

# THE $\mathbb{Q}$ -PICARD GROUP OF THE MODULI SPACE OF CURVES IN POSITIVE CHARACTERISTIC

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ABSTRACT. In this note, we prove that the  $\mathbb{Q}$ -Picard group of the moduli space of  $n$ -pointed stable curves of genus  $g$  over an algebraically closed field is generated by the tautological classes. We also prove that the cycle map to the 2nd étale cohomology group is bijective.

## INTRODUCTION

Let  $k$  be an algebraically closed field,  $g$  and  $n$  non-negative integers with  $2g - 2 + n > 0$ , and  $\bar{M}_{g,n}$  (resp.  $M_{g,n}$ ) the moduli space of  $n$ -pointed stable (resp. smooth) curves of genus  $g$  over  $k$ . We denote by  $\text{Pic}(\bar{M}_{g,n})_{\mathbb{Q}}$  the  $\mathbb{Q}$ -Picard group of  $\bar{M}_{g,n}$ ; that is,  $\text{Pic}(\bar{M}_{g,n})_{\mathbb{Q}} = \text{Pic}(\bar{M}_{g,n}) \otimes \mathbb{Q}$ . Let  $\lambda$  be the Hodge class,  $\psi_1, \dots, \psi_n$  the classes of  $\mathbb{Q}$ -line bundles given by the pull-back of the relative dualizing sheaf of the universal curve over  $\bar{M}_{g,n}$  in terms of  $n$  sections, and  $\{\delta_t\}_{t \in T}$  the boundary classes in  $\text{Pic}(\bar{M}_{g,n})_{\mathbb{Q}}$  (for details, see §§1.6). These classes

$$\lambda, \psi_1, \dots, \psi_n \text{ and } \delta_t \text{'s } (t \in T)$$

are called *the tautological classes of  $\text{Pic}(\bar{M}_{g,n})_{\mathbb{Q}}$* . It is well known (due to Harer) that  $\text{Pic}(\bar{M}_{g,n})_{\mathbb{Q}}$  is generated by the tautological classes if the characteristic of  $k$  is zero (see Arbarello-Cornalba [1] for its proof by means of algebraic geometry). In this note, we would like to show that this still holds even if the characteristic of  $k$  is positive. Namely, we have the following.

**Theorem** (cf. Theorem 5.1).  *$\text{Pic}(\bar{M}_{g,n})_{\mathbb{Q}}$  is generated by the tautological classes*

$$\lambda, \psi_1, \dots, \psi_n \text{ and } \delta_t \text{'s } (t \in T)$$

*for any algebraically closed field  $k$ . Moreover, the cycle map*

$$\text{Pic}(\bar{M}_{g,n}) \otimes \mathbb{Q}_{\ell} \rightarrow H_{\text{et}}^2(\bar{M}_{g,n}, \mathbb{Q}_{\ell})$$

*is bijective for every prime  $\ell$  invertible in  $k$ .*

We prove the above theorem by using modulo  $p$  reduction. The outline of the proof is as follows: Let  $\bar{\mathcal{M}}_{g,n}$  be the algebraic stack classifying  $n$ -pointed stable curves of genus  $g$ . We compare the 2nd étale cohomology group  $H_{\text{et}}^2(\bar{\mathcal{M}}_{g,n}(k), \mathbb{Q}_{\ell})$  over  $k$  with the 2nd cohomology group  $H^2(\bar{\mathcal{M}}_{g,n}(\mathbb{C}), \mathbb{Q})$  of the analytic space  $\bar{\mathcal{M}}_{g,n}(\mathbb{C})$  via a smooth Galois covering of  $\bar{\mathcal{M}}_{g,n}$  in terms of a Teichmüller level structure due to Looijenga-Pikaart-de Jong-Boggi; namely,

$$\dim_{\mathbb{Q}_{\ell}} H_{\text{et}}^2(\bar{\mathcal{M}}_{g,n}(k), \mathbb{Q}_{\ell}) \leq \dim_{\mathbb{Q}} H^2(\bar{\mathcal{M}}_{g,n}(\mathbb{C}), \mathbb{Q}).$$

Moreover, using the simple connectedness of the moduli space of curves with a level  $m \geq 3$  (due to Boggi-Pikaart), we see that the cycle map

$$\text{Pic}(\bar{\mathcal{M}}_{g,n}(k)) \otimes \mathbb{Q}_{\ell} \rightarrow H_{\text{et}}^2(\bar{\mathcal{M}}_{g,n}(k), \mathbb{Q}_{\ell})$$

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is injective. In this way, we obtain our theorem together with the linear independence of the tautological classes.

After writing this note, Prof. Keel informed me that Prof. de Jong had the same idea for the proof of the above theorem.

## 1. NOTATIONS AND CONVENTIONS

1.1. For a finite set  $S$ , we denote the number of it by  $|S|$ .

1.2. If a group  $G$  acts on a certain kind of a mathematical object, the induced automorphism by  $g \in G$  is denoted by  $[g]$ .

1.3. Let  $X$  be a proper algebraic space over an algebraically closed field  $k$ . Let  $L_1$  and  $L_2$  be line bundles on  $X$ . We say  $L_1$  is *numerically equivalent* to  $L_2$ , denoted by  $L_1 \equiv L_2$ , if  $(L_1 \cdot C) = (L_2 \cdot C)$  for all curves  $C$  on  $X$ . The group  $\text{Pic}(X)$  modulo the numerical equivalence, denoted by  $\text{NS}^\nu(X)$ , is called *the numerical Néron-Severi group of  $X$* . Moreover, we denote  $\text{NS}^\nu(X) \otimes \mathbb{Q}$  by  $\text{NS}^\nu(X)_\mathbb{Q}$ .

1.4. Let  $X$  be an algebraic scheme over an algebraically closed field  $k$ . We define the  $\mathbb{Q}$ -Picard group  $\text{Pic}(X)_\mathbb{Q}$  of  $X$  to be  $\text{Pic}(X)_\mathbb{Q} = \text{Pic}(X) \otimes \mathbb{Q}$ . For  $L_1, L_2 \in \text{Pic}(X)$ , we say  $L_1$  is *algebraically equivalent* to  $L_2$ , denoted by  $L_1 \sim_{alg} L_2$ , if there are a connected and smooth algebraic scheme  $T$  over  $k$ , a line bundle  $\mathcal{L}$  on  $T \times X$  and  $t_1, t_2 \in T$  such that  $\mathcal{L}|_{\{t_1\} \times X} \simeq L_1$  and  $\mathcal{L}|_{\{t_2\} \times X} \simeq L_2$ . The Néron-Severi group  $\text{NS}(X)$  of  $X$  is defined by  $\text{Pic}(X)$  modulo the algebraic equivalence. In other words,  $\text{NS}(X) = \text{Pic}_X(k)/(\text{Pic}_X^0)_{red}(k)$ , where  $\text{Pic}_X$  is the Picard scheme of  $X$  and  $\text{Pic}_X^0$  is the connected component containing 0. The group  $\text{NS}(X) \otimes \mathbb{Q}$ , denoted by  $\text{NS}(X)_\mathbb{Q}$ , is called *the  $\mathbb{Q}$ -Néron-Severi group of  $X$* . We assume that  $X$  is projective over  $k$ . It is not difficult to see that  $L_1 \sim_{alg} L_2$  implies  $L_1 \equiv L_2$ , so that we have the natural surjective homomorphism  $\text{NS}(X) \rightarrow \text{NS}^\nu(X)$ . It is well known (due to Matsusaka) that the kernel of  $\text{NS}(X) \rightarrow \text{NS}^\nu(X)$  is a finite group. Thus, we can identify  $\text{NS}(X)_\mathbb{Q}$  with  $\text{NS}^\nu(X)_\mathbb{Q}$ .

1.5. In this note, an algebraic stack always means a separated algebraic stack over a locally noetherian scheme in the sense of Deligne-Mumford. Let  $X$  be a algebraic stack over a locally noetherian scheme  $S$ . For an algebraically closed field  $L$  and a morphism  $\text{Spec}(L) \rightarrow S$ , the coarse moduli space of  $X \times_S \text{Spec}(L)$  is denoted by  $X_L$  (cf. [4, Chapter I, Theorem 4.10] and [9, Corollary 1.3]).

1.6. Let  $g$  and  $n$  be non-negative integers with  $2g - 2 + n > 0$ , and  $\bar{M}_{g,n}$  (resp.  $M_{g,n}$ ) the moduli space of  $n$ -pointed stable (resp. smooth) curves of genus  $g$  over an algebraically closed field. Roughly speaking, the  $\mathbb{Q}$ -line bundles  $\lambda$  and  $\psi_1, \dots, \psi_n$  on  $\bar{M}_{g,n}$  are defined as follows: Let  $\pi : \bar{M}_{g,n+1} \rightarrow \bar{M}_{g,n}$  be the universal curve of  $\bar{M}_{g,n}$ , and  $s_1, \dots, s_n : \bar{M}_{g,n} \rightarrow \bar{M}_{g,n+1}$  the sections of  $\pi$  arising from the  $n$ -points of  $\bar{M}_{g,n}$ . Then,  $\lambda = \det(\pi_*(\omega_{\bar{M}_{g,n+1}/\bar{M}_{g,n}}))$  and  $\psi_i = s_i^*(\omega_{\bar{M}_{g,n+1}/\bar{M}_{g,n}})$  for  $i = 1, \dots, n$ . Here we set

$$\begin{aligned} [n] &= \{1, \dots, n\} \quad (\text{note that } [0] = \emptyset), \\ \Upsilon_{g,n} &= \{(i, I) \mid i \in \mathbb{Z}, 0 \leq i \leq g \text{ and } I \subseteq [n] \setminus \{(0, \emptyset), (0, \{1\}), \dots, (0, \{n\})\}\}, \\ \bar{\Upsilon}_{g,n} &= \{(i, I), (j, J) \mid (i, I), (j, J) \in \Upsilon_{g,n}, i + j = g, I \cap J = \emptyset, I \cup J = [n]\}. \end{aligned}$$

The boundary  $\Delta = \bar{M}_{g,n} \setminus M_{g,n}$  has the following irreducible decomposition:

$$\Delta = \Delta_{irr} \cup \bigcup_{\{(i,I),(j,J)\} \in \bar{\Upsilon}_{g,n}} \Delta_{\{(i,I),(j,J)\}}.$$

A general point of  $\Delta_{irr}$  represents an  $n$ -pointed irreducible stable curve with one node. A general point of  $\Delta_{\{(i,I),(j,J)\}}$  represents an  $n$ -pointed stable curve consisting of an  $|I|$ -pointed smooth curve  $C_1$  of genus  $i$  and a  $|J|$ -pointed smooth curve  $C_2$  of genus  $j$  meeting transversally at one point, where  $|I|$ -points on  $C_1$  (resp.  $|J|$ -points on  $C_2$ ) arise from  $\{s_t\}_{t \in I}$  (resp.  $\{s_l\}_{l \in J}$ ). Let  $\delta_{irr}$  and  $\delta_{\{(i,I),(j,J)\}}$  be the classes of  $\Delta_{irr}$  and  $\Delta_{\{(i,I),(j,J)\}}$  in  $\text{Pic}(\bar{M}_{g,n})_{\mathbb{Q}}$  respectively. For our convenience, we denote  $\{(i, I), (g - i, [n] \setminus I)\}$  by  $[i, I]$ . Moreover, we set

$$\bar{\Upsilon}_{g,n}^e = \bar{\Upsilon}_{g,n} \cup \{[0, \{1\}], \dots, [0, \{n\}]\}$$

and  $\delta_{[0, \{i\}]} = -\psi_i$  for  $i = 1, \dots, n$ .

## 2. COMPARISON OF COHOMOLOGY GROUPS

In this section, we would like to show the following theorem, which is crucial for our note.

**Theorem 2.1.** *Let  $R$  be a discrete valuation ring with  $R \subset \mathbb{C}$ , and  $X$  a proper algebraic stack over  $R$  (see §§1.5 for assumptions of stacks in this note). We assume that there are (i) a finite group  $G$ , (ii) a smooth, proper and pure dimensional scheme  $Y$  over  $R$ , and (iii) a surjective morphism  $\pi : Y \rightarrow X$  over  $R$  with the following properties:*

- (a)  $G$  acts on  $Y$  over  $X$ , i.e.,  $\pi \cdot [g] = \pi$  for all  $g \in G$ .
- (b)  $X(\mathbb{C}) \simeq Y(\mathbb{C})/G$  as analytic spaces.

*Let  $\text{Spec}(k) \rightarrow \text{Spec}(R)$  be a geometric point of  $\text{Spec}(R)$  (i.e.,  $k$  is an algebraically closed field), and  $X_k$  the coarse moduli space of  $X \times_{\text{Spec}(R)} \text{Spec}(k)$ . If  $X_k$  is a normal algebraic scheme over  $k$ , then*

$$\dim_{\mathbb{Q}_\ell} H_{et}^i(X_k, \mathbb{Q}_\ell) \leq \dim_{\mathbb{Q}} H^i(X(\mathbb{C}), \mathbb{Q})$$

for every non-negative integer  $i$  and every prime  $\ell$  invertible in  $k$ .

*Proof.* We need two lemmas for the proof of the above theorem.

**Lemma 2.2.** *Let  $f : Y \rightarrow X$  be a finite surjective morphism of normal noetherian schemes. Then, the natural homomorphism*

$$f^* : H_{et}^i(X, \mathbb{Q}_\ell) \rightarrow H_{et}^i(Y, \mathbb{Q}_\ell)$$

*is injective for every non-negative integer  $i$  and every prime  $\ell$  invertible in  $H^0(X, \mathcal{O}_X)$ .*

*Proof.* Clearly, we may assume that  $X$  and  $Y$  are connected. Here we claim the following.

**Claim 2.2.1.** *For every abelian group  $\Lambda$ , there is a homomorphism*

$$\rho_f(\Lambda) : f_*(\Lambda_Y) \rightarrow \Lambda_X$$

*with the following properties:*

- (i)  $\rho_f(\Lambda) \cdot f^* = \text{deg}(f) \text{id}$ , where  $f^* : \Lambda_X \rightarrow f_*(\Lambda_Y)$  is the natural injective homomorphism.

(ii) Let  $\phi : \Lambda' \rightarrow \Lambda$  be a homomorphism of abelian groups. Then, the following diagram is commutative.

$$\begin{array}{ccc} f_*(\Lambda'_Y) & \xrightarrow{\rho_{f'}(\Lambda')} & \Lambda'_X \\ \downarrow & & \downarrow \\ f_*(\Lambda_Y) & \xrightarrow{\rho_f(\Lambda)} & \Lambda_X \end{array}$$

Let  $K'$  and  $K$  be the function fields of  $Y$  and  $X$  respectively. First, we assume that  $K'$  is separable over  $K$ . Let  $K''$  be the Galois closure of  $K'$  over  $K$ , and  $G$  the Galois group of  $K''/K$ . Moreover, let  $\tilde{Y}$  be the normalization of  $Y$  in  $K''$ ,  $g : \tilde{Y} \rightarrow Y$  the induced morphism, and  $\tilde{f} : \tilde{Y} \xrightarrow{g} Y \xrightarrow{f} X$  the composition of morphisms  $g$  and  $f$ . We denote by  $(K'/K)(K'')$  the set of embeddings of  $K'$  into  $K''$  over  $K$ ; that is,

$$(K'/K)(K'') = \{\sigma : K' \hookrightarrow K'' \mid \sigma|_K = \text{id}\}.$$

For each  $\sigma \in (K'/K)(K'')$ , there is a morphism  $\tilde{\sigma} : \tilde{Y} \rightarrow Y$  over  $X$  such that the induced map of function fields is  $\sigma$ . Here let us consider a homomorphism  $\rho' : f_*(\Lambda_Y) \rightarrow \tilde{f}_*(\Lambda_{\tilde{Y}})$  given by  $\rho'(x) = \sum_{\sigma \in (K'/K)(K'')} \tilde{\sigma}^*(x)$ . It is easy to see that

$$\text{Im}(\rho') \subseteq \tilde{f}_*(\Lambda_{\tilde{Y}})^G.$$

Moreover, since  $G$  acts transitively on the fibers of  $\tilde{Y} \rightarrow X$ , we can see

$$\tilde{f}_*(\Lambda_{\tilde{Y}})^G \subseteq \text{Im}(\tilde{f}^* : \Lambda_X \rightarrow \tilde{f}_*(\Lambda_{\tilde{Y}})).$$

Thus  $\rho'$  gives rise to a homomorphism

$$\rho_f(\Lambda) : f_*(\Lambda_Y) \rightarrow \Lambda_X.$$

Next, let us consider a general case. Let  $K_1$  be the separable closure of  $K$  in  $K'$ , and  $Y_1$  the normalization of  $X$  in  $K_1$ . Then, there are finite morphisms  $g : Y \rightarrow Y_1$  and  $h : Y_1 \rightarrow X$  with  $f = h \cdot g$ . Since  $g$  is purely inseparable,  $\Lambda_{Y_1} \xrightarrow{\sim} g_*(\Lambda_Y)$ . Thus,  $h_*(\Lambda_{Y_1}) \xrightarrow{\sim} f_*(\Lambda_Y)$ . Let  $\rho_\Lambda(h) : h_*(\Lambda_{Y_1}) \rightarrow \Lambda_X$  be a homomorphism as above. Then,  $\rho_\Lambda(f)$  is given by  $\text{deg}(g)\rho_\Lambda(h)$ .

The properties (i) and (ii) are obvious by our construction.

Let us go back to the proof of our lemma. Since  $f$  is finite,  $H^i(Y_{et}, \mathbb{Z}/\ell^m\mathbb{Z}) = H^i(X_{et}, f_*(\mathbb{Z}/\ell^m\mathbb{Z}))$ . Thus, by the above claim, we have a homomorphism  $\varrho_m : H^i(Y_{et}, \mathbb{Z}/\ell^m\mathbb{Z}) \rightarrow H^i(X_{et}, \mathbb{Z}/\ell^m\mathbb{Z})$  such that  $\varrho_m \cdot f^* = \text{deg}(f) \text{id}$ . Here the following diagram is commutative by the property (ii):

$$\begin{array}{ccc} H^i(Y_{et}, \mathbb{Z}/\ell^{m+1}\mathbb{Z}) & \xrightarrow{\varrho_{m+1}} & H^i(X_{et}, \mathbb{Z}/\ell^{m+1}\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^i(Y_{et}, \mathbb{Z}/\ell^m\mathbb{Z}) & \xrightarrow{\varrho_m} & H^i(X_{et}, \mathbb{Z}/\ell^m\mathbb{Z}) \end{array}$$

Thus, we have  $\varrho : H^i_{et}(Y, \mathbb{Q}_\ell) \rightarrow H^i_{et}(X, \mathbb{Q}_\ell)$  with  $\varrho \cdot f^* = \text{deg}(f) \text{id}$ . Therefore,  $f^*$  is injective.  $\square$

**Lemma 2.3.** *Let  $Y$  be a complex manifold, and  $G$  a finite group acting on  $Y$  holomorphically. Let  $X$  be the quotient analytic space  $Y/G$  of  $Y$  by the action of  $G$ , and  $\pi : Y \rightarrow X$  the canonical morphism. Let  $\Lambda$  be an abelian group such that  $|G| \text{id}_\Lambda : \Lambda \rightarrow \Lambda$  is bijective. Let  $\pi^* : H^i(X, \Lambda) \rightarrow$*

$H^i(Y, \Lambda)$  be the natural homomorphism of the  $i$ -th cohomology groups. Then,  $\pi^*$  is injective and its image is the  $G$ -invariant part  $H^i(Y, \Lambda)^G$  of  $H^i(Y, \Lambda)$ .

*Proof.* By abuse of notation, let  $\pi^* : \Lambda_X \rightarrow \pi_*(\Lambda_Y)$  be the natural homomorphism. In the same way as in Lemma 2.2, if we define  $\rho : \pi_*(\Lambda_Y) \rightarrow \Lambda_X$  to be

$$\pi^*(\rho(x)) = \sum_{g \in G} [g]^*(x),$$

then  $\rho \cdot \pi^* = |G| \text{id}$ . Thus, there is a homomorphism  $\varrho : H^i(Y, \Lambda) \rightarrow H^i(X, \Lambda)$  with  $\varrho \cdot \pi^* = |G| \text{id}$ . In particular,  $\pi^*$  is injective. Clearly,  $\pi^*(H^i(X, \Lambda)) \subseteq H^i(Y, \Lambda)^G$ . Note that  $\pi^*(\varrho(\phi)) = \sum_{g \in G} [g]^*(\phi)$  for  $\phi \in H^i(Y, \Lambda)$ . Thus, if  $\phi \in H^i(Y, \Lambda)^G$ , then  $\pi^*(\varrho(\phi)) = |G|\phi$ . Here  $|G| \text{id}_\Lambda : \Lambda \rightarrow \Lambda$  is bijective. Therefore,  $\phi = \pi^*(\varrho(\phi)/|G|)$ .  $\square$

Let us start the proof of Theorem 2.1. Let  $\bar{\eta}$  be the geometric generic point of  $\text{Spec}(R)$ , and  $\bar{t}$  the geometric closed point of  $\text{Spec}(R)$ . Here we consider two cases:

- (i) The image of  $\text{Spec}(k) \rightarrow \text{Spec}(R)$  is the generic point.
- (ii) The image of  $\text{Spec}(k) \rightarrow \text{Spec}(R)$  is the closed point.

In the first case, by the proper base change theorem (cf. [5, Chapter I, Theorem 6.1]),

$$H_{et}^i(Y_k, \mathbb{Q}_\ell) = H_{et}^i(Y_{\bar{\eta}}, \mathbb{Q}_\ell).$$

Here, by virtue of Lemma 2.2, the natural homomorphism  $H_{et}^i(X_k, \mathbb{Q}_\ell) \rightarrow H_{et}^i(Y_k, \mathbb{Q}_\ell)$  is injective and its image is contained in  $H_{et}^i(Y_k, \mathbb{Q}_\ell)^G$  because the action of  $G$  is given over  $X_k$ . Thus,

$$\dim_{\mathbb{Q}_\ell} H_{et}^i(X_k, \mathbb{Q}_\ell) \leq \dim_{\mathbb{Q}_\ell} H_{et}^i(Y_k, \mathbb{Q}_\ell)^G = \dim_{\mathbb{Q}_\ell} H_{et}^i(Y_{\bar{\eta}}, \mathbb{Q}_\ell)^G.$$

Further, by the proper base change theorem, the comparison theorem (cf. [5, Chapter I, Theorem 11.6]) and Lemma 2.3,

$$\begin{aligned} \dim_{\mathbb{Q}_\ell} H_{et}^i(Y_{\bar{\eta}}, \mathbb{Q}_\ell)^G &= \dim_{\mathbb{Q}_\ell} H_{et}^i(Y_{\mathbb{C}}, \mathbb{Q}_\ell)^G = \dim_{\mathbb{Q}_\ell} H^i(Y(\mathbb{C}), \mathbb{Q}_\ell)^G \\ &= \dim_{\mathbb{Q}_\ell} H^i(X(\mathbb{C}), \mathbb{Q}_\ell) = \dim_{\mathbb{Q}} H^i(X(\mathbb{C}), \mathbb{Q}). \end{aligned}$$

Therefore, we get our assertion.

In the second case, by using the proper base change theorem and Lemma 2.2 as before, we have

$$\dim_{\mathbb{Q}_\ell} H_{et}^i(X_k, \mathbb{Q}_\ell) \leq \dim_{\mathbb{Q}_\ell} H_{et}^i(Y_k, \mathbb{Q}_\ell)^G = \dim_{\mathbb{Q}_\ell} H_{et}^i(Y_{\bar{t}}, \mathbb{Q}_\ell)^G$$

and

$$\dim_{\mathbb{Q}_\ell} H_{et}^i(Y_{\bar{\eta}}, \mathbb{Q}_\ell)^G = \dim_{\mathbb{Q}} H^i(X(\mathbb{C}), \mathbb{Q}).$$

Therefore, it is sufficient to show that

$$(2.4) \quad \dim_{\mathbb{Q}_\ell} H_{et}^i(Y_{\bar{t}}, \mathbb{Q}_\ell)^G = \dim_{\mathbb{Q}_\ell} H_{et}^i(Y_{\bar{\eta}}, \mathbb{Q}_\ell)^G.$$

Indeed, let  $f : Y \rightarrow \text{Spec}(R)$  be the canonical morphism, and we set  $F = R_{et}^i f_*(\mathbb{Q}_\ell)$ . Then,  $G$  acts on the sheaf  $F$  of étale topology. Namely, for any étale neighborhood  $U$  of  $\text{Spec}(R)$ ,  $G$  acts on  $F(U)$ , and for any étale morphism  $V \rightarrow U$  of étale neighborhoods of  $\text{Spec}(R)$ , the canonical homomorphism  $F(U) \rightarrow F(V)$  is a  $G$ -homomorphism. Thus, the specialization map

$$s : H_{et}^i(Y_{\bar{t}}, \mathbb{Q}_\ell) = F_{\bar{t}} \rightarrow F_{\bar{\eta}} = H_{et}^i(Y_{\bar{\eta}}, \mathbb{Q}_\ell)$$

is a  $G$ -homomorphism. On the other hand, by virtue of the proper-smooth base change theorem (cf. [5, Chapter I, Lemma 8.13]),  $s$  is bijective. Thus, we get (2.4), which completes the proof of Theorem 2.1.  $\square$

**Corollary 2.5.** *Let  $g$  and  $n$  be non-negative integers with  $2g - 2 + n > 0$ , and  $\bar{\mathcal{M}}_{g,n}$  the algebraic stack classifying  $n$ -pointed stable curves of genus  $g$ . Then*

$$\dim_{\mathbb{Q}_\ell} H_{\text{et}}^i((\bar{\mathcal{M}}_{g,n})_k, \mathbb{Q}_\ell) \leq \dim_{\mathbb{Q}} H^i(\bar{\mathcal{M}}_{g,n}(\mathbb{C}), \mathbb{Q})$$

for every algebraically closed field  $k$ , every non-negative integer  $i$  and every prime  $\ell$  invertible in  $k$ .

*Proof.* By virtue of smoothness of the moduli of curves with non-abelian level structure due to Looijenga-Pikaart-de Jong-Boggi ([11], [13], [2]), especially by [2, Proposition 2.6], there are (1) a positive integer  $m$ , (2) a finite group  $G$ , (3) a smooth, proper and pure dimensional scheme  $Y$  over  $\mathbb{Z}[1/m]$ , and (4) a surjective morphism  $\pi : Y \rightarrow \bar{\mathcal{M}}_{g,n} \otimes \mathbb{Z}[1/m]$  over  $\mathbb{Z}[1/m]$  such that (a)  $m$  is invertible in  $k$ , (b)  $G$  acts on  $Y$  over  $\bar{\mathcal{M}}_{g,n} \otimes \mathbb{Z}[1/m]$  (i.e.  $\pi \cdot [g] = \pi$  for all  $g \in G$ ), and that (c)  $\bar{\mathcal{M}}_{g,n}(\mathbb{C}) \simeq Y(\mathbb{C})/G$  as analytic spaces. Here,  $(\bar{\mathcal{M}}_{g,n})_k$  is projective. Therefore, Theorem 2.1 implies our corollary.  $\square$

### 3. COMPARISON OF THE $\mathbb{Q}$ -PICARD GROUP WITH THE $\mathbb{Q}$ -NÉRON-SEVERI GROUP

In this section, we prove the following theorem.

**Theorem 3.1.** *Let  $g$  and  $n$  be non-negative integers with  $2g - 2 + n > 0$ , and  $\bar{M}_{g,n}$  the moduli space of  $n$ -pointed stable curves of genus  $g$  over an algebraically closed field  $k$ . Then, the natural homomorphism  $\text{Pic}(\bar{M}_{g,n})_{\mathbb{Q}} \rightarrow \text{NS}(\bar{M}_{g,n})_{\mathbb{Q}}$  is bijective.*

*Proof.* We need to prepare several lemmas.

**Lemma 3.2.** *Let  $f : Y \rightarrow X$  be a finite and surjective morphism of normal noetherian schemes. Then, there is a homomorphism  $\text{Nm}_{X/Y} : \text{Pic}(Y) \rightarrow \text{Pic}(X)$  such that  $\text{Nm}_{X/Y}(f^*(L)) = L^{\otimes \deg(f)}$  for all  $L \in \text{Pic}(X)$ .*

*Proof.* Let  $K'$  and  $K$  be the function fields of  $Y$  and  $X$  respectively. Let  $\text{Nm} : K' \rightarrow K$  be the norm map of  $K'$  over  $K$ . Here we claim that  $\text{Nm} : K' \rightarrow K$  gives rise to  $\text{Nm} : f_*(\mathcal{O}_Y^\times) \rightarrow \mathcal{O}_X^\times$ . This is a local question, so that we may assume that  $Y = \text{Spec}(B)$  and  $X = \text{Spec}(A)$ . In this case, our assertion means that  $\text{Nm}(x) \in A$  for all  $x \in B$ . Since  $A$  is normal, to see  $\text{Nm}(x) \in A$ , it is sufficient to check that  $\text{Nm}(x) \in A_P$  for all  $P \in \text{Spec}(A)$  with  $\text{ht}(P) = 1$ . Here  $B_P$  is flat over  $A_P$ . Thus,  $B_P$  is free as  $A_P$ -module. Hence we can see  $\text{Nm}(B_P) \subseteq A_P$ . Therefore, we get our claim.

Let  $L \in \text{Pic}(Y)$ . Then, by [12, Lecture 10, Lemma B], there is an open covering  $\{U_\alpha\}_{\alpha \in I}$  of  $X$  such that  $L|_{f^{-1}(U_\alpha)}$  is a trivial line bundle; i.e., there is  $\omega_\alpha \in L(f^{-1}(U_\alpha))$  with  $L|_{f^{-1}(U_\alpha)} = \mathcal{O}_{f^{-1}(U_\alpha)}\omega_\alpha$ . Thus, if we set  $g_{\alpha\beta} = \omega_\beta/\omega_\alpha$  for  $\alpha, \beta \in I$ , then  $g_{\alpha\beta} \in \mathcal{O}_Y^\times(f^{-1}(U_\alpha \cap U_\beta))$ , so that  $\text{Nm}(g_{\alpha\beta}) \in \mathcal{O}_X^\times(U_\alpha \cap U_\beta)$ . Therefore,  $\{\text{Nm}(g_{\alpha\beta})\}$  gives rise to a line bundle  $M$  on  $X$ . This is the definition of  $\text{Nm}_{X/Y} : \text{Pic}(Y) \rightarrow \text{Pic}(X)$ . The remaining assertion is obvious by our construction.  $\square$

**Lemma 3.3.** *Let  $f : Y \rightarrow X$  be a finite and surjective morphism of normal varieties over an algebraically closed field  $k$ . Then, we have the following.*

- (1)  $f^* : \text{Pic}(X)_{\mathbb{Q}} \rightarrow \text{Pic}(Y)_{\mathbb{Q}}$  is injective.
- (2) If  $\text{Pic}(Y)_{\mathbb{Q}} \rightarrow \text{NS}(Y)_{\mathbb{Q}}$  is injective, then so is  $\text{Pic}(X)_{\mathbb{Q}} \rightarrow \text{NS}(X)_{\mathbb{Q}}$ .

*Proof.* (1) This is a consequence of Lemma 3.2.

(2) Let us consider the following commutative diagram:

$$\begin{array}{ccc} \text{Pic}(X)_{\mathbb{Q}} & \longrightarrow & \text{NS}(X)_{\mathbb{Q}} \\ f^* \downarrow & & \downarrow f^* \\ \text{Pic}(Y)_{\mathbb{Q}} & \longrightarrow & \text{NS}(Y)_{\mathbb{Q}} \end{array}$$

By (1),  $f^* : \text{Pic}(X)_{\mathbb{Q}} \rightarrow \text{Pic}(Y)_{\mathbb{Q}}$  is injective. Thus, we have (2) using the above diagram.  $\square$

**Lemma 3.4.** *Let  $f : Y \rightarrow X$  be a finite and surjective morphism of normal noetherian schemes. If  $Y$  is locally factorial (i.e.,  $\mathcal{O}_{Y,y}$  is UFD for all  $y \in Y$ ), then, for any Weil divisor  $D$  on  $X$ ,  $\deg(f)D$  is a Cartier divisor.*

*Proof.* Clearly, we may assume that  $D$  is a prime divisor. Let  $D'$  be a Weil divisor associated with the scheme  $f^{-1}(D)$ . Then,  $D'$  is a Cartier divisor. Thus,  $\text{Nm}_{Y/X}(\mathcal{O}_Y(D')) \in \text{Pic}(X)$ . Let  $X_0$  be a Zariski open set of  $X$  such that  $D$  is a Cartier divisor on  $X_0$  and  $\text{codim}(X \setminus X_0) \geq 2$ . Then,  $\text{Nm}_{Y/X}(\mathcal{O}_Y(D')) = \mathcal{O}_X(\deg(f)D)$  on  $X_0$ . Thus,  $\text{Nm}_{Y/X}(\mathcal{O}_Y(D')) = \mathcal{O}_X(\deg(f)D)$  on  $X$ . In particular,  $\mathcal{O}_X(\deg(f)D)$  is locally free, which means that  $\deg(f)D$  is a Cartier divisor.  $\square$

**Lemma 3.5.** *Let  $Y$  be a normal projective variety over an algebraically closed field  $k$ . If  $Y$  is simply connected and there is a finite and surjective morphism  $f : Z \rightarrow Y$  of normal projective varieties such that  $Z$  is smooth over  $k$ , then the natural homomorphism*

$$\text{Pic}(Y) \otimes \mathbb{Z}[1/\deg(f)] \rightarrow \text{NS}(Y) \otimes \mathbb{Z}[1/\deg(f)]$$

*is bijective. Moreover, if  $\deg(f)$  is invertible in  $k$ , then  $\text{Pic}(Y) \rightarrow \text{NS}(Y)$  is bijective.*

*Proof.* We set  $n = \deg(f)$  and  $P = (\text{Pic}_Y^0)_{red}$ , which is a subgroup scheme of  $\text{Pic}_Y$ . For a positive integer  $\ell$ , let  $[\ell] : P \rightarrow P$  be a homomorphism given by  $[\ell](x) = \ell x$ . First we claim the following.

**Claim 3.5.1.**  $[n](P)$  is proper over  $k$ .

For this purpose, it is sufficient to see that every closed irreducible curve  $C$  in  $[n](P)$  is proper over  $k$ . For a curve  $C$  as above, there are a proper and smooth curve  $T$  over  $k$ , a non-empty Zariski open set  $T_0$  of  $T$ , and a morphism  $\phi : T_0 \rightarrow P$  such that the Zariski closure of the image  $T_0 \xrightarrow{\phi} P \xrightarrow{[n]} P$  is  $C$ . The morphism  $\phi : T_0 \rightarrow P \subseteq \text{Pic}_Y$  gives rise to a line bundle  $L_0$  on  $T_0 \times Y$  such that  $\phi(x)$  is the class of  $L_0|_{\{x\} \times Y}$  for all  $x \in T_0$ . Let us take a Cartier divisor  $D_0$  such that  $\mathcal{O}_{T_0 \times Y}(D_0) = L_0$ . Then, there is a Weil divisor  $D$  on  $T \times Y$  with  $D|_{T_0 \times Y} = D_0$ . Here  $\text{id}_T \times f : T \times Z \rightarrow T \times Y$  is a finite and surjective morphism of normal varieties such that  $T \times Z$  is smooth over  $k$ . Note that  $\deg(\text{id}_T \times f) = \deg(f) = n$ . Thus, by Lemma 3.4,  $nD$  is a Cartier divisor. Let  $\phi' : T \rightarrow P$  be a morphism given by a line bundle  $\mathcal{O}_{T \times Y}(nD)$ . Then,  $\phi'|_{T_0} = [n] \cdot \phi$ .

Here  $C$  is closed in  $P$  because  $[n](P)$  is closed in  $P$ . Thus, we can see that  $\phi'(T) = C$ . Hence  $C$  is proper over  $k$ .

Next we claim the following.

**Claim 3.5.2.**  $[\ell] : P(k) \rightarrow P(k)$  is injective for every positive integer  $\ell$  invertible in  $k$ .

Since  $Y$  is simply connected, by [5, Proposition 2.11],  $H^1(Y, (\mu_\ell)_Y) = 0$ . Thus, the Kummer exact sequence:

$$1 \rightarrow (\mu_\ell)_Y \rightarrow \mathcal{O}_Y^\times \xrightarrow{\ell} \mathcal{O}_Y^\times \rightarrow 1$$

yields an injection  $[\ell] : \text{Pic}(Y) \rightarrow \text{Pic}(Y)$ . Hence  $[\ell] : P(k) \rightarrow P(k)$  is injective.

By Claim 3.5.2,  $[\ell] : [n](P)(k) \rightarrow [n](P)(k)$  is injective for every positive integer  $\ell$  invertible in  $k$ . Thus,  $[n](P) = \{0\}$  because  $[n](P)$  is an abelian variety by Claim 3.5.1. Hence

$$P(k) \otimes \mathbb{Z}[1/n] = \{0\}.$$

Therefore, we get the first assertion because  $\text{NS}(X) \otimes \mathbb{Z}[1/n] = \text{Pic}_Y(k) \otimes \mathbb{Z}[1/n] / P(k) \otimes \mathbb{Z}[1/n]$ .

Moreover, if  $n$  is invertible in  $k$ , then  $[n] : P(k) \rightarrow P(k)$  is injective. Therefore,  $P(k) = \{0\}$ . Hence, we have the second assertion.  $\square$

**Lemma 3.6.** *Let  $f : Y \rightarrow X$  be a morphism of projective normal varieties over an algebraically closed field  $k$ . We assume that (1)  $X$  and  $Y$  are  $\mathbb{Q}$ -factorial, (2)  $\dim f^{-1}(x) = 1$  for all  $x \in X$ , and that (3) there is a non-empty open set  $X_0$  such that  $f^{-1}(x)$  is a smooth rational curve for every  $x \in X_0(k)$ . If  $D$  is a  $\mathbb{Q}$ -divisor on  $Y$  with  $D \equiv 0$ , then there is a  $\mathbb{Q}$ -divisor  $E$  on  $X$  such that  $f^*(E) \sim_{\mathbb{Q}} D$  and  $E \equiv 0$ .*

*Proof.* Clearly, we may assume that  $D$  is a Cartier divisor. Then,  $f_*(\mathcal{O}_Y(D))$  is a torsion free sheaf of rank 1 because  $D|_{f^{-1}(x)} = \mathcal{O}_{f^{-1}(x)}$  for all  $x \in X_0(k)$ . Thus, there is a divisor  $E$  on  $X$  such that  $f_*(\mathcal{O}_Y(D))^{\vee\vee} = \mathcal{O}_X(E)$ . Considering the natural homomorphism  $f^*f_*(\mathcal{O}_Y(D)) \rightarrow \mathcal{O}_Y(D)$ . We can find an effective divisor  $T$  on  $Y$  such that  $D \sim_{\mathbb{Q}} f^*(E) + T$  and  $f(T) \subseteq X \setminus X_0$ . Here  $(T \cdot C) = 0$  for all curve  $C$  with  $\dim f(C) = 0$ . Thus, using Zariski's Lemma (cf. Lemma A.1), there are a  $\mathbb{Q}$ -divisor  $S$  on  $X$  and a Zariski open set  $X_1$  of  $X$  such that  $f^*(S) \sim_{\mathbb{Q}} T$  on  $f^{-1}(X_1)$  and  $\text{codim}(X \setminus X_1) \geq 2$ . Therefore,  $D \sim_{\mathbb{Q}} f^*(E + S)$  on  $f^{-1}(X_1)$ . Here  $\text{codim}(Y \setminus f^{-1}(X_1)) \geq 2$ . Thus,  $D \sim_{\mathbb{Q}} f^*(E + S)$  on  $Y$ . It is easy to see that  $E + S \equiv 0$  using  $D \equiv 0$ .  $\square$

Let us start the proof of Theorem 3.1. First, let us consider the case  $g \geq 2$ . Then, by [2, Proposition 2.6 and Proposition 3.3], there are finite and surjective morphisms  $Z \rightarrow Y$  and  $Y \rightarrow \bar{M}_{g,n}$  of normal projective varieties over  $k$  such that  $Y$  is simply connected and  $Z$  is smooth over  $k$ . By Lemma 3.3.2 and Lemma 3.5,  $\text{Pic}(\bar{M}_{g,n})_{\mathbb{Q}} \rightarrow \text{NS}(\bar{M}_{g,n})_{\mathbb{Q}}$  is bijective.

Next let us consider the case  $g = 0, 1$ . In order to see our assertion, it is sufficient to show that if  $D \equiv 0$  for a  $\mathbb{Q}$ -divisor  $D$  on  $\bar{M}_{g,n}$ , then  $D \sim_{\mathbb{Q}} 0$ . We prove this by induction on  $n$ . First, note that  $\dim \bar{M}_{0,3} = 0$ ,  $\bar{M}_{0,4} = \mathbb{P}_k^1$  and  $\bar{M}_{1,1} = \mathbb{P}_k^1$ . Let us consider  $\pi : \bar{M}_{g,n} \rightarrow \bar{M}_{g,n-1}$ . If  $g = 0, 1$ , then a general fiber of  $\pi$  is a smooth rational curve. Moreover,  $\bar{M}_{g,n}$  and  $\bar{M}_{g,n-1}$  are  $\mathbb{Q}$ -factorial and  $\dim \pi^{-1}(x) = 1$  for all  $x \in \bar{M}_{g,n-1}$ . Thus, by Lemma 3.6, there is a  $\mathbb{Q}$ -divisor  $E$  on  $\bar{M}_{g,n-1}$  such that  $D \sim_{\mathbb{Q}} \pi^*(E)$  and  $E \equiv 0$ . By the hypothesis of induction, we have  $E \sim_{\mathbb{Q}} 0$ . Thus  $D \sim_{\mathbb{Q}} 0$ .  $\square$



**Corollary 3.7.** *Let  $g, n, \bar{M}_{g,n}$  and  $k$  be the same as in Theorem 3.1. Then, the cycle map*

$$\text{cl}^1 : \text{Pic}(\bar{M}_{g,n}) \otimes \mathbb{Q}_\ell \rightarrow H_{\text{et}}^2(\bar{M}_{g,n}, \mathbb{Q}_\ell)$$

*is injective for every prime  $\ell$  invertible in  $k$ .*

*Proof.* Since  $\bar{M}_{g,n}$  is projective, the kernel of  $\text{NS}(\bar{M}_{g,n}) \rightarrow \text{NS}^\nu(\bar{M}_{g,n})$  is finite. Thus, by Theorem 3.1, we have

$$\text{Pic}(\bar{M}_{g,n})_{\mathbb{Q}} \xrightarrow{\sim} \text{NS}(\bar{M}_{g,n})_{\mathbb{Q}} \xrightarrow{\sim} \text{NS}^\nu(\bar{M}_{g,n})_{\mathbb{Q}}.$$

Therefore, it is sufficient to show the following lemma. □

**Lemma 3.8.** *Let  $X$  be a proper algebraic spaces over an algebraically closed field  $k$ , and  $\ell$  a prime invertible in  $k$ . Let  $\pi : \text{Pic}(X) \otimes \mathbb{Z}_\ell \rightarrow \text{NS}^\nu(X) \otimes \mathbb{Z}_\ell$  be the natural homomorphism and  $\text{cl}^1 : \text{Pic}(X) \otimes \mathbb{Z}_\ell \rightarrow H_{\text{et}}^2(X, \mathbb{Z}_\ell)$  the cycle map. Then,  $\text{Ker}(\text{cl}^1) \subseteq \text{Ker}(\pi)$ . In particular,*

$$\text{Ker}(\text{Pic}(X) \otimes \mathbb{Q}_\ell \rightarrow H_{\text{et}}^2(X, \mathbb{Q}_\ell)) \subseteq \text{Ker}(\text{Pic}(X) \otimes \mathbb{Q}_\ell \rightarrow \text{NS}^\nu(X) \otimes \mathbb{Q}_\ell).$$

*Proof.* Let us consider an exact sequence

$$\text{Pic}(X) \xrightarrow{\ell^m} \text{Pic}(X) \rightarrow H^2(X_{\text{et}}, \mathbb{Z}/\ell^m\mathbb{Z})$$

arising from the Kummer exact sequence

$$0 \rightarrow \mathbb{Z}/\ell^m\mathbb{Z} \rightarrow \mathcal{O}_X^\times \xrightarrow{\ell^m} \mathcal{O}_X^\times \rightarrow 0.$$

Since  $\mathbb{Z}_\ell$  is flat over  $\mathbb{Z}$ , we have an exact sequence

$$\text{Pic}(X) \otimes \mathbb{Z}_\ell \xrightarrow{\ell^m} \text{Pic}(X) \otimes \mathbb{Z}_\ell \xrightarrow{\rho_m} H^2(X_{\text{et}}, \mathbb{Z}/\ell^m\mathbb{Z}) \otimes \mathbb{Z}_\ell = H^2(X_{\text{et}}, \mathbb{Z}/\ell^m\mathbb{Z})$$

Note that  $\text{cl}^1$  is given by  $\varinjlim \rho_m : \text{Pic}(X) \otimes \mathbb{Z}_\ell \rightarrow \varinjlim H^2(X_{\text{et}}, \mathbb{Z}/\ell^m\mathbb{Z})$ . Thus, if  $x \in \text{Ker}(\text{cl}^1)$ , then  $\rho_m(x) = 0$  for all  $m$ . Therefore, there is  $y_m \in \text{Pic}(X) \otimes \mathbb{Z}_\ell$  with  $\ell^m y_m = x$ . Let  $C$  be an irreducible curve on  $X$ . Then,  $(x \cdot C) = \ell^m (y_m \cdot C)$ . Here  $(y_m \cdot C) \in \mathbb{Z}_\ell$ . Thus,

$$(x \cdot C) \in \bigcap_m \ell^m \mathbb{Z}_\ell = \{0\}.$$

Thus,  $x \in \text{Ker}(\pi)$ . □

#### 4. LINEAR INDEPENDENCE OF THE TAUTOLOGICAL CLASSES

Let  $k$  be an algebraically closed field,  $g$  and  $n$  non-negative integers with  $2g - 2 + n > 0$ , and  $\bar{M}_{g,n}$  the moduli space of  $n$ -pointed stable curves of genus  $g$  over  $k$ . Then, we have the following (see §§1.6 for the definition of  $\bar{\Upsilon}_{g,n}$ ,  $\bar{\Upsilon}_{g,n}^e$  and the classes  $\delta_v$  ( $v \in \bar{\Upsilon}_{g,n}^e$ )):

**Proposition 4.1.** (1) *If  $g \geq 3$ , then  $\lambda, \delta_{\text{irr}}$  and  $\delta_v$ 's ( $v \in \bar{\Upsilon}_{g,n}^e$ ) are linearly independent in  $\text{NS}(\bar{M}_{g,n})_{\mathbb{Q}}$ .*

(2) *If  $g = 2$ , then  $\lambda$  and  $\delta_v$ 's ( $v \in \bar{\Upsilon}_{2,n}^e$ ) are linearly independent in  $\text{NS}(\bar{M}_{2,n})_{\mathbb{Q}}$ .*

(3) *If  $g = 1$ , then  $\lambda$  and  $\delta_v$ 's ( $v \in \bar{\Upsilon}_{1,n}^e$ ) are linearly independent in  $\text{NS}(\bar{M}_{1,n})_{\mathbb{Q}}$ .*

*Proof.* First of all, by [7, Theorem 2.2], we have the following.

(a) *If  $g \geq 3$ , then there is a morphism  $\varphi_{\text{irr}} : \mathbb{P}_k^1 \rightarrow \bar{M}_{g,n}$  such that  $\deg(\varphi_{\text{irr}}^*(\lambda)) = 0$ ,  $\deg(\varphi_{\text{irr}}^*(\delta_{\text{irr}})) = -1$  and  $\deg(\varphi_{\text{irr}}^*(\delta_v)) = 0$  ( $\forall v \in \bar{\Upsilon}_{g,n}^e$ ).*

(b) If  $g \geq 2$ , then, for every  $0 \leq i \leq g - 2$  and every  $I \subset \{1, \dots, n\}$  with  $[i, I] \in \overline{\Upsilon}_{g,n}^e$ , there is a morphism  $\varphi_{i,I} : \mathbb{P}_k^1 \rightarrow \overline{M}_{g,n}$  such that  $\deg(\varphi_{i,I}^*(\lambda)) = 0$ ,  $\deg(\varphi_{i,I}^*(\delta_{irr})) = 0$  and

$$\deg(\varphi_{i,I}^*(\delta_v)) = \begin{cases} -1 & \text{if } v = [i, I] \\ 0 & \text{if } v \neq [i, I] \end{cases} \quad (\forall v \in \overline{\Upsilon}_{g,n}^e).$$

(c) If  $g \geq 2$ , then, for every  $1 \leq i \leq g - 1$  and every  $I \subset \{1, \dots, n\}$  with  $[i, I] \in \overline{\Upsilon}_{g,n}^e$ , there is a morphism  $\varphi_{i,I} : \mathbb{P}_k^1 \rightarrow \overline{M}_{g,n}$  such that  $\deg(\varphi_{i,I}^*(\lambda)) = 0$ ,  $\deg(\varphi_{i,I}^*(\delta_{irr})) = -2$  and

$$\deg(\varphi_{i,I}^*(\delta_v)) = \begin{cases} 1 & \text{if } v = [i, I] \\ 0 & \text{if } v \neq [i, I] \end{cases} \quad (\forall v \in \overline{\Upsilon}_{g,n}^e).$$

(d) If  $g \geq 1$ , then, for every  $i, j \geq 0$  and every  $I, J \subseteq \{1, \dots, n\}$  with  $i + j \leq g - 1$ ,  $I \cap J = \emptyset$  and  $[i, I] \in \overline{\Upsilon}_{g,n}^e$ , there is a morphism  $\varphi_{i,j,I,J} : \mathbb{P}_k^1 \rightarrow \overline{M}_{g,n}$  such that  $\deg(\varphi_{i,j,I,J}^*(\lambda)) = 0$ ,  $\deg(\varphi_{i,j,I,J}^*(\delta_{irr})) = 0$  and

$$\deg(\varphi_{i,j,I,J}^*(\delta_v)) = \begin{cases} 1 & \text{if } v = [i + j, I \cup J] \\ -1 & \text{if } v = [i, I], [j, J] \\ 0 & \text{otherwise} \end{cases}$$

for all  $v \in \overline{\Upsilon}_{g,n}^e$ .

(1) First, let us consider the case  $g \geq 3$ . We assume that

$$D = a\lambda + b_{irr}\delta_{irr} + \sum_{v \in \overline{\Upsilon}_{g,n}^e} b_v \delta_v \equiv 0.$$

Then, since  $\deg(\varphi_{irr}^*(D)) = 0$ , we have  $b_{irr} = 0$ . Here  $g \geq 3$ . Thus, for every  $v \in \overline{\Upsilon}_{g,n}^e$ , we can find  $i, I$  such that  $0 \leq i \leq g - 2$ ,  $I \subseteq \{1, \dots, n\}$  and  $v = [i, I]$ . Thus, by the above (b), we can see  $b_v = 0$  for all  $v \in \overline{\Upsilon}_{g,n}^e$ . Hence  $D = a\lambda \equiv 0$ . Therefore,  $a = 0$ .

(2) Next, let us consider the case  $g = 2$ . We assume that

$$D = a\lambda + \sum_{v \in \overline{\Upsilon}_{2,n}^e} b_v \delta_v \equiv 0.$$

If  $v = [0, I]$  for some  $I \subseteq \{1, \dots, n\}$ , then, using (b), we can see  $b_v = 0$ . Otherwise, we can set  $v = [1, I']$  for some  $I' \subseteq \{1, \dots, n\}$ . Then,  $b_v = 0$  by (c). Thus  $D = a\lambda \equiv 0$ . Hence,  $a = 0$ .

(3) Finally, let us consider the case  $g = 1$ . We assume

$$D = \lambda + \sum_{v \in \overline{\Upsilon}_{1,n}^e} b_v \delta_v \equiv 0,$$

where  $b_{[0,\{i\}]} = 0$  for all  $i = 1, \dots, n$ . Then, by virtue of (d), For every non-empty  $I, J \subseteq \{1, \dots, n\}$  with  $I \cap J = \emptyset$ ,  $b_{[0,I \cup J]} = b_{[0,I]} + b_{[0,J]}$ . Therefore,

$$b_{[0,I]} = \sum_{i \in I} b_{[0,\{i\}]} = 0$$

for all non-empty  $I \subseteq \{1, \dots, n\}$ . Hence  $D = a\lambda \equiv 0$ . Therefore,  $a = 0$ .  $\square$

5. GENERATORS OF THE  $\mathbb{Q}$ -PICARD GROUP AND THE CYCLE MAP

Let  $g$  and  $n$  be non-negative integers with  $2g - 2 + n > 0$ , and  $\bar{\mathcal{M}}_{g,n}$  the algebraic stack classifying  $n$ -pointed stable curves of genus  $g$ . For an algebraically closed field  $k$ , let  $(\bar{\mathcal{M}}_{g,n})_k$  be the coarse moduli scheme of  $\bar{\mathcal{M}}_{g,n} \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(k)$ .

**Theorem 5.1.**  $\text{Pic}((\bar{\mathcal{M}}_{g,n})_k)_{\mathbb{Q}}$  is generated by

$$\lambda, \psi_1, \dots, \psi_n, \delta_{irr} \text{ and } \delta_v \text{'s } (v \in \bar{\Upsilon}_{g,n})$$

for any algebraically closed field  $k$ . Moreover, the cycle map

$$\text{Pic}((\bar{\mathcal{M}}_{g,n})_k) \otimes \mathbb{Q}_{\ell} \rightarrow H_{et}^2((\bar{\mathcal{M}}_{g,n})_k, \mathbb{Q}_{\ell})$$

is bijective for every prime  $\ell$  invertible in  $k$ .

*Proof.* By Corollary 3.7, the cycle map

$$\text{Pic}((\bar{\mathcal{M}}_{g,n})_k) \otimes \mathbb{Q}_{\ell} \rightarrow H_{et}^2((\bar{\mathcal{M}}_{g,n})_k, \mathbb{Q}_{\ell})$$

is injective. Hence, we get

$$\dim_{\mathbb{Q}} \text{Pic}((\bar{\mathcal{M}}_{g,n})_k)_{\mathbb{Q}} \leq \dim_{\mathbb{Q}_{\ell}} H_{et}^2((\bar{\mathcal{M}}_{g,n})_k, \mathbb{Q}_{\ell}).$$

Moreover, by Corollary 2.5,

$$\dim_{\mathbb{Q}_{\ell}} H_{et}^2((\bar{\mathcal{M}}_{g,n})_k, \mathbb{Q}_{\ell}) \leq \dim_{\mathbb{Q}} H^2(\bar{\mathcal{M}}_{g,n}(\mathbb{C}), \mathbb{Q}).$$

Therefore,

$$(5.1.1) \quad \dim_{\mathbb{Q}} \text{Pic}((\bar{\mathcal{M}}_{g,n})_k)_{\mathbb{Q}} \leq \dim_{\mathbb{Q}} H^2(\bar{\mathcal{M}}_{g,n}(\mathbb{C}), \mathbb{Q})$$

and if the equation holds, then the cycle map is bijective.

In the case  $g = 0$ , the assertions of our theorem are well known (for example, see [8]), so that it is sufficient to show the following (a)–(c) and  $\dim_{\mathbb{Q}} \text{Pic}((\bar{\mathcal{M}}_{g,n})_k)_{\mathbb{Q}} = \dim_{\mathbb{Q}} H^2(\bar{\mathcal{M}}_{g,n}(\mathbb{C}), \mathbb{Q})$  for each case.

- (a) If  $g \geq 3$ , then  $\lambda, \psi_1, \dots, \psi_n, \delta_{irr}$  and  $\delta_v$ 's ( $v \in \bar{\Upsilon}_{g,n}$ ) form a basis of  $\text{Pic}((\bar{\mathcal{M}}_{g,n})_k)_{\mathbb{Q}}$ .
- (b) If  $g = 2$ , then  $\lambda, \psi_1, \dots, \psi_n$  and  $\delta_v$ 's ( $v \in \bar{\Upsilon}_{2,n}$ ) form a basis of  $\text{Pic}((\bar{\mathcal{M}}_{2,n})_k)_{\mathbb{Q}}$ .
- (c) If  $g = 1$ , then  $\lambda$  and  $\delta_v$ 's ( $v \in \bar{\Upsilon}_{1,n}$ ) form a basis of  $\text{Pic}((\bar{\mathcal{M}}_{1,n})_k)_{\mathbb{Q}}$ .

First of all, it is well known (cf. [1]) that  $H^2(\bar{\mathcal{M}}_{g,n}(\mathbb{C}), \mathbb{Q})$  is generated by

$$\lambda, \psi_1, \dots, \psi_n, \delta_{irr} \text{ and } \delta_v \text{'s } (v \in \bar{\Upsilon}_{g,n}).$$

(a) By Proposition 4.1.1,  $\lambda, \psi_1, \dots, \psi_n, \delta_{irr}$  and  $\delta_v$ 's ( $v \in \bar{\Upsilon}_{g,n}$ ) are linearly independent in  $H^2(\bar{\mathcal{M}}_{g,n}(\mathbb{C}), \mathbb{Q})$ , and these are also linearly independent in  $\text{Pic}((\bar{\mathcal{M}}_{g,n})_k)_{\mathbb{Q}}$ . Thus, by (5.1.1),

$$\lambda, \psi_1, \dots, \psi_n, \delta_{irr} \text{ and } \delta_v \text{'s } (v \in \bar{\Upsilon}_{g,n})$$

gives rise to a basis of  $\text{Pic}((\bar{\mathcal{M}}_{g,n})_k)_{\mathbb{Q}}$ , and  $\dim_{\mathbb{Q}} \text{Pic}((\bar{\mathcal{M}}_{g,n})_k)_{\mathbb{Q}} = \dim_{\mathbb{Q}} H^2(\bar{\mathcal{M}}_{g,n}(\mathbb{C}), \mathbb{Q})$ .

(b) We know that  $10\lambda = \delta_{irr} + 2\delta_1$  on  $\bar{\mathcal{M}}_2(\mathbb{C})$ . Thus,  $H^2(\bar{\mathcal{M}}_{2,n}(\mathbb{C}), \mathbb{Q})$  is generated by

$$\lambda, \psi_1, \dots, \psi_n \text{ and } \delta_v \text{'s } (v \in \bar{\Upsilon}_{2,n}).$$

By Proposition 4.1.2, these are linearly independent in  $H^2(\bar{\mathcal{M}}_{2,n}(\mathbb{C}), \mathbb{Q})$ . Moreover, these are also linearly independent in  $\text{Pic}((\bar{\mathcal{M}}_{2,n})_k)_{\mathbb{Q}}$ . Thus, (5.1.1) shows us that these form a basis of  $\text{NS}((\bar{\mathcal{M}}_{2,n})_k)_{\mathbb{Q}}$  and that  $\dim_{\mathbb{Q}} \text{Pic}((\bar{\mathcal{M}}_{2,n})_k)_{\mathbb{Q}} = \dim_{\mathbb{Q}} H^2(\bar{\mathcal{M}}_{2,n}(\mathbb{C}), \mathbb{Q})$ .

(c) First, we know  $\delta_{irr} = 12\lambda$  on  $\bar{\mathcal{M}}_{1,1}(\mathbb{C})$  and  $\psi_i = \lambda + \sum_{i \in I, |I| \geq 2} \delta_{[0,I]}$  for all  $i$  on  $\bar{\mathcal{M}}_{1,n}(\mathbb{C})$ . Thus,  $H^2(\bar{\mathcal{M}}_{1,n}(\mathbb{C}), \mathbb{Q})$  is generated by  $\lambda$  and  $\delta_v$ 's ( $v \in \bar{\mathcal{Y}}_{1,n}$ ). Therefore, by using Proposition 4.1.3 and (5.1.1),  $\lambda$  and  $\delta_v$ 's ( $v \in \bar{\mathcal{Y}}_{1,n}$ ) form a basis of  $\text{Pic}((\bar{\mathcal{M}}_{1,n})_k)_{\mathbb{Q}}$  and  $\dim_{\mathbb{Q}} \text{Pic}((\bar{\mathcal{M}}_{1,n})_k)_{\mathbb{Q}} = \dim_{\mathbb{Q}} H^2(\bar{\mathcal{M}}_{1,n}(\mathbb{C}), \mathbb{Q})$ .  $\square$

**Corollary 5.2.** *Let  $g$  and  $n$  be non-negative integers with  $2g - 2 + n > 0$ , and  $M_{g,n}$  the moduli space of  $n$ -pointed smooth curves of genus  $g$  over an algebraically closed field  $k$ . Then,  $\text{Pic}(M_{g,n})_{\mathbb{Q}}$  is generated by  $\lambda, \psi_1, \dots, \psi_n$ .*

*Proof.* Let us consider the restriction map  $\text{Pic}(\bar{M}_{g,n})_{\mathbb{Q}} \rightarrow \text{Pic}(M_{g,n})_{\mathbb{Q}}$ . Since  $\bar{M}_{g,n}$  is  $\mathbb{Q}$ -factorial, it is surjective. Thus, Theorem 5.1 implies our corollary.  $\square$

### Appendix A. ZARISKI'S LEMMA FOR INTEGRAL SCHEME

Let  $R$  be a discrete valuation ring, and  $f : Y \rightarrow \text{Spec}(R)$  a flat and projective integral scheme over  $R$ . Let  $\eta$  be the generic point of  $\text{Spec}(R)$  and  $o$  the closed point of  $\text{Spec}(R)$ . We assume that the generic fiber  $Y_{\eta}$  of  $f$  is geometrically reduced and irreducible curve. Let  $Y_o$  be the special fiber of  $f$ , i.e.,  $Y_o = f^*(o)$ . Let us consider a pairing

$$\text{Pic}(Y) \otimes \text{CH}^0(Y_o) \rightarrow \text{CH}^1(Y_o)$$

given by the composition of homomorphisms

$$\text{Pic}(Y) \otimes \text{CH}^0(Y_o) \rightarrow \text{Pic}(Y_o) \otimes \text{CH}^0(Y_o) \rightarrow \text{CH}^1(Y_o).$$

We denote by  $x \cdot z$  the image of  $x \otimes z$  by the above homomorphism. For a Cartier divisor  $D$  on  $Y$ , the associated cycle of  $D$  is denoted by  $[D]$ , which is an element of  $Z^1(Y)$ . Let us consider the following subgroup  $F_c(Y)$  of  $Z^0(Y_o)$ :

$$F_c(Y) = \{x \in Z^0(Y_o) \mid x = [D] \text{ for some Cartier divisor } D \text{ on } Y\}.$$

For a Cartier divisor  $D$  on  $Y$  with  $[D] \in F_c(Y)$ , and  $y \in F_c(Y)$ ,  $D \cdot y$  depend only on  $[D]$ . For, if  $D'$  is a Cartier divisor on  $Y$  with  $[D'] = [D]$ , and  $E$  is a Cartier divisor on  $Y$  with  $y = [E]$ , then, by [6, Theorem 2.4],

$$D \cdot y = E \cdot [D] = E \cdot [D'] = D' \cdot y.$$

Thus, we can define a bi-linear map

$$q : F_c(Y) \times F_c(Y) \rightarrow \text{CH}^1(Y_o)$$

by  $q([D], y) = D \cdot y$ . Moreover, [6, Theorem 2.4] says us that  $q$  is symmetric; i.e.,  $q(x, y) = q(y, x)$  for all  $x, y \in F_c(Y)$ . Then, we define the quadratic form  $Q$  on  $F_c(Y)$  by

$$Q(x, y) = \deg(q(x, y)).$$

Then, we have the following Zariski's lemma on integral schemes.

**Lemma A.1** (Zariski's lemma for integral scheme). (1)  $Q([Y_o], x) = 0$  for all  $x \in F_c(Y)_{\mathbb{Q}}$ .  
 (2)  $Q(x, x) \leq 0$  for any  $x \in F_c(Y)_{\mathbb{Q}}$ .  
 (3)  $Q(x, x) = 0$  if and only if  $x \in \mathbb{Q} \cdot [Y_o]$ .

*Proof.* (1): This is obvious because  $\mathcal{O}_Y(Y_o) \simeq \mathcal{O}_Y$ .

(2) and (3): If  $x \in \mathbb{Q} \cdot [Y_o]$ , then by (1),  $Q(x, x) = 0$ . Thus, it is sufficient to prove that (a)  $Q(x, x) \leq 0$  for any  $x \in F_c(Y)_{\mathbb{Q}}$ , and that (b) if  $Q(x, x) = 0$ , then  $x \in \mathbb{Q} \cdot [Y_o]$ . Here we need the following sublemma.

**Sublemma A.1.1.** *Let  $V$  be a finite dimensional vector space over  $\mathbb{R}$ , and  $Q$  a quadratic form on  $V$ . We assume that there are  $e \in V$  and a basis  $\{e_1, \dots, e_n\}$  of  $V$  with the following properties:*

- (i) *If we set  $e = a_1 e_1 + \dots + a_n e_n$ , then  $a_i > 0$  for all  $i$ .*
- (ii)  *$Q(x, e) \leq 0$  for all  $x \in V$ .*
- (iii)  *$Q(e_i, e_j) \geq 0$  for all  $i \neq j$ .*
- (iv) *If we set  $S = \{(i, j) \mid i \neq j \text{ and } Q(e_i, e_j) > 0\}$ , then, for any  $i \neq j$ , there is a sequence  $i_1, \dots, i_l$  such that  $i_1 = i$ ,  $i_l = j$ , and  $(i_t, i_{t+1}) \in S$  for all  $1 \leq t < l$ .*

*Then,  $Q(x, x) \leq 0$  for all  $x \in V$ . Moreover, if  $Q(x, x) = 0$  for some  $x \neq 0$ , then  $x \in \mathbb{R}e$  and  $Q(y, e) = 0$  for all  $y \in V$ .*

*Proof.* Replacing  $e_i$  by  $a_i e_i$ , we may assume that  $a_1 = \dots = a_n = 1$ . If we set  $x = x_1 e_1 + \dots + x_n e_n$ , then, by an easy calculation, we can show

$$Q(x, x) = \sum_i x_i^2 Q(e_i, e) - \sum_{i < j} (x_i - x_j)^2 Q(e_i, e_j).$$

Thus, we can easily see our assertions. □

Let us go back to the proof of Lemma A.1. First, we assume that  $Y$  is regular. Let  $(Y_o)_{red} = E_1 + \dots + E_n$  be the irreducible decomposition of  $(Y_o)_{red}$ . Since  $Y$  is regular,  $E_i$ 's are Cartier divisors on  $Y$  and  $[E_i] \in F_c(Y)$  for all  $i$ . Moreover, we can set  $Y_o = a_1 E_1 + \dots + a_n E_n$  for some positive integers  $a_1, \dots, a_n$ . Thus, if we set  $e = [Y_o]$  and  $e_i = [E_i]$  for  $i = 1, \dots, n$ , then (i), (ii) and (iii) in the above sublemma hold. Moreover, since  $Y_o$  is geometrically connected, (iv) also holds. Thus, we have our assertion in the case where  $Y$  is regular.

Next, let us consider a general case. Clearly we may assume that  $x \in F_c(Y)$ ; i.e.,  $x = [D]$  for some Cartier divisor  $D$  on  $Y$ . By virtue of [10], there is a birational morphism  $\mu : Y' \rightarrow Y$  of projective schemes over  $R$  such that  $Y'$  is regular. Using the projection formula (cf. [6, (c) of Proposition 2.4]),

$$\deg(\mathcal{O}_{Y'}(\mu^*(D)) \cdot [\mu^*(D)]) = \deg(\mathcal{O}_Y(D) \cdot [D]).$$

Thus, if  $Q([\mu^*(D)], [\mu^*(H)]) \leq 0$ , then  $Q([D], [D]) \leq 0$ . Moreover, if there is a rational number  $\alpha$  such that  $[\mu^*(D)] = \alpha[Y'_o]$ , then  $[\mu^*(D)] = \alpha[\mu^*(Y_o)]$ . Thus, taking the push-forward  $\mu_*$ , we can see that  $[D] = \alpha[Y_o]$  in  $Z^1(Y)_{\mathbb{Q}}$ . Hence, we get our lemma. □

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