 12]). Thus, the naturality of the one-to-one correspondence between $H^2(\widehat{G})$ and the conjugacy classes of full multiplicity ergodic actions of G shows $[\omega] = [\omega_1]$ in $H^2(\widehat{G})$.

Now, we assume that (1) holds. Then, since ω_1 is non-degenerate, $(\mathbb{C}N)^{\omega_1}$ being cocommmutative implies that N is commutative ([32, Theorem 12]). Since δ_G^ω is cocommutative, we have $F\omega_1^*\delta_G(g)\omega_1F = \omega_1^*\delta_G(g)\omega_1$ $(g \in G)$, which is, by simple computation, equivalent to

$$(g^{-1} \otimes g^{-1})\beta_{\omega_1}(g \otimes g) = \beta_{\omega_1} \quad \text{for each } g \in G. \tag{4.1}$$

Set $A = \cap_{g\in G}\mathbb{C}gNg^{-1}$. Then, (4.1) implies $\beta_{\omega_1} \in A \otimes A$, and so $A = \mathbb{C}N$ because ω_1 is non-degenerate (see the condition (C_1) in [32, Theorem 12]). Thus, N is normal. As N is abelian, it is well known that the cohomology class $[\omega_1] \in H^2(\widehat{N}, \mathbb{T})$ is determined by β_{ω_1}, and so $[\omega_1] = [{}^g\omega_1]$ in $H^2(\widehat{N})$.

The converse implication follows from the same computation. □

Remark 4.2. Let G be a finite group with a cocycle $\omega \in Z^2(\widehat{G})$ satisfying the condition of Lemma 4.1. Then, there exists a permutation symmetry $\mathcal{E}^\omega$ of the tensor category $\mathrm{Rep}(G)$ of the unitary representations of G coming from the canonical permutation symmetry of $\mathrm{Rep}(G_\omega)$ through the isomorphism between $\mathrm{Rep}(G)$ and $\mathrm{Rep}(G_\omega)$ (see Corollary 3.6,(4)). For $(\pi_i, \mathcal{H}_i) \in \mathrm{Rep}(G_\omega)$, $i = 1, 2$, $\mathcal{E}^\omega : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2 \otimes \mathcal{H}_1$ is given by

$$\mathcal{E}^\omega = (\pi_2 \otimes \pi_1)(\omega)\mathcal{E}(\pi_1 \otimes \pi_2)(\omega^*) = (\pi_2 \otimes \pi_1)(\beta_{\omega_1})\mathcal{E},$$

where $\mathcal{E}$ is the canonical permutation symmetry of $\mathrm{Rep}(G)$ and $\omega_1 \in Z^2(\widehat{N})$ is a cocycle equivalent to ω in $H^2(\widehat{G})$. Although this follows from a purely category theoretical argument, we present a proof based on our model for the convenience of those not familiar with category theory (including the authors).

First, we assume $\omega = \omega_1$. Let $(L, \ell^2(G))$ be the left regular representation of G, and V be the multiplicative unitary on $\ell^2(G)\otimes\ell^2(G)$ defined by $Vf(g, h) = f(hg, h)$, $g, h \in G$. Then, the image of $\overline{\rho}_V$ is the fixed-point algebra $\mathcal{O}_n^\alpha$ under the G-action α defined by $\alpha_g = \lambda_{L_g}$, $g \in G$. We denote $\overline{\rho}_V$ by θ when regarded as an isomorphism from $\mathcal{O}_n$ onto $\mathcal{O}_n^\alpha$. Let σ_G be the restriction of $\sigma_{\mathcal{H}}$ to $\mathcal{O}_n^\alpha$. Then, $\theta\lambda_{VF} = \sigma_G\theta$ holds, showing that λ_{VF} is conjugate to σ_G. By simple computation based on the pentagon equation and $V_{13}V_{23} = V_{23}V_{13}$, we get $\theta(VFV^*) = F$, which shows that VFV^* is the canonical permutation symmetry for λ_{VF} coming from σ_G through θ. Thus, $\mathcal{E}^\omega$ acting on $\ell^2(G) \otimes \ell^2(G)$ is given by $\theta(V_PF_\mathcal{K}V_P^*)$, where V_P and $F_\mathcal{K}$ are as in §3. By straightforward computation using the cocycle relation of ω' and the fact that N is abelian, we get $V_PF_\mathcal{K}V_P^* = V\beta_\omega FV^*$. Since β_ω commutes with $L_g \otimes L_g$, we can conclude $\theta(V_PF_\mathcal{K}V_P^*) = \beta_\omega F$. Assume now that $\omega = \delta_G(v)\omega_1(v^* \otimes v^*)$ with a unitary $v \in \mathbb{C}G$. From the above computation we get $\mathcal{E}^\omega = (\pi_2\otimes\pi_1)(\omega)\mathcal{E}(\pi_1\otimes\pi_2)(\omega^*)$, which is equal to $(\pi_2 \otimes \pi_1)(\beta_{\omega_1})\mathcal{E}$ as β_{ω_1} commutes with $\delta_G(\mathbb{C}G)$.

We assume that N and $\omega = \omega_1$ satisfy the condition (2) in Lemma 4.1. Since N is abelian, by the duality we can regard ω as a group 2-cocycle in $Z^2(\widehat{N}, \mathbb{T})$. In what follows, we use the following notation: For $n \in N$, $g \in G$, and $\tau \in \widehat{N}$, we set $n^g = g^{-1}ng$ and $\tau^g(n) = \tau(n^{g^{-1}})$. Also, for $\xi : \widehat{N} \to \mathbb{T}$, we set $\partial\xi(\sigma, \tau) = \xi(\sigma)\xi(\tau)\overline{\xi(\sigma\tau)}$.

Lemma 4.3. *With these notations we have*

(1) *There exist $\xi_g : \widehat{N} \longrightarrow \mathbb{T}$ for each $g \in G$ and $\eta \in Z^2(G, N)$ such that for any pair $\sigma, \tau \in \widehat{N}$ we have*

$$\omega(\sigma^g, \tau^g) = \partial\xi_g(\sigma, \tau)\omega(\sigma, \tau),$$

$$\xi_{g_1}(\sigma)\xi_{g_2}(\sigma^{g_1}) = \sigma(\eta(g_1,g_2))\xi_{g_1g_2}(\sigma).$$

(2) *We may take ξ_g so that $\xi_{gn} = \xi_g$, $\xi_e = 1$ hold for all $n \in N$, $g \in G$, and η is regarded as an element in $Z^2(G/N, N)$. Then $\{\xi_g g\}_{g\in G}$ is the set of group-like elements of $(\mathbb{C}G)^\omega$, and G_ω is given by the extension*

$$1 \to N \to G_\omega \to G/N \to 1$$

whose difference from the extension G is given by η.

Proof. (1). Since the cohomology class of ω is invariant under the G-action, ξ_g satisfying the first equality exists. The associativity of G implies

$$\partial\xi_{g_1}(\sigma,\tau)\partial\xi_{g_2}(\sigma^{g_1},\tau^{g_1}) = \partial\xi_{g_1g_2}(\sigma,\tau),$$

showing the existence of $\eta(g_1,g_2) \in N$ as above.
(2). Since N is abelian, $\omega(\sigma^n,\tau^n) = \omega(\sigma,\tau)$ holds for each $n \in N$ and we may assume $\xi_{gn} = \xi_g$, $\xi_e = 1$. Note that through the duality we have $\ell^\infty(\widehat{N}) = \mathbb{C}N \subset \mathbb{C}G$. Let $\tilde{g} = \xi_g g \in \mathbb{C}G$ for $g \in G$. Then,

$$\delta_G^\omega(\tilde{g}) = \omega^*\delta_G(\xi_g)(g\otimes g)\omega = \omega^*\delta_G(\xi_g)\partial\xi_g\omega(g\otimes g) = \tilde{g}\otimes\tilde{g},$$

which shows that $\{\tilde{g}\}_{g\in G}$ is the set of group-like elements of $(\mathbb{C}G)^\omega$, and so $G_\omega = \{\tilde{g}\}_{g\in G}$ by definition. For $g,h \in G$, we have

$$\tilde{g}\tilde{h} = \xi_g(g\xi_h g^{-1})gh = \xi_g(g\xi_h g^{-1})\xi_{gh}^*\tilde{gh} = \eta(g,h)\tilde{gh},$$

which proves the lemma. ∎

Remark 4.4. Since $\widehat{N}$ is a finite abelian group, it has a unique direct product decomposition $\widehat{N}_e \times \widehat{N}_o$, where $\widehat{N}_e$ is a 2-group and $\widehat{N}_o$ is an odd group. Due to $H^2(\widehat{N},\mathbb{T}) = H^2(\widehat{N}_e,\mathbb{T}) \times H^2(\widehat{N}_o,\mathbb{T})$ (see [29]) we may assume $\omega = \omega_e\omega_o$ with $\omega_e \in Z^2(\widehat{N}_e,\mathbb{T})$, $\omega_o \in Z^2(\widehat{N}_o,\mathbb{T})$. Since β_{ω_o} has odd order, the class $[\omega_o]$ has a representative of the form $\beta_{\omega_o}^m$ for some $m \in \mathbb{N}$. Therefore, if $[\omega_o]$ is fixed by the G-action, we can choose a representative ω_o invariant under the G-action. Thus, ω_o has no contribution to η.

From now on, we consider the case $G = N \rtimes H$ with $N = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ and $H \subset SL(2,\mathbb{Z}/m\mathbb{Z})$. As shown in Remark 4.4, the case where m is a power of 2 is essential for our purpose and we assume so. We identify G/N with H and regard η above as an element of $Z^2(H,N)$. We identify $\widehat{N}$ with $\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/m\mathbb{Z}$ via the paring $\langle(n_1,n_2),(\tau_1,\tau_2)\rangle = \zeta_m^{n_1\tau_1+n_2\tau_2}$ where $\zeta_m = \exp(\frac{2\pi i}{m})$. For $n \in N$, $\tau \in \widehat{N}$, and

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}/m\mathbb{Z}),$$

we use the convention ${}^t(n^h) = h^{-1}\cdot {}^t n$ so that $\tau^h = \tau\cdot h = (a\tau_1 + c\tau_2, b\tau_1 + d\tau_2)$. We define a cocycle $\omega \in Z^2(\widehat{N},\mathbb{T})$ by $\omega((x_1,x_2),(y_1,y_2)) = \zeta_m^{x_1y_2}$. Then, since β_ω is invariant under the $SL(2,\mathbb{Z}/m\mathbb{Z})$-action, G, N, and ω satisfy the assumption of Lemma 4.3. It is a routine work to show the following:

Lemma 4.5. *For* $(s,t) \in \widehat{N}$ *and*

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}/m\mathbb{Z}),$$

a general solution of the equation ${}^h\omega = \partial\xi_h\omega$ *in Lemma 4.3, (1) is given by*

$$\xi_h(s,t) = \xi_h(1,0)^s\xi_h(0,1)^t\zeta_m^{-\left[\frac{abs(s-1)}{2}+\frac{cdt(t-1)}{2}+bcst\right]},$$

where $\xi_h(1,0)$ *and* $\xi_h(0,1)$ *satisfy* $\xi_h(1,0)^m = (-1)^{ab}$, $\xi_h(0,1)^m = (-1)^{cd}$. *In particular, the following is a solution*:

$$\xi_h(s,t) = \zeta_{2m}^{-(abs^2+cdt^2+2bcst)} \quad \text{with } 0 \le a,b,c,d < m.$$

We fix ξ_h as in the last formula in the above lemma. We consider the case $m = 4$ with $H = \{1, h_1, h_2, h_3\} \subset SL(2, \mathbb{Z}/4\mathbb{Z})$, where

$$h_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then, direct computation shows

$$\xi_{h_1}(s,t) = \zeta_4^{-s^2}, \quad \xi_{h_2}(s,t) = \zeta_4^{-t^2}, \quad \xi_{h_3}(s,t) = \zeta_4^{-(s^2+t^2)}.$$

Let $\{n_1, n_2\}$ be the natural basis of N, and we set $n_3 = n_1n_2$. We have

$$\eta(h_i, h_i) = n_i^2 \quad (i = 1,2,3),$$
$$\eta(h_1, h_2) = \eta(h_2, h_1) = 1,$$
$$\eta(h_1, h_3) = \eta(h_3, h_1) = n_1^2, \quad \eta(h_2, h_3) = \eta(h_3, h_2) = n_2^2.$$

We set $k_i = \tilde{h}_i$, $i = 1,2,3$. Then, G_ω is generated by $\{n_i\}\bigcup\{k_i\}$, and they satisfy the following relations:

$$k_1n_1 = n_1k_1, \quad k_2n_2 = n_2k_2,$$
$$k_1n_2 = n_1^2n_2k_1, \quad k_2n_1 = n_1n_2^2k_2,$$
$$k_i^2 = n_i^2 \quad (i = 1,2,3),$$
$$k_1k_2 = k_2k_1 = k_3,$$
$$k_1k_3 = k_3k_1 = n_1^2, \quad k_2k_3 = k_3k_2 = n_2^2.$$

Theorem 4.6. *With the 2-cocycle* ω *constructed so far the two groups* G *and* G_ω *are non-isomorphic.*

Proof. Let $Z(G)$, $Z(G_\omega)$ be the centers of G and G_ω respectively. Then, we have $Z(G) = Z(G_\omega) = \{1, n_1^2, n_2^2, n_3^2\}$. We show that there exists no subgroup $K \subset G_\omega$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with trivial intersection with $Z(G_\omega)$. Note that this will prove the theorem as G does have such a subgroup.

Let $G_\omega^{(2)}$ be the set of order 2 elements of G_ω. Then, direct computation shows

$$G_\omega^{(2)} = (Z(G_\omega) \setminus \{e\}) \cup c_1Z(G_\omega) \cup c_2Z(G_\omega) \cup c_3Z(G_\omega) \cup c_4Z(G_\omega),$$

where $c_1 = n_1k_1$, $c_2 = n_2k_2$, $c_3 = n_1k_3$, $c_4 = n_2k_3$. Suppose that there exists a subgroup $K \subset G_\omega$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with trivial intersection with $Z(G_\omega)$. Note that K cannot include two elements of the forms z_3c_3 and z_4c_4 with $z_3, z_4 \in$

$Z(G_\omega)$ because $c_3c_4 = n_1n_2^3$. Thus, K should include two elements z_1c_1 and z_2c_2 for some $z_1, z_2 \in Z(G_\omega)$. On the other hand, we have $c_1c_2 = n_1^3n_2k_3$, $c_2c_1 = n_1n_2^3k_3$, which shows that this is impossible. Therefore, there is no such K. □

Remark 4.7. A similar strategy works for $H = SL(2, \mathbb{Z}/4\mathbb{Z})$. Let Γ and Γ_ω be the subgroup generated by the order 2 elements of G and G_ω respectively. Then, we have $Z(\Gamma) = Z(\Gamma_\omega) = \{1, n_1^2, n_2^2, n_3^2\}$. It is possible to show that there is no subgroup of Γ_ω isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with trivial intersection with $Z(\Gamma_\omega)$, and hence we can conclude that G_ω is not isomorphic to G.

5. Group-subgroup subfactors

Let R be an infinite factor equipped with an outer action α of a finite group G. We assume that a subgroup H of G contains no non-trivial normal subgroup of G (see [19, Proposition 3.1] for its meaning), and R^H, R^G be the fixed-point algebras under the H and G-actions respectively. In this section, we determine the structure of $H^2(R^H \supset R^G)$, which will be parameterized by relative cohomology-like objects.

Definition 5.1. Let $G \supset H$ be a pair of finite groups. We say that a 2-cocycle $\omega \in Z^2(G, \mathbb{T})$ is *H-invariant* if it satisfies

$$\omega(hg_1, g_2) = \omega(g_1, g_2h) = \omega(g_1h, h^{-1}g_2) = \omega(g_1, g_2) \text{ for } g_1, g_2 \in G \text{ and } h \in H.$$

We say that two H-invariant 2-cocycles ω_1, ω_2 are *equivalent* if there exists a two-sided H-invariant function $\eta : G \longrightarrow \mathbb{T}$ satisfying

$$\omega_2(g_1, g_2) = \omega_1(g_1, g_2)\eta(g_1g_2)\overline{\eta(g_1)\eta(g_2)}.$$

We denote by $Z_H^2(G, \mathbb{T})$ and $B_H^2(G, \mathbb{T})$ the set of H-invariant 2-cocycles and the set of H-invariant 2-cocycles equivalent to 1 respectively. The *H-invariant cohomology group* $H_H^2(G, \mathbb{T})$ is the quotient group of $Z_H^2(G, \mathbb{T})$ by $B_H^2(G, \mathbb{T})$.

Remark 5.2. When G is a semi-direct product $G = N \rtimes H$, an H-invariant cocycle $\omega \in Z_H^2(G, \mathbb{T})$ is determined by its restriction $\omega|_N \in Z^2(N, \mathbb{T})$ to N and $Z_H^2(G, \mathbb{T})$ can be identified with the subset of $Z^2(N, \mathbb{T})$ consisting of those elements ω satisfying

$$\omega(n_1^h, n_2^h) = \omega(n_1, n_2) \text{ for } n_1, n_2 \in N \text{ and } h \in H.$$

The corresponding coboundaries are given by functions $\eta : N \longrightarrow \mathbb{T}$ satisfying

$$\eta(n^h) = \eta(n) \quad (n \in N,\ h \in H).$$

Therefore, $H_H^2(G, \mathbb{T})$ is isomorphic to a subgroup of the relative cohomology group $H^2(G/N, G; \mathbb{T}, \mathbb{T})$ (see [10, Introduction] for instance for the definition).

We will use the model inclusion for $R^H \supset R^G$ constructed in [13]. We denote by $\dot{g}$ $(g \in G)$ the left coset gH. We fix a left transversal Ω of G/H with $e \in \Omega$, and let $\mathcal{O}_n$ (with $n = [G : H]$) be the Cuntz algebra with generators $\{S_{\dot{g}}\}_{g \in \Omega}$. We denote by $\mathcal{H}$ the linear span of $\{S_{\dot{g}}\}_{g \in \Omega}$ as before. Let θ be an action of G on $\mathcal{O}_n$ defined through the left translation of G on G/H, and $\mathcal{O}_H, \mathcal{O}_G$ be the fixed-point algebras of $\mathcal{O}_n$ under the H and G-actions. We define an endomorphism $\gamma_{G/H}$ of $\mathcal{O}_H$ by

$$\gamma_{G/H}(x) = \sum_{g\in\Omega} S_{\dot{g}}\theta_g(x)S_{\dot{g}}^*. \tag{5.1}$$

Note that this expression makes sense only for $x \in \mathcal{O}_H$ and does not depend on the choice of Ω. Let $\sigma_{\mathcal{H}}$ be the canonical shift of $\mathcal{O}_n$ and σ_G be its restriction to $\mathcal{O}_G$. Note that the restriction of $\gamma_{G/H}$ to $\mathcal{O}_G$ coincides with σ_G and so $\gamma_{G/H}^2 = \sigma_G \cdot \gamma_{G/H}$. The following statements are either well-known or easily deduced from well-known facts (see [7], [13] for example):

Lemma 5.3. (1) *The relative commutant $\mathcal{O}_n \cap \mathcal{O}_G'$ is trivial.*
(2) $(\sigma_G^m, \sigma_G^n) = (\mathcal{H}^n\mathcal{H}^{*m})_G$, *where $(\mathcal{H}^n\mathcal{H}^{*m})_G$ is the G-invariant part of $\mathcal{H}^n\mathcal{H}^{*m}$.*
(3) $(\gamma_{G/H}^m, \gamma_{G/H}^n) \subset \mathcal{H}^n\mathcal{H}^{*m}$.
(4) *The subspace $\{X \in \mathcal{O}_n;\ X\theta_{g_1}(x) = \theta_{g_2}(x)X$ for each $x \in \mathcal{O}_H\}$ (with $g_1, g_2 \in G$) is non-zero if and only if $g_1H = g_2H$.*

We set

$$S_0 = S_{\dot{e}} \quad \text{and} \quad T_0 = \frac{1}{\sqrt{n}}\sum_{g\in\Omega} S_{\dot{g}}.$$

As in §3, we can show that $(\gamma_{G/H}, S_0, T_0)$ is a canonical model of the Q-system for the inclusion $R^H \supset R^G$ (see [13]).

Theorem 5.4. *For a Q-system $(\gamma_{G/H}, S, T) \in Q(\gamma_{G/H})$, there exists a unique H-invariant 2-cocycle $\omega \in Z_H^2(G, \mathbb{T})$ satisfying*

$$S = \omega(e,e)S_{\dot{e}} \quad \text{and} \quad T = \frac{1}{\sqrt{n}}\sum_{g,k\in\Omega} \omega(g, g^{-1}k)S_{\dot{g}}S_{\dot{k}}S_{\dot{k}}^*,$$

and every $\omega \in Z_H^2(G, \mathbb{T})$ is obtained in this way. Furthermore, two Q-systems are equivalent if and only if the corresponding H-invariant 2-cocycles are equivalent. Therefore, we can identify $H^2(R^H \supset R^G)$ with $H_H^2(G, \mathbb{T})$.

Proof. Let $(\gamma_{G/H}, S, T)$ be a Q-system. The intertwining property $\gamma_{G/H}(x)S_{\dot{g}} = S_{\dot{g}}\theta_g(x)$ for $x \in \mathcal{O}_H$ (see (5.1)) and $T \in (\gamma_{G/H}, \gamma_{G/H}^2)$ yield

$$S_{\dot{g}_2}^*S_{\dot{g}_1}^*TS_{\dot{g}_3}\theta_{g_3}(x) = \theta_{g_2}(x)S_{\dot{g}_2}^*S_{\dot{g}_1}^*TS_{\dot{g}_3}\ (x \in \mathcal{O}_H) \text{ for } g_1, g_2, g_3 \in \Omega.$$

Thus, $S_{\dot{g}_2}^*S_{\dot{g}_1}^*TS_{\dot{g}_3}$ is either 0 or a scalar (when $g_2 = g_3$) by Lemma 5.4, and one gets scalars $\omega_0(\dot{g}, \dot{k})$ $(g, k \in \Omega)$ satisfying

$$T = \frac{1}{\sqrt{n}}\sum_{g,k\in\Omega} \omega_0(\dot{g}, \dot{k})S_{\dot{g}}S_{\dot{k}}S_{\dot{k}}^*.$$

Since T is an isometry, we must have

$$\sum_{g\in\Omega} |\omega_0(g,k)|^2 = n \quad (k \in G). \tag{5.2}$$

On the other hand, since $T \in \mathcal{O}_H$, we must have the invariance $\omega_0((\dot{hg}), (\dot{hk})) = \omega_0(\dot{g}, \dot{k})$ for $h \in H$. Hence, the function $\omega(g,k) = \omega_0(\dot{g}, \dot{gk})$ on $G \times G$ satisfies

$$\omega(hg,k) = \omega(g,kh) = \omega(gh,h^{-1}k) = \omega(g,k) \quad \text{for each } g,k \in G \text{ and } h \in H,$$

and T is expressed as

$$\begin{aligned} T = \frac{1}{\#H\sqrt{n}} \sum_{g\in\Omega, k\in G} \omega(g,g^{-1}k) S_{\dot{g}} S_{\dot{k}} S_{\dot{k}}^* &= \frac{1}{\#H\sqrt{n}} \sum_{g\in\Omega, k\in G} \omega(g,k) S_{\dot{g}} S_{\dot{gk}} S_{\dot{gk}}^* \\ &= \frac{1}{\sqrt{n}} \sum_{g,k\in\Omega} \omega(g,k) S_{\dot{g}} S_{\dot{gk}} S_{\dot{gk}}^*. \end{aligned}$$

Since $(\mathrm{Id}, \gamma_{G/H}) = \mathbb{C}S_0$, we have $S = cS_0$ for some $c \in \mathbb{T}$. Thus, the requirement (2.1) in Definition 2.1 is equivalent to $\omega(e,g) = \omega(g,e) = c$. We next show that (2.2) and (2.3) are equivalent to the 2-cocycle relation. Indeed, we compute

$$\begin{aligned} \gamma_{G/H}(T)T &= \frac{1}{\sqrt{n}} \sum_{g,k\in\Omega} \omega(g,k) S_{\dot{g}} \theta_g(T) S_{\dot{gk}} S_{\dot{gk}}^* = \frac{1}{\sqrt{n}} \sum_{g,k\in\Omega} \omega(g,k) \theta_g(S_{\dot{e}} T S_{\dot{k}} S_{\dot{k}}^*) \\ &= \frac{1}{n} \sum_{g,k,l,m\in\Omega} \omega(g,k)\omega(l,m) \theta_g(S_{\dot{e}} S_{\dot{l}} S_{\dot{lm}} S_{\dot{lm}}^* S_{\dot{k}} S_{\dot{k}}^*) \\ &= \frac{1}{n} \sum_{g,l,m\in\Omega} \omega(g,lm)\omega(l,m) \theta_g(S_{\dot{e}} S_{\dot{l}} S_{\dot{lm}} S_{\dot{lm}}^*). \end{aligned}$$

On the other hand, we have

$$T^2 = \frac{1}{\sqrt{n}} \sum_{g,l\in\Omega} \omega(g,l) \theta_g(S_{\dot{e}} S_{\dot{l}} S_{\dot{l}}^*) T = \frac{1}{n} \sum_{g,l,m\in\Omega} \omega(g,l)\omega(gl,m) \theta_g(S_{\dot{e}} S_{\dot{l}} S_{\dot{lm}} S_{\dot{lm}}^*)$$

thanks to $\omega(nh,m) = \omega(n,hm)$ $(m,n \in G,\ h \in H)$. Hence, the requirement (2.3) means the cocycle relation

$$\omega(g,lm)\omega(l,m) = \omega(g,l)\omega(gl,m).$$

Note that we must show $|\omega(g,h)| = 1$ $(g,h \in G)$. But, in a similar way as above $TT^* = \gamma_{G/H}(T)^*T$ (i.e., (2.2)) implies

$$\omega(l,m)\overline{\omega(g,l)} = \omega(gl,m)\overline{\omega(g,lm)}.$$

By setting $l = e$ here, we conclude $|\omega(g,m)| = 1$ thanks to (5.2).

The above computation also shows that every H-invariant cocycle gives a Q-system and this correspondence is one-to-one. Also, close examination of the above computations shows that the correspondence preserves the equivalence relations. □

Remark 5.5. A few remarks are in order.
(1) For $\omega \in Z_H^2(G, \mathbb{T})$ we may always assume $\omega(g,e) = \omega(e,g) = 1$ for each $g \in G$ up to equivalence. Then, the H-invariance of ω shows

$$\omega(h,g) = \omega(g,h) = 1 \text{ for each } g \in G,\ h \in H.$$

which is, in fact, equivalent to the H-invariance for $\omega \in Z^2(G,\mathbb{T})$ satisfying the above requirement. For such a cocycle, we can find unitaries $\{u_g\}_{g\in G} \subset R$ satisfying

$$u_{gk} = \omega(g,k)u_g\alpha_g(u_k)\ (g,k \in G) \quad \text{and} \quad u_h = 1 \text{ for } h \in H.$$

Let P be the fixed-point algebra under the perturbed action $\beta = Ad(u_g)\cdot\alpha_g$,. The class of the Q-system for the inclusion R^H $(= R^{(\alpha,H)} = R^{(\beta,H)}) \supset P$ probably corresponds to the class of ω in the theorem.

(2) The kernel of the natural homomorphism from $H^2_H(G,\mathbb{T})$ to $H^2(G,\mathbb{T})$ is not trivial in general. Indeed, let us assume $\omega(g,k) = \xi(g)\xi(k)\overline{\xi(gk)}$ with a function $\xi : G \to \mathbb{T}$. Then, ω falls into $Z^2_H(G,\mathbb{T})$ if and only if ξ is a character on H and

$$\xi(hg) = \xi(gh) = \xi(h)\xi(g), \quad g \in G,\ h \in H.$$

On the other hand, $\omega \in B^2_H(G,\mathbb{T})$ if and only if ξ extends to a character of G. Let G be a finite group with $[G,G] = G$ and H be a non-normal subgroup with $\#H = 2$. The non-trivial character on H cannot be extended to the whole group G (since G does not admit any), and hence there is a non-trivial H-invariant cocycle that is a coboundary in $B^2(G,\mathbb{T})$.

(3) When G is a semi-direct product $G = N \rtimes H$, there is a natural homomorphism from $H^2_H(G,\mathbb{T})$ to $H^2(N,\mathbb{T})$ (recall Remark 5.2). We now describe the kernel. Assume that $\omega \in Z^2_H(G,\mathbb{T})$ and that there is $\eta : N \longrightarrow \mathbb{T}$ satisfying

$$\omega(n_1,n_2) = \eta(n_1)\eta(n_2)\overline{\eta(n_1n_2)}, \quad n_1,n_2 \in N.$$

By the invariance $\omega(n_1^h,n_2^h) = \omega(n_1,n_2)$, $h \in H$ (Remark 5.2) one can choose a character χ_h (for each $h \in H$) of N satisfying

$$\eta(n^h) = \chi_h(n)\eta(n), \quad n \in N,\ h \in H.$$

This $\chi = \{\chi_h\}_{h\in H}$ gives an element of $Z^1(H,\mathrm{Hom}(N,\mathbb{T}))$. If there exists an H-invariant function $\zeta : N \longrightarrow \mathbb{T}$ satisfying

$$\omega(n_1,n_2) = \zeta(n_1)\zeta(n_2)\overline{\zeta(n_1n_2)}, \quad n_1,n_2 \in N,$$

then there exists a character τ of N satisfying $\zeta(n) = \tau(n)\zeta(n)$, $n \in N$. This happens if and only if $\chi \in B^1(H,\mathrm{Hom}(N,\mathbb{T}))$. Therefore, there exists an injection from the kernel into $H^1(H,\mathrm{Hom}(N,\mathbb{T}))$. This actually comes from the exact sequence

$$0 \to H^1(G/N,\mathrm{Hom}(N,\mathbb{T})) \to H^2(G/N,G;\mathbb{T},\mathbb{T}) \to H^2(N,\mathbb{T}).$$

Corollary 5.6. *For an inclusion $M \supset N$ with the principal graph $E_6^{(1)}$ we have $\#H^2(M \supset N) = 2$.*

Proof. The alternative group $\mathfrak{A}_4$ of degree 4 is a semi-direct product $(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z})\rtimes\mathbb{Z}/3\mathbb{Z}$, where the generator of $\mathbb{Z}/3\mathbb{Z}$ acts as the cyclic permutation of the non-trivial elements in $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. It is known ([11, 13]) that $M \supset N$ is always of the form $R^{\mathbb{Z}/3\mathbb{Z}} \supset R^{\mathfrak{A}_4}$. Since we can choose a $\mathbb{Z}/3\mathbb{Z}$-invariant representative of the unique non-trivial element in $H^2(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},\mathbb{T})$, we get a non-trivial element in $H^2_{\mathbb{Z}/3\mathbb{Z}}(\mathfrak{A}_4,\mathbb{T})$ thanks to Remark 5.2. The result now follows from Remark 5.2 and Remark 5.5,(3) as $H^1(\mathbb{Z}/3\mathbb{Z},\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},\mathbb{T}))$ is easily shown to be trivial. $\square$

Remark 5.7. For the dual inclusion of $R^H \supset R^G$ or a little broader class of subfactors the following conjecture is plausible: Let θ be an ergodic action of G on a finite-dimensional von Neumann algebra A, and we set $M = (A \otimes R)^G \supset N = (\mathbb{C} \otimes R^G)$. The basic extension M_1 for $M \supset N$ is identified with $(\mathbb{B}(L^2(A)) \otimes R)^G$ with the GNS Hilbert space $L^2(A)$ of the unique G-invariant trace on A. In view of [27], we conjecture that $H^2(M_1 \supset M)$ is in one-to-one correspondence with the set of conjugacy classes of ergodic G-actions on B such that the natural representation of G on $L^2(B)$ is equivalent to that on $L^2(A)$. Note that for $A = \ell^2(G/H)$ (with the natural G-action), $M \supset N$ is isomorphic to $R^H \supset R^G$, and so it is well-known that $M_1 \supset M$ is isomorphic to $P \rtimes G \supset P \rtimes H$ for some factor P. For the special case $H = \{e\}$, the conjecture is certainly true thanks to what was mentioned in the paragraph before Corollary 3.4 and the one-to-one correspondence (in [32]) between the full multiplicity ergodic actions and $H^2(\widehat{G})$.

6. Small Index Cases

In this section, we determine the structure of $H^2(M \supset N)$ for subfactors of index less than or equal to 4.

Lemma 6.1. *Let $M \supset N$ be an irreducible inclusion of factors of finite index, and*

$$N \subset M \subset M_1 \subset M_2 \subset \cdots$$

be the corresponding Jones tower. If $M_1 \cap N' \cong \mathbb{C} \oplus \mathbb{C}$ and $M_2 \cap N' \cong M_2(\mathbb{C}) \oplus \mathbb{C}$, then $H^2(M \supset N)$ is trivial.

Proof. The assumption $M_2 \cap N' \cong M_2(\mathbb{C}) \oplus \mathbb{C}$ means that the inclusion map ρ of N into M satisfies $[\rho\overline{\rho}\rho] = 2[\rho] \oplus [\sigma]$ with an irreducible sector σ. Let P be a subfactor of N whose canonical endomorphism is equivalent to that of N, and ρ' be its inclusion map into M. The assumption $M_1 \cap N' \cong \mathbb{C} \oplus \mathbb{C}$ and the Frobenius reciprocity guarantee

$$\dim(\overline{\rho}'\rho, \overline{\rho}'\rho) = \dim(\rho\overline{\rho}, \rho'\overline{\rho}') = \dim(\rho\overline{\rho}, \rho\overline{\rho}) = (\overline{\rho}\rho, \overline{\rho}\rho) = 2, \tag{6.1}$$

$$[\overline{\rho}'\rho] = [\mu_1] \oplus [\mu_2] \tag{6.2}$$

with irreducible sectors μ_1, μ_2. Another use of the Frobenius reciprocity shows that both of $\rho'\mu_1$ and $\rho'\mu_2$ contain ρ with multiplicity one. On the other hand, we have

$$[\rho'\mu_1] \oplus [\rho'\mu_2] = [\rho'\overline{\rho}'\rho] = [\rho\overline{\rho}\rho] = 2[\rho] \oplus [\sigma]$$

so that one of μ_i's, say μ_1, must satisfy $[\rho'\mu_1] = [\rho]$. This means $d(\mu_1) = 1$ and $[\rho'] = [\rho\overline{\mu_1}]$ ($\in \mathrm{Sect}(M, P)$), showing that P is inner conjugate to N. $\square$

Among irreducible subfactors with indices less than or equal to 4 (the reducible cases can be easily treated), those not covered by the above lemma are the ones with the principal graphs D_4, E_6, $D_n^{(1)}$, and $E_6^{(1)}$. For D_4, $D_4^{(1)}$, and $E_6^{(1)}$, we already know the answers: the first two cases are given by fixed-point algebras under finite group actions and the last was treated in Corollary 5.6. For E_6 and $D_n^{(1)}$ ($n > 4$) we have

Proposition 6.2. *Let $M \supset N$ be an inclusion of factors with the principal graph either E_6 or $D_n^{(1)}$, $n > 4$. Then, $H^2(M \supset N)$ is trivial.*

Proof. We begin with the principal graph E_6. The same notations as in the proof of the previous lemma are used ((6.1),(6.2) are still valid), but we have the irreducible decomposition $[\rho\overline{\rho}\rho] = 2[\rho] \oplus [\sigma_1] \oplus [\sigma_2]$ with dimensions

$$d(\rho) = d(\sigma_1) = \frac{\sqrt{3}+1}{\sqrt{2}}, \quad d(\sigma_2) = \sqrt{2}.$$

As before, $\rho'\mu_1$ and $\rho'\mu_2$ contain ρ with multiplicity one. We claim that either $\rho'\mu_1$ or $\rho'\mu_2$ must be ρ (which shows inner conjugacy as before). Suppose we had $[\rho'\mu_1] = [\rho] \oplus [\sigma_1]$ and $[\rho'\mu_2] = [\rho] \oplus [\sigma_2]$. Then, $\overline{\rho}'\sigma_2$ would contain μ_2 with multiplicity one by the Frobenius reciprocity, and $d(\mu_2) = \sqrt{3}$. Comparing dimensions, we conclude $[\overline{\rho}'\sigma_2] = [\mu_2] \oplus [\theta]$ with $d(\theta) = 1$. But, the Frobenius reciprocity again shows $[\sigma_2] = [\rho'\theta]$, contradicting $d(\sigma_2) \neq d(\rho')$. Therefore, $H^2(M \supset N)$ is trivial for the E_6 case.

Now we assume that the principal graph is $D_n^{(1)}$, $n > 4$, and let ρ, ρ' be as above. Note that (6.1) is replaced by

$$\dim(\overline{\rho}'\rho, \overline{\rho}'\rho) = \dim(\rho\overline{\rho}, \rho'\overline{\rho}') = \dim(\rho\overline{\rho}, \rho\overline{\rho}) = 3$$

so that $\overline{\rho}'\rho$ is decomposed into 3 irreducible sectors. Since $d(\overline{\rho}'\rho) = 4$ $(< 3\sqrt{2})$, $\overline{\rho}'\rho$ must contain a sector of dimension 1, showing that N and P are inner conjugate. □

Summing up the arguments so far, we have shown

Theorem 6.3. *Let $M \supset N$ be an inclusion of factors of index less than or equal to 4. Except for the following two cases, $H^2(M \supset N)$ is trivial:*

(1) *If the principal graph is $D_4^{(1)}$ and N is the fixed point-algebra under a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-action, then $H^2(M \supset N)$ has exactly two elements.*
(2) *If the principal graph is $E_6^{(1)}$, then $H^2(M \supset N)$ has exactly two elements.*

For (1) we note $H^2(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{T}) = \mathbb{Z}/2\mathbb{Z}$ while $H^2(\mathbb{Z}/4\mathbb{Z}, \mathbb{T})$ is trivial (see the paragraph after Definition 2.3). Before ending the final section, we present an easy but handy criterion, showing non-triviality of $H^2(M \supset N)$ for generic $SU(N)_k$-Hecke algebra subfactors (see [33]) for $N \geq 3$ (apply the criterion below to the fundamental representation).

Proposition 6.4. *Let ρ be an irreducible endomorphism of a factor M satisfying $d(\rho) < \infty$. When $[\rho\overline{\rho}] = [\overline{\rho}\rho]$ and ρ^2 contains no sector of dimension 1, we have $\#H^2(M \supset \rho(M)) \geq 2$.*

Proof. It suffices to show that $\rho(M)$ and $\overline{\rho}(M)$ are not inner conjugate. Indeed, if they were, i.e, $\overline{\rho} \cdot \alpha = \mathrm{Ad}(u) \cdot \rho$ with some unitary $u \in M$ and $\alpha \in Aut(M)$, then ρ^2 would contain α thanks to the Frobenius reciprocity, a contradiction. □

References

[1] M. Asaeda and U. Haagerup, *Exotic subfactors of finite depth with Jones index $(5+\sqrt{13})/2$ and $(5+\sqrt{17})/2$*, Comm. Math. Phys., **202** (1999), 1–63.
[2] S. Baaj and G. Skandalis, *Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algébres*, Ann. Sci. École Norm. Sup. (4), **26** (1993), 425–488.

[3] A. Connes, *Periodic automorphisms of the hyperfinite factor of type* II_1, Acta Sci. Math. (Szeged), **39** (1977), 39–66.
[4] J. Cuntz, *Regular actions of Hopf algebras on the C^*-algebra generated by a Hilbert space*, in "Operator Algebras, Mathematical Physics, and Low-dimensional Topology (Istanbul, 1991)", (pp. 87–100), Res. Notes Math. 5, A K Peters, Wellesley MA, 1993.
[5] M.-C. David, *Paragroup d'Adrian Ocneanu et algèbra de Kac*, Pacific J. Math., **172** (1996), 331-363.
[6] S. Doplicher and J. E. Roberts, *A new duality theory for compact groups*, Invent. Math., **98** (1989), 157–218.
[7] S. Doplicher and J. E. Roberts, *Endomorphisms of C^*-algebras, cross products and duality for compact groups*, Ann. of Math., **130** (1989), 75–119.
[8] D. E. Evans and Y. Kawahigashi, *Quantum Symmetries on Operator Algebras*, Oxford Math. Monographs Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1998.
[9] M. Enock and J.-M. Schwartz, *Kac Algebras and Duality of Locally Compact Groups*, Springer-Verlag, Berlin, 1992.
[10] N. Habegger, V. F. R. Jones, O. Pino Oritz and J. Ratcliffe, *Relative cohomology of groups*, Comment. Math. Helvetici, **59** (1984), 149-164.
[11] Jeong Hee Hong, *Subfactors with principal graph $E_6^{(1)}$*, Acta Appl. Math., **40** (1995), 255–264.
[12] M. Izumi, *Application of fusion rules to classification of subfactors*, Publ. Res. Inst. Math. Sci., **27** (1991), 953–994.
[13] M. Izumi, *Goldman's type theorems in index theory*, in "Operator Algebras and Quantum Field Theory (Rome, 1996)", (pp. 249–269), Internat. Press, Cambridge MA, 1997.
[14] M. Izumi, *Subalgebras of infinite C^*-algebras with finite Watatani indices II. Cuntz-Krieger algebras*, Duke Math. J., **91** (1998), 409–461.
[15] M. Izumi, *The structure of sectors associated with Longo-Rehren inclusions II. Examples*, Rev. Math. Phys., **13** (2001), 603-674.
[16] M. Izumi and H. Kosaki, *Kac algebras arising from composition of subfactors: General theory and classification*, to appear in Memoirs Amer. Math. Soc.
[17] V. F. R. Jones, *Actions of finite groups on the hyperfinite type* II_1 *factor*, Memoirs Amer. Math. Soc., **28** (1980), no. 237.
[18] H. Kosaki, *Extension of Jones' theory on index to arbitrary factors*, J. Funct. Anal., **66** (1986), 123–140.
[19] H. Kosaki and S. Yamagami, *Irreducible bimodules associated with crossed product algebras*, Internat. J. Math., **3** (1992), 661–676.
[20] R. Longo, *Index of subfactors and statistics of quantum fields I and II*, Comm. Math. Phys., **126** (1989), 217–247 and **130** (1990), 285–309.
[21] R. Longo, *A duality for Hopf algebras and for subfactors I*, Comm. Math. Phys., **159** (1994), 133–150.
[22] R. Longo and K.-H. Rehren, *Nets of subfactors*, Workshop on Algebraic Quantum Field Theory and Jones Theory (Berlin, 1994), Rev. Math. Phys., **7** (1995), 567–597.
[23] R. Longo and J. E. Roberts, *A theory of dimension*, K-Theory, **11** (1997), 103–159.
[24] Toshihiko Masuda, *An analogue of Longo's canonical endomorphism for bimodule theory and its application to asymptotic inclusions*, Internat. J. Math., **8** (1997), 249–265.
[25] M. Pimsner and S. Popa, *Sur les sous-facteurs d'indice fini d'un facteur de type* II_1 *ayant la propriété T*, C. R. Acad. Sci. Paris Sér. I Math., **303** (1986), 359–361.
[26] S. Popa, *Symmetric enveloping algebras, amenability and AFD properties for subfactors*, Math. Res. Lett., **1** (1994), 409–425.
[27] R. Schaflitzel, II_1*-subfactors associated with the C^*-tensor category of a finite group*, Pacific J. Math., **184** (1998), 333–348.
[28] C. E. Sutherland, *Cohomology and extensions of von Neumann algebras II*, Publ. Res. Inst. Math. Sci., **16** (1980), 135–174.

[29] M. Suzuki, *Group Theory I*, Grundlehren der Mathematischen Wissenschaften 247, Springer-Verlag, Berlin-New York, 1982.
[30] W. Szymański, *Finite index subfactors and Hopf algebra crossed products*, Proc. Amer. Math. Soc., **120** (1994), 519-528.
[31] D. Tambara and S. Yamagami, *Tensor categories with fusion rules of self-duality for finite abelian groups*, J. Algebra, **209** (1998), 692–707.
[32] A. Wassermann, *Ergodic actions of compact groups on operator algebras II. Classification of full multiplicity ergodic actions*, Canad. J. Math., **40** (1988), 1482–1527.
[33] H. Wenzl, *Hecke algebras of type A_n and subfactors*, Invent. Math., **92** (1988), 349–383.
[34] S. Yamagami, *Group symmetry in tensor categories and duality for orbifold*, preprint.

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