Type III Factors arising from Cuntz-Krieger Algebras

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Abstract

We determine the types of factors arising from GNS-representations of quasi-free KMS states on Cuntz-Krieger algebras. Applying our result to the Cuntz-Krieger algebras arising from the boundary actions of some amalgamated free product groups, we also determine the types of the harmonic measures on the boundaries.

1 Introduction

The Cuntz algebra $\mathcal{O}_n$ [Cun] and the Cuntz-Krieger algebra $\mathcal{O}_A$ [CK], a generalization of $\mathcal{O}_n$, are important examples of C^*-algebras. The Cuntz-Krieger algebra $\mathcal{O}_A$, associated with a 0-1 matrix A, is the universal C^*-algebra generated by the family of partial isometries $\{S_i\}_{i=1}^N$ satisfying the Cuntz-Krieger relations. The universal property of $\mathcal{O}_A$ allows us to define the so-called gauge action α on $\mathcal{O}_A$ by

$$\alpha_t(S_i) = e^{\sqrt{-1}t} S_i \quad \text{for } t \in \mathbb{R}.$$

The existence of KMS states for one-parameter automorphisms is one of the natural questions. The KMS states for the gauge actions on $\mathcal{O}_n$ and $\mathcal{O}_A$ were obtained by D. Olesen and G. K. Pedersen [OP], and M. Enomoto, M. Fujii and Y. Watatani [EFW], respectively. More generally, D. E. Evans determined the KMS states on $\mathcal{O}_n$ for the quasi-free actions in [Eva]. In order to construct examples of subfactors, M. Izumi determined the types of factors obtained by the GNS-representations of quasi-free KMS states in [Izu]. One of the purposes in this paper is to generalize his result to Cuntz-Krieger algebras. The existence and the uniqueness of quasi-free KMS states on Cuntz-Krieger algebras were proved by R. Exel and M. Laca in [EL]. It implies that the von Neumann algebras arising from their GNS-representations are factors. We will compute the Connes spectrum of the modular automorphism group and determine the types of quasi-free KMS states.

As an application, we can give a construction of type III factors from geometric objects. J. Spielberg proved in [Spi] that some Cuntz-Krieger algebras can be obtained by the crossed product construction of the boundary action $(\partial\Gamma, \Gamma)$, where Γ is the free product of cyclic groups and $\partial\Gamma$ is the hyperbolic boundary as a hyperbolic group. This construction was generalized to amalgamated free product groups in [Oka]. Under this identification, it was shown that there is one-to-one correspondence between quasi-free KMS states and some class of random walks on Γ. Namely, by identifying $\partial\Gamma$ with the Poisson boundary, harmonic measures on $\partial\Gamma$ induce quasi-free KMS states. We will apply the main result to the harmonic measures and determine the types of them. It turns out that the resulting factors are either of type III_1 or of type III_λ $(0 < \lambda < 1)$, where λ is some algebraic number. Therefore, by combining these results, we can make type III factors from boundary actions and harmonic measures on the boundary, which generalizes J. Ramagge and G. Robertson's result in [RR].

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2 Preliminaries

2.1 Perron-Frobenius theorem

Let $A = [A(i,j)]_{i,j=1}^N$ be an $N \times N$ matrix with non-negative entries. We denote the (i,j)-entry of A^m by $A^m(i,j)$. A matrix A is *irreducible* if for every pair of indices i and j there is an $m > 0$ with $A^m(i,j) > 0$. For $1 \leq i,j \leq N$, put $E(i,j) = \{m \in \mathbb{N} \mid A^m(i,j) > 0\}$ and $p(i) = \text{g.c.d.}\{m \in \mathbb{N} \mid A^m(i,i) \neq 0\}$. Note that if A is irreducible, then $p \equiv p(i)$ for any i and we call it the *period* of A. An irreducible matrix A is said to be *periodic of period* p if $p > 1$ and *aperiodic* if $p = 1$. Set $I_k = \{i \mid 1 \leq i \leq N, E(i,1) = k-1 \pmod{p}\}$ for $k = 1, \ldots, p$. If A is periodic, then the index set $\{1, \ldots, N\}$ can be decomposed into distinct subsets $I_1, \ldots, I_p$ such that the matrix A translates from I_k into I_{k+1}, (I_p into I_1), and the restriction of A^p to I_k is aperiodic. If A is irreducible, the Perron-Frobenius theorem guarantees the existence of the strictly positive eigenvector with respect to the simple root α of the characteristic polynomial such that $\alpha \geq |\beta|$ for any other eigenvalue β. Moreover, the following theorem is known.

Theorem 2.1 ([Kit, Theorem 1.3.8]) *Let A be an irreducible matrix with non-negative entries and p the period of A. If $x = {}^T(x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$ are the right and left Perron eigenvectors of the Perron eigenvalue α such that $\sum_{i=1}^N x_i y_i = p$, then*

$$\lim_{n\to\infty} A^{pn}(i,j)/\alpha^{pn} = x_i y_j,$$

for any $i, j = 1, \ldots, N$.

2.2 Cuntz-Krieger algebras

Let A be an $N \times N$ matrix with 0-1 entries. Then the Cuntz-Krieger algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by the family of partial isometries $S_1, \ldots, S_N$ satisfying: $S_i^* S_i = \sum_{j=1}^N A(i,j) S_j S_j^*$, and $1 = \sum_{i=1}^N S_i S_i^*$. For $i = 1, \ldots, N$, let us denote the initial projection of S_i by Q_i and the range projection by P_i. We say that $\xi = (\xi_1, \ldots, \xi_n) \in \prod_{i=1}^n \{1, \ldots, N\}$ with $A(\xi_i, \xi_{i+1}) \neq 0$ is an *admissible word* and denote the set of all admissible words by $\mathcal{W}_A$. We define two maps s and r by $s(\xi) = \xi_1$ and $r(\xi) = \xi_n$. For $\xi = (\xi_1, \ldots, \xi_n), \eta = (\eta_1, \ldots, \eta_m) \in \mathcal{W}_A$ with $A(\xi_n, \eta_1) = 1$, we define the concatenation $\xi \cdot \eta$ in $\mathcal{W}_A$ by $(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m)$. Let us say that an admissible word $\xi = (\xi_1, \ldots, \xi_n)$ is a *loop* if $A(\xi_n, \xi_1) = 1$. We say that a loop ξ is a *circle* if $\xi_k \neq \xi_l$ for any $1 \leq k, l \leq n$, ($k \neq l$).

Let $\omega = (\omega_1, \ldots, \omega_N) \in \mathbb{R}_+^N$. We define the action α^ω of $\mathbb{R}$ on $\mathcal{O}_A$ by

$$\alpha_t^\omega(S_i) = e^{\sqrt{-1}\omega_i t} S_i$$

for $t \in \mathbb{R}$ and $i = 1, \ldots, N$. If $\omega = (1, \ldots, 1)$, then α^ω is the gauge action. We define two word-length functions. For $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{W}_A$, we denote the canonical one by $|\xi| = n$ and the other by $\omega_\xi = \omega_{\xi_1} + \cdots + \omega_{\xi_n}$. Let $\Omega_A = \{(a_k)_{i=k}^\infty \mid A(a_k, a_{k+1}) = 1\}$ be the set of all one-sided infinite admissible words. Note that there is the faithful conditional expectation Φ from $\mathcal{O}_A$ onto $\overline{\mathrm{span}}\{S_\xi S_\xi^* \mid \xi \in \mathcal{W}_A\} \simeq C(\Omega_A)$ (see [CK]). We assume that there is $\beta \in \mathbb{R}_+$ and $x_i > 0$ that satisfies:

$$x_i = \sum_{j=1}^{N} e^{-\beta\omega_i} A(i,j) x_j,$$

and

$$1 = x_1 + \cdots + x_N.$$

We can define a probability measure ν on Ω_A by

$$\nu(\Omega_A(\xi_1, \ldots, \xi_{n-1}, \xi_n)) = e^{-\beta\omega_{\xi_1}} \cdots e^{-\beta\omega_{\xi_{n-1}}} x_{\xi_n},$$

where $\Omega_A(\xi_1, \ldots, \xi_n)$ is the cylinder set $\{(a_k)_{k=1}^\infty \in \Omega_A \mid a_1 = \xi_1, \ldots, a_n = \xi_n\}$. This probability measure induces a β-KMS state for α^ω on $\mathcal{O}_A$ by $\phi^\omega = \nu \circ \Phi$. Set $A_\omega(i,j) = e^{-\beta\omega_i} A(i,j)$. Note that the vector $x = {}^T(x_1, \ldots, x_N)$ is the right Perron eigenvector of the matrix A_ω with respect to the Perron eigenvalue 1. R. Exel and M. Laca, in fact, showed the following in [EL].

Theorem 2.2 ([EL, Theorem 18.5]) *If A is irreducible, then there exists the unique β-KMS state ϕ^ω of the Cuntz-Krieger algebra $\mathcal{O}_A$ for the action α^ω and the inverse temperature β is also unique.*

Throughout this paper, we assume that A is irreducible and not a permutation matrix. Let $(\pi_{\phi^\omega}, H_{\phi^\omega}, \xi_{\phi^\omega})$ be the GNS-triple of ϕ^ω. The above theorem, in particular, says that the von Neumann algebra $M = \pi_{\phi^\omega}(\mathcal{O}_A)''$ becomes a factor.

2.3 AF-algebras

The following results are based on [SV, Theorem I.3.12.]. Consider an AF-algebra $B = \overline{\bigcup_{n\geq 0} B_n}$, where $\{B_n\}_{n=0}^\infty$ is an increasing family of finite C^*-subalgebras. We assume that $B_0 = \mathbb{C}1$. We define a maximal abelian subalgebra C of B as follows. Let $C_0 = B_0$ and C_{n+1} the C^*-subalgebra generated by C_n and D_{n+1}, where D_{n+1} is a masa of B_{n+1}, containing C_n. We define $C = \overline{\bigcup_{n\geq 0} C_n}$. One can check that C is a masa of B. There is a conditional expectation Ψ from B onto C, and there is a topological dynamical system (Ω, Γ) such that $C \simeq C(\Omega)$, $B = \overline{\mathrm{span}}\{fu \mid f \in C(\Omega), u \in \Gamma\}$ and $\Gamma = \bigcup_{n\geq 0} \Gamma_n$, where Γ_n consists of all unitaries u in B_n with $uC_nu^* = C_n$. Let ν be a Γ-quasi-invariant probability measure on Ω. It induces a state $\psi = \nu \circ \Psi$ of B. Let $(\pi_\psi, H_\psi, \xi_\psi)$ be the GNS-triple of ψ. Then we obtain the following:

(1) $\pi_\psi(C)''$ is a masa in $\pi_\psi(B)''$.

(2) $\pi_\psi(C)'' \simeq L^\infty(\Omega, \nu)$.

(3) The conditional expectation Ψ can extend to $\pi_\psi(B)''$ whose image is $\pi_\psi(C)''$.

3 Lemmata

We denote by $\mathcal{O}_A^{\alpha^\omega}$ the fixed-point algebra under α^ω. We first introduce an equivalence relation on the index set $I = \{1, \ldots, N\}$. We say that i is equivalent to j if there are $\xi, \eta \in \mathcal{W}_A$ such that $s(\xi) = i, s(\eta) = j, r(\xi) = r(\eta)$ and $\omega_\xi = \omega_\eta$. It is easy to check that this is an equivalence relation. We obtain the corresponding disjoint union $I = I_1^\omega \cup \cdots \cup I_{n_\omega}^\omega$. Note that if α^ω is the gauge action, then this decomposition coincides with the one with respect to the period of A. Set $P_{I_k^\omega} = \sum_{i \in I_k^\omega} P_i$. Our goal in this section is to prove the following lemma.

Lemma 3.1

$$Z(\pi_{\phi^\omega}(\mathcal{O}_A^{\alpha^\omega})'') = \pi_{\phi^\omega}(\mathcal{O}_A^{\alpha^\omega})'' \cap \pi_{\phi^\omega}(\mathcal{O}_A^{\alpha^\omega})' = \bigoplus_{k=1}^{n_\omega} \mathbb{C} P_{I_k^\omega}.$$

We need some lemmata to show the above.

Lemma 3.2 *The fixed-point algebra $\mathcal{O}_A^{\alpha^\omega}$ is an AF-algebra.*

Proof. Set $F_t^i = \mathrm{span}\{S_\xi P_i S_\eta^* \mid \omega_\xi = \omega_\eta = t\}$ for $t \in \{\omega_\xi \mid \xi \in \mathcal{W}_A\}$. Since $\{S_\xi P_i S_\eta^*\}$ gives the matrix units, F_t^i is a simple finite-dimensional C^*-algebra.

We can define finite-dimensional C^*-algebras F_n as follows:

$$\begin{aligned} F_{-1} &= \mathbb{C}1, \\ F_n &= \bigvee_{i\in I}\bigvee_{t\in K_n^i} F_t^i \\ &= \bigoplus_{i\in I}\bigoplus_{t\in K_n^i} F_t^i \quad \text{for } n\geq 0, \end{aligned}$$

where $\omega_{\min} = \min\{\omega_i \mid i\in I\}$ and

$$K_n^i = \{\omega_\xi \mid \xi\in\mathcal{W}_A, A(r(\xi),i)=1, n\omega_{\min}-\omega_i < \omega_\xi \leq n\omega_{\min}\}.$$

Indeed, let $S_{\xi_1}P_iS_{\eta_1}^* \in F_t^i$ for $t\in K_n^i$ and $S_{\xi_2}P_jS_{\eta_2}^*\in F_s^j$ for $s\in K_n^j$. We assume that $S_{\xi_1}P_iS_{\eta_1}^*S_{\xi_2}P_jS_{\eta_2}^*\neq 0$. If $|\eta_1|=|\xi_2|$, then $\eta_1=\xi_2$ and thus $s=t$ and $i=j$. We now suppose that $|\eta_1|\neq|\xi_2|$. Without loss of generality, we may assume that $|\eta_1|<|\xi_2|$. Since $P_iS_{\eta_1}^*S_{\xi_2}P_j\neq 0$, we have $P_iS_{\eta_1}^*S_{\xi_2}P_j = S_\xi P_j$ for some ξ with $\xi_2=\eta_1\cdot\xi$ and $s(\xi)=i$. Hence we obtain $\omega_i\leq\omega_\xi$. However,

$$\omega_{\xi_2} = \omega_{\eta_1}+\omega_\xi > n\omega_{\min}-\omega_i+\omega_i = n\omega_{\min}.$$

Thus $n\omega_{\min}<\omega_{\xi_2}\leq n\omega_{\min}$ and this is a contradiction.

We next show that F_n is a C^*-subalgebra of F_{n+1}. Let $S_\xi P_iS_\eta^*\in F_t^i$ with $t\in K_n^i$. If $(n+1)\omega_{\min}-\omega_i<t$, then we have $S_\xi P_iS_\eta^*\in F_{n+1}$. If $t\leq(n+1)\omega_{\min}-\omega_i$, then we have

$$S_\xi P_iS_\eta^* = \sum_{j\in I} A(i,j)S_\xi S_iP_jS_i^*S_\eta^*\in F_{n+1}.$$

We can define an AF-algbera $F=\overline{\bigcup_n F_n}$.

We claim that $F=\mathcal{O}_A^{\alpha^\omega}$. It is clear that $F\subseteq\mathcal{O}_A^{\alpha^\omega}$. To show the converse, we need the conditional expextation. If $\omega_i/\omega_j\in\mathbb{Q}$ for any $i,j\in I$, then we can define the faithful conditional expectation from $\mathcal{O}_A$ onto $\mathcal{O}_A^{\alpha^\omega}$ by the integration on $\mathbb{T}$. If not, we consider an action $\tilde{\alpha}$ of $\mathbb{T}^N$ such that $\tilde{\alpha}_z(S_i)=z_iS_i$ for $z\in(z_1,\ldots,z_N)\in\mathbb{T}^N$. Since there is the embedding of $\mathbb{R}$ into $\mathbb{T}^N$, $t\mapsto(e^{\sqrt{-1}\omega_1t},\ldots,e^{\sqrt{-1}\omega_Nt})$, we can consider the closure of $\mathbb{R}$ in $\mathbb{T}^N$ via this embedding. Therefore the conditional expectation is given by the integration on the compact group $\overline{\mathbb{R}}$. One can easily check that the fixed-point algebra under $\tilde{\alpha}|_{\overline{\mathbb{R}}}$ coinsides with $\mathcal{O}_A^{\alpha^\omega}$ and thus we can show that $\mathcal{O}_A^{\alpha^\omega}=F$ by using this conditional expectation. $\square$

We will need one more lemma. Let p be the period of the matrix A. We define partial isometries for $m \in \mathbb{N}$, $i \in I$ by

$$\theta_m^{(i)} = \sum_{\xi,\eta \in L_i(mp)} S_\xi S_\eta P_i S_\xi^* S_\eta^*,$$

where $L_i(n) = \{\xi \in \mathcal{W}_A \mid s(\xi) = i, A(r(\xi), i) = 1, |\xi| = n\}$ is the set of all loops of i with the canonical length n. Note that $\theta_m^{(i)}$ is self-adjoint. We define the tracial state by $\psi^\omega = \phi^\omega|_{\mathcal{O}_A^{\alpha^\omega}}$ on $\mathcal{O}_A^{\alpha^\omega}$, and use the same symbol ψ^ω for its normal extension to $\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$ for simplicity.

Lemma 3.3 *Let $f \in \pi_{\psi^\omega}(C(\Omega_A))''$ and $a \in \pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$. Then for any $i \in I$,*

$$\lim_{m\to\infty} \psi^\omega(\theta_m^{(i)} f \theta_m^{(i)} a) = \psi^\omega(P_i f)\psi^\omega(P_i a) x_i y_i^2,$$

where $y = (y_1, \ldots, y_N)$ is the left Perron eigenvector of A_ω with $\sum_{i\in I} x_i y_i = p$.

Proof. Note that $C(\Omega_A) \simeq \overline{\mathrm{span}}\{S_\xi S_\xi^* \mid \xi \in \mathcal{W}_A\}$ is a masa in the AF-algebra $\mathcal{O}_A^{\alpha^\omega}$. We denote by Ψ the conditional expectation from $\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$ onto $\pi_{\psi^\omega}(C(\Omega_A))'' \simeq L^\infty(\Omega_A, \nu)$. We first prove the lemma for $f \in C(\Omega_A)$ and $a \in \mathcal{O}_A^{\alpha^\omega}$. Remark that $\psi^\omega = \nu \circ \Psi$. We may assume that $a \in C(\Omega_A)$. Indeed, if the statement holds for $\Psi(a)$ instead of a, then since $\theta_m^{(i)} f \theta_m^{(i)} \in C(\Omega_A)$, we will have

$$\begin{aligned}
\lim_{m\to\infty} \psi^\omega(\theta_m^{(i)} f \theta_m^{(i)} a) &= \lim_{m\to\infty} \psi^\omega(\theta_m^{(i)} f \theta_m^{(i)} \Psi(a)) \\
&= \psi^\omega(P_i f)\psi^\omega(P_i \Psi(a)) x_i y_i^2 \\
&= \psi^\omega(P_i f)\psi^\omega(P_i a) x_i y_i^2.
\end{aligned}$$

It suffices to check the statement for $f = S_{\zeta_1} P_{j_1} S_{\zeta_1}^*, a = S_{\zeta_2} P_{j_2} S_{\zeta_2}^*$ with $|\zeta_1| = kp, |\zeta_2| = lp$ and $s(\zeta_1) = s(\zeta_2) = i$. In this case, for sufficiently large m we have

$$\theta_m^{(i)} f \theta_m^{(i)} a = \sum_{\xi',\eta'} S_{\zeta_2} S_{\xi'} S_{\zeta_1} S_{\eta'} P_i S_{\eta'}^* S_{\zeta_1}^* S_{\xi'}^* S_{\zeta_2}^*,$$

where ξ' and η' run over all admissible words from j_2, j_1 to i with $|\xi'| = (m-l)p, |\eta'| = (m-k)p$. Therefore

$$\begin{aligned}
\psi^\omega(\theta_m^{(i)} f \theta_m^{(i)} a) &= e^{-\beta\omega_{\zeta_2}} A_\omega^{(m-l)p}(j_2, i) e^{-\beta\omega_{\zeta_1}} A_\omega^{(m-k)p}(j_1, i) x_i \\
&\longrightarrow e^{-\beta\omega_{\zeta_2}} x_{j_2} y_i e^{-\beta\omega_{\zeta_1}} x_{j_1} y_i x_i \quad (m \to \infty) \\
&= \psi^\omega(f)\psi^\omega(a) x_i y_i^2.
\end{aligned}$$

Next let $f \in L^\infty(\Omega_A, \nu)$. We choose $g \in C(\Omega_A)$ with $||(f-g)\xi_{\psi^\omega}|| < \varepsilon$. Then

$$\begin{aligned} |\psi^\omega(\theta_m^{(i)} f \theta_m^{(i)} a) - \psi^\omega(P_i f)\psi^\omega(P_i a)x_i y_i^2| &\leq |\psi^\omega(\theta_m^{(i)}(f-g)\theta_m^{(i)} a)| \\ &+ |\psi^\omega(\theta_m^{(i)} g \theta_m^{(i)} a) - \psi^\omega(P_i g)\psi^\omega(P_i a)x_i y_i^2| \\ &+ |\psi^\omega(P_i(f-g))\psi^\omega(P_i a)x_i y_i^2|, \end{aligned}$$

and we get the following estimate of the first term:

$$\begin{aligned} |\psi^\omega(\theta_m^{(i)}(f-g)\theta_m^{(i)} a)| &= |\psi^\omega(\theta_m^{(i)} a \theta_m^{(i)}(f-g))| \\ &\leq \psi^\omega(\theta_m^{(i)} a \theta_m^{(i)} \theta_m^{(i)} a^* \theta_m^{(i)})^{1/2} \psi^\omega((f-g)^*(f-g))^{1/2} \\ &\leq ||a||||(f-g)\xi_{\psi^\omega}||, \end{aligned}$$

because ψ^ω is tracial. In a similar way, we can show the statement for $a \in \pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$. □

We will use the following folklore among specialists, (e.g. see [Izu]).

Lemma 3.4 ([Izu, Lemma 4.1]) *Let B be a unital C^*-algebra, ϕ a state of B and $(\pi_\phi, H_\phi, \xi_\phi)$ the GNS-triple of ϕ. We assume that the cyclic vector ξ_ϕ is a separating vector for $\pi_\phi(B)''$. Let C be a unital C^*-sublagebra of B and ψ the restriction of ϕ to C. Then $(\pi_\phi|_C, H_\phi)$ is quasi-equivalent to the GNS-representation (ϕ_ψ, H_ψ) of ψ.*

Now we have the necessary ingredients for the proof of Lemma 3.1.

Proof of Lemma 3.1 It is easy to show that $P_{I_k^\omega} \in Z(\pi_{\phi^\omega}(\mathcal{O}_A^{\alpha^\omega})'')$ for $k = 1, \ldots, n_\omega$. By Lemma 3.4, $\pi_{\phi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$ is isomorphic to $\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$. It therefore suffices to show that $Z(\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})'') = \bigoplus_{k=1}^{n_\omega} \mathbb{C}P_{I_k^\omega}$. Let $z \in Z(\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})'')$ be a non-trivial projection. Since $L^\infty(\Omega_A)$ is a masa in $\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$, we have $z \in L^\infty(\Omega_A)$. We can apply Lemma 3.3 to $f = P_i z$:

$$\lim_{m\to\infty} \psi^\omega(\theta_m^{(i)} z \theta_m^{(i)} a) = \psi^\omega(P_i z)\psi^\omega(P_i a)x_i y_i^2.$$

On the other hand, since z is centered, we get

$$\lim_{m\to\infty} \psi^\omega(\theta_m^{(i)} z \theta_m^{(i)} a) = \lim_{m\to\infty} \psi^\omega(\theta_m^{(i)} \theta_m^{(i)} z a) = \psi^\omega(P_i)\psi^\omega(P_i z a)x_i y_i^2.$$

Therefore

$$\psi^\omega(P_i)\psi^\omega(P_i z a) = \psi^\omega(P_i z)\psi^\omega(P_i a).$$

Since ψ^ω is faithful on $\pi_{\psi^\omega}(\mathcal{O}_A^{\alpha^\omega})''$ and a is arbitrary, we get

$$P_i z = \frac{\psi^\omega(P_i z)}{\psi^\omega(P_i)} P_i.$$

Therefore we obtain $z = \sum_{i \in I} c_i P_i$ for $c_i \in \{0, 1\}$. If $c_i \neq 0$ for $i \in I_k^\omega$, then there are admissible words ξ, η with $s(\xi) = i, s(\eta) = j, r(\xi) = r(\eta)$ and $\omega_\xi = \omega_\eta$ for $j \in I_k^\omega$, and z must commute with $S_\xi S_\eta^* \in \mathcal{O}_A^{\alpha^\omega}$. Hence we have $z = \sum_{k=1}^{n_\omega} c_k P_{I_k^\omega}$ for $c_k \in \{0, 1\}$. □

4 Main theorem

We first review some notations in [Con]. Let $(M, \mathbb{R}, \sigma)$ be a W^*-dynamical system. For $f \in L^1(\mathbb{R})$, we define a σ-weakly continuous linear map on M by

$$\sigma_f(x) = \int f(t) \sigma_t(x) dt \quad \text{for } x \in M.$$

The *Arveson spectrum* of σ is defined by

$$\mathrm{Sp}(\sigma) = \bigcap \{Z(f) \mid f \in L^1(\mathbb{R}),\ \sigma_f = 0\},$$

where $Z(f) = \{r \in \hat{\mathbb{R}} \mid \hat{f}(r) = 0\}$ and $\hat{\mathbb{R}}$ is the dual group of $\mathbb{R}$. Then the *Connes spectrum* of σ is defined by

$$\Gamma(\sigma) = \bigcap_p \mathrm{Sp}(\sigma|_{pMp}),$$

where p runs over all non-zero projections in $Z(M^\sigma) = M^\sigma \cap (M^\sigma)'$. Note that $\Gamma(\sigma) \subseteq \mathrm{Sp}(\sigma|_{pMp})$ holds for any non-zero projection p in M^σ.

For each $i \in I$, let G_i be the closed additive subgroup of $\mathbb{R}$ generated by $\beta\omega_\xi$ for all loops ξ with $s(\xi) = i$ and G the closed additive subgroup generated by $\beta\omega_\xi$ for all circles ξ.

Lemma 4.1 *For any $i \in I$, $G = G_i$.*

Proof. It is clear that $G \subseteq G_i$. Conversely, let ξ be a loop with $s(\xi) = i$. Then there are circles $\xi(1), \ldots, \xi(n)$ such that $\omega_\xi = \omega_{\xi(1)} + \cdots + \omega_{\xi(n)}$. Thus $G_i \subseteq G$. □

We will prove the following main theorem.

Theorem 4.2 (1) *If $\omega_\xi/\omega_\eta \in \mathbb{Q}$ for any circles ξ, η, then $M = \pi_{\psi^\omega}(\mathcal{O}_A)''$ is the AFD type III_λ factor for $\lambda = e^{-r}$, where $G = r\mathbb{Z}$ for some $r \in \mathbb{R}_+$.*

(2) *If $\omega_\xi/\omega_\eta \notin \mathbb{Q}$ for some circles ξ, η, then $M = \pi_{\psi^\omega}(\mathcal{O}_A)''$ is the AFD type III_1 factor.*

Proof. Since ϕ^ω is α^ω-invariant, α^ω can be extended to an action on M. We use the same symbol ϕ^ω for its normal extension. Let σ^{ϕ^ω} be the modular automorphism group for ϕ^ω , which satisfies $\sigma_t^{\phi^\omega} = \alpha^\omega_{-\beta t}$ for $t \in \mathbb{R}$. We first claim that $M^\sigma = \pi_{\phi^\omega}(\mathcal{O}_A^\sigma)''$. One can check that the conditional expectation form $\mathcal{O}_A$ onto $\mathcal{O}_A^{\alpha^\omega}$ in the proof of Lemma 3.2 can extend to the one on $\pi_{\phi^\omega}(\mathcal{O}_A)''$. Thus by the approximation arguments, we can obtain our claim.

By Lemma 3.1, we obtain $\Gamma(\sigma^{\phi^\omega}) = \bigcap_k \mathrm{Sp}(\sigma^{\phi^\omega}|_{P_{I_k^\omega}MP_{I_k^\omega}})$. Moreover, since $\mathrm{Sp}(\sigma^{\phi^\omega}|_{P_iMP_i}) \subseteq \mathrm{Sp}(\sigma^{\phi^\omega}|_{P_{I_k^\omega}MP_{I_k^\omega}})$ for $i \in I_k^\omega$, we have

$$\Gamma(\sigma^{\phi^\omega}) = \bigcap_{i\in I} \mathrm{Sp}(\sigma^{\phi^\omega}|_{P_iMP_i}).$$

We now claim that $\mathrm{Sp}(\sigma^{\phi^\omega}|_{P_iMP_i}) = G_i$ for each $i \in I$. Let ξ, η be loops with $s(\xi) = s(\eta) = i$. If $f \in \mathrm{Ker}\sigma^{\phi_\omega}|_{P_iMP_i}$, then

$$0 = \sigma_f^{\phi_\omega}(P_iS_\xi S_\eta^* P_i) = \hat{f}(\beta(\omega_\xi - \omega_\eta))P_iS_\xi S_\eta^* P_i.$$

Since $P_iS_\xi S_\eta^*P_i \neq 0$, we have $\beta(\omega_\xi - \omega_\eta) \in \mathrm{Sp}(\sigma^{\phi^\omega}|_{P_iMP_i})$. Thus a group generated by $\beta\omega_\xi$ for all loops ξ with $s(\xi) = i$ is contained in $\mathrm{Sp}(\sigma^{\phi^\omega}|_{P_iMP_i})$. Since $\mathrm{Sp}(\sigma^{\phi^\omega}|_{P_iMP_i})$ is closed, $G_i \subseteq \mathrm{Sp}(\sigma^{\phi^\omega}|_{P_iMP_i})$ holds. Conversely let $r \in \mathbb{R} \setminus G_i$. Choose a function $f \in L^1(\mathbb{R})$ with $\hat{f}(r) \neq 0$ and $\hat{f}|_{G_i} = 0$. We have

$$\sigma_f^{\phi_\omega}|_{P_iMP_i}(P_iS_\xi S_\eta^*P_i) = \hat{f}(\beta(\omega_\xi - \omega_\eta))P_iS_\xi S_\eta^*P_i.$$

If $P_iS_\xi S_\eta^*P_i \neq 0$, then we have $s(\xi) = s(\eta) = i$ and $A(r(\xi), j) = A(r(\eta), j) = 1$ for some $j \in I$. Since A is irreducible, there is an admissible word ζ with $s(\zeta) = j$ and $A(r(\zeta), i) = 1$. Two admissible words $\xi \cdot \zeta$, $\eta \cdot \zeta$ are loops with $s(\xi \cdot \zeta) = s(\eta \cdot \zeta) = i$. Hence

$$\beta(\omega_\xi - \omega_\eta) = \beta(\omega_\xi + \omega_\zeta - \omega_\eta - \omega_\zeta) = \beta(\omega_{\xi\cdot\zeta} - \omega_{\eta\cdot\zeta}) \in G_i.$$

We therefore obtain $f \in \mathrm{Ker}\sigma^{\phi_\omega}|_{P_iMP_i}$. It follows from Lemma 4.1 that $\Gamma(\sigma^{\phi^\omega}) = G$. In the case (1), we have $G = r\mathbb{Z}$ for some $r \in \mathbb{R}_+$ and λ is determined by e^{-r}. $\square$

Example 4.3 Let $\mathbb{F}_n$ be the free group with generators $a_1, \ldots, a_n$ and $S = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$ the generating set. The corresponding Cayley graph is the homogeneous tree with degree $2n$. We define a compact space by

$$\Omega = \{(x_i)_{i=1}^{\infty} \mid x_i \neq x_{i+1}^{-1}\} \subseteq \prod_{i=1}^{\infty} S.$$

Note that Ω is compact and Γ acts on Ω by left multiplications. We remark that Ω coinsides with the hyperbolic boundary $\partial\mathbb{F}_n$ of $\mathbb{F}_n$. In [Spi], Spielberg showed the identification $\mathcal{O}_A \simeq C(\Omega) \rtimes \mathbb{F}_n$, where

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 1 \end{pmatrix} \quad (2n \times 2n\text{-matrix}).$$

We now apply Theorem 4.2 to $\mathcal{O}_A \simeq C(\Omega) \rtimes \mathbb{F}_n$. Note that the canonical masa $C(\Omega_A)$ of $\mathcal{O}_A$ coinsides with $C(\Omega)$. Let $\omega = (\omega_x)_{x \in S} \in \mathbb{R}_+^{2n}$ and ν the corresponding probability measure on Ω, which induces the KMS state for α^ω. By Theorem 4.2, we have the following:

(1) *If $\omega_x/\omega_y \in \mathbb{Q}$ for any $x, y \in S$, then $L^\infty(\Omega, \nu) \rtimes \mathbb{F}_n$ is the AFD type* III_λ *factor for some $0 < \lambda < 1$.*

(2) *If $\omega_x/\omega_y \notin \mathbb{Q}$ for some $x, y \in S$, then $L^\infty(\Omega, \nu) \rtimes \mathbb{F}_n$ is the AFD type* III_1 *factor.*

Let μ be a probability measure on $\mathbb{F}_n$ with $\mathrm{supp}\mu = S$. By [Oka], the random walk with law μ induces the harmonic measure ν on Ω such that the modular automorphism group of the state $\nu \circ \Phi$ has the form α^ω for some $\omega = (\omega_x)_{x \in S} \in \mathbb{R}_+^{2n}$. Therefore the above result also means that we determine the type of harmonic measures on Ω (cf. [RR]).

Remark 4.4 We can also prove the same results for $\mathcal{O}_\Gamma$ in [Oka] in the same way, where Γ is an amalgamated free product group $*_H G_i$. Here, we will give a sketch of the proof.

Let I be a finite index set and G_i a group containing a copy of a group H as a subgroup for $i \in I$. We assume that G_i is finite for simplicity. The C^*-algebra $\mathcal{O}_\Gamma$ is the universal C^*-algebra generated by partial isometries S_g, $g \in \bigcup_{i \in I} G_i \setminus H$ and unitaries U_h, $h \in H$ satisfying certain conditions (see [Oka]). We use some symbols in [Oka]. For $\omega = (\omega_i)_{i \in I} \in \mathbb{R}_+^{|I|}$, we consider the action α^ω of $\mathbb{R}$ given by

$$\begin{array}{rclll} \alpha_t^\omega(S_g) & = & e^{\sqrt{-1}\omega_i t} S_g & \text{for} & i \in I,\ g \in G_i \setminus H, \\ \alpha_t^\omega(U_h) & = & U_h & \text{for} & h \in H, \end{array}$$

where $|I|$ is the cardinality of I. Remark that there is an identification $\mathcal{O}_\Gamma \simeq C(\Omega) \rtimes \Gamma$ for some compact space Ω ([Oka, Theorem 5.3]). Let Φ be the canonical conditional expectation from $C(\Omega) \rtimes \Gamma$ onto $C(\Omega)$. It was shown that there is the unique β-KMS state $\phi = \nu \circ \Phi$ for α^ω, where ν is the corresponding probability measure on Ω. However the difference from the above example is that $C(\Omega)$ may be not a masa of the fixed-point algebra under α^ω. Therefore we need some arguments to obtain the similar result for $\mathcal{O}_\Gamma$. Choose a masa C, containing $C(\Omega)$. We assume that $\Gamma =_H G_i$ satisfies the following condition:

For any $i \in I$, there is an erement $\gamma_i = g_1 \cdots g_n \in \Gamma$ such that $h\gamma_i H \neq \gamma_i H$ for any $(e \neq) h \in H$, where $g_k \in G_{i_k} \setminus H$ with $i = i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$.

We remark that the above assumption holds if $\Gamma = *_H G_i$ satisfies the condition of [Oka, Corollary 6.4]. Fix γ_i satisfying the above. Let ψ be the restriction of ϕ on the fixed-point algebra under $\mathcal{O}_\Gamma^{\alpha^\omega}$. For $g \in G_i \setminus H$, we set

$$\theta_m^{(g)} = \sum_{\xi,\eta} S_\xi S_{\gamma_i} S_\eta P_g S_\xi^* S_{\gamma_i}^* S_\eta^*,$$

where ξ, η run over all words from g to an element, which is not in G_i, with length m if $|I| > 2$ and length $2m$ if $|I| = 2$. Let π_ψ be the GNS-representation of ψ. Then we will get the similar result of Lemma 3.3.

Lemma 4.5 *For $f \in \pi_\psi(C)''$ and $a \in \pi_\psi(\mathcal{O}_\Gamma^{\alpha^\omega})''$, we have*

$$\lim_{m \to \infty} \psi(\theta_m^{(g)} f \theta_m^{(g)} a) = \psi(P_g f)\psi(P_g a) x_g y_g^2 z_{\gamma_i},$$

where x_g, y_g, z_{γ_i} are some constants.

Using this lemma, we can prove the following similarly.

Proposition 4.6

$$Z(\pi_\psi(\mathcal{O}_\Gamma^{\alpha^\omega})'') = \begin{cases} \bigoplus_{i\in I} \mathbb{C}P_i & \text{if} \quad |I| = 2, \\ \mathbb{C}1 & \text{if} \quad |I| > 2. \end{cases}$$

Hence we can compute the Connes spectrum of the modular automorphism group in the similar way. This gives a generalization on [RR].

Corollary 4.7 *Let $\mathcal{O}_\Gamma, \omega, \phi, \nu$ be the above and π_ϕ the GNS-representation of ϕ. Then*

(1) *If $\omega_i/\omega_j \in \mathbb{Q}$ for any $i, j \in I$, then $\pi_\phi(\mathcal{O}_\Gamma)'' \simeq L^\infty(\Omega, \nu) \rtimes \Gamma$ is the AFD type* III_λ *factor for some $0 < \lambda < 1$.*

(2) *If $\omega_i/\omega_j \notin \mathbb{Q}$ for some $i, j \in I$, then $\pi_\phi(\mathcal{O}_\Gamma)'' \simeq L^\infty(\Omega, \nu) \rtimes \Gamma$ is the AFD type* III_1 *factor.*

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