On monads and vector bundles over the projective plane

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Let k be an algebraically closed field of any characteristic and $\mathbb{P}^2$ the projective plane over k. We discuss vector bundles over $\mathbb{P}^2$ with $c_1 = 0$ throughout this paper. The main theorem is that there is a 1-1 correspondence between μ-stable vector bundles and θ-stable modules over a certain 5-dimensional algebra. These subjects were discussed in [5] and the existence of the 1-1 correspondence was proved only in the rank 2 case or the rank 3 and $c_2 \leq 7$ case. In the present paper we prove that the 1-1 correspondence exists for every rank and c_2.

We refer to related works of others: there exist correspondences between Gieseker-Maruyama stable sheaves and θ-stable modules over some algebras, which allow us to describe the moduli spaces of stable vector bundles (see [12], [1] and so on). The present paper deals with narrow objects compared with [12] and [1], but it is useful that we have various styles but these papers of others to describe vector bundles. Indeed, we can easily calculate several representations of the 5-dimensional algebra in the present paper and we can construct several μ-stable vector bundles with $c_1 = 0$ explicitly, although it is hard to develop the representation theory on the 5-dimensional algebra. As an application of the main theorem in this paper, we can determine μ-stability of several vector bundles (see [10]).

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1. Preliminaries

Throughout this paper we assume that k is an algebraically closed field. Let V be a 3-dimensional vector space over k and V^* the dual space of V. Set $\mathbb{P}^2 := \operatorname{Proj} S(V^*)$, that is, the projective plane over k. In this section we will use Hulek [5] as our main reference.

Definition 1.1 (K.Hulek [5]). Let $\mathcal{F}$ be a vector bundle over $\mathbb{P}^2$ with $c_1(\mathcal{F}) = 0$. We say that $\mathcal{F}$ is *s-stable* if $h^0(\mathcal{F}) = h^0(\mathcal{F}^\vee) = 0$.

Remark 1.2. Every μ-stable vector bundle with $c_1 = 0$ and $\mathrm{rk} \geq 2$ is s-stable. Conversely any s-stable vector bundle with rank 2 or 3 is μ-stable. However it is not true that s-stability implies μ-stability for the case rank ≥ 4. Indeed, any direct sum of s-stable vector bundles is again s-stable, but not μ-stable.

Definition 1.3. A complex of vector bundles

$$0 \to \mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C} \to 0$$

is called *a monad* if f is injective, g is surjective, $g \circ f = 0$, and $\mathrm{Im} f$ is a subbundle of $\mathcal{B}$. We say that $\mathcal{F} := \mathrm{Ker}(g)/\mathrm{Im}(f)$ is the *cohomology bundle* of the monad.

Definition 1.4. Let n be an integer with $n \geq 2$. We call a k-linear map $\alpha : k^n \otimes V^* \to k^n$ a *Kronecker module.* Fix a basis $z_0, z_1, z_2 \in V^*$. For a Kronecker module α, we define the k-linear maps α_i by

$$\begin{array}{rcccc} \alpha_i & : & k^n & \to & k^n \\ & & \phi & \mapsto & \alpha(\phi \otimes z_i), \end{array}$$

for $i = 0, 1, 2$. Note that giving a Kronecker module α is equivalent to giving three matrices $(\alpha_0, \alpha_1, \alpha_2) \in \mathrm{M}_n(k)^{\oplus 3}$.

Definition 1.5. We say that a Kronecker module α is *prestable* if $\dim\langle \alpha_0(\phi), \alpha_1(\phi), \alpha_2(\phi)\rangle \geq 2$ and $\dim\langle {}^t\alpha_0(\phi), {}^t\alpha_1(\phi), {}^t\alpha_2(\phi)\rangle \geq 2$ for each $0 \neq \phi \in k^n$. Here we denote by $\langle v_0, v_1, v_2\rangle$ the subspace of k^n generated by v_0, v_1, and $v_2 \in k^n$.

For two Kronecker modules α, β we say that α and β are *equivalent* if there exist $\omega_1, \omega_2 \in \mathrm{GL}_n(k)$ such that $\omega_1 \alpha_i \omega_2 = \beta_i$ for $i = 0, 1, 2$. If α and β are equivalent, then we simply write $\alpha \sim \beta$. If $\alpha \sim \beta$ and if α is prestable, then β is also prestable.

For a Kronecker module α, we define

$$A(\alpha) := \begin{pmatrix} 0 & \alpha_2 & -\alpha_1 \\ -\alpha_2 & 0 & \alpha_0 \\ \alpha_1 & -\alpha_0 & 0 \end{pmatrix} \in \mathrm{M}_{3n}(k)$$

Let $v_0, v_1, v_2 \in V$ be the dual basis of z_0, z_1, z_2. From a Kronecker module α we construct a monad as follows (for details, see [5]):

$$\begin{array}{ccccccc} & \mathcal{A} & & & & \mathcal{B} := \mathrm{Im}A(\alpha) \otimes \mathcal{O} & \\ & \| & & & \nearrow & \downarrow & \\ 0 \longrightarrow & k^n \otimes \mathcal{O}(-1) & \longrightarrow & k^n \otimes V \otimes \mathcal{O} & \xrightarrow{A(\alpha)} & k^n \otimes V^* \otimes \mathcal{O} & \\ & & & \phi \otimes v_i & \longmapsto & \alpha_{i+1}(\phi) \otimes z_{i-1} & \\ & & & & & -\alpha_{i-1}(\phi) \otimes z_{i+1} & \end{array}$$

$$\begin{array}{ccccccccc} & & \mathcal{B} & & & & \mathcal{C} & & \\ & & \| & & & & \| & & \\ 0 & \longrightarrow & \mathrm{Im}A(\alpha)\otimes\mathcal{O} & \longrightarrow & k^n\otimes V^*\otimes\mathcal{O} & \longrightarrow & k^n\otimes\mathcal{O}(1) & \longrightarrow & 0. \end{array}$$

Thus we get the monad

$$0 \longrightarrow k^n \otimes \mathcal{O}(-1) \longrightarrow \mathrm{Im}A(\alpha) \otimes \mathcal{O} \longrightarrow k^n \otimes \mathcal{O}(1) \longrightarrow 0.$$

We denote by $\mathcal{F}(\alpha)$ the cohomology bundle obtained from the above monad.

Here we quote the following theorem from Hulek [5].

Theorem 1.6 (Hulek [5]). *Let $n \geq r \geq 2$. There exists a one to one correspondence from the set of equivalence classes of prestable Kronecker modules $k^n \otimes V^* \to k^n$ with $\mathrm{rk}A(\alpha) = 2n + r$ to the set of isomorphism classes of s-stable vector bundles with $c_2 = n$ and $\mathrm{rk} = r$.*

$$\begin{array}{ccc} \{\alpha : k^n \otimes V^* \to k^n \mid \textit{ prestable and } \mathrm{rk}A(\alpha) = 2n + r\}/\sim & \ni & \alpha \\ \downarrow & & \downarrow \\ \{\mathcal{F} \mid \textit{ an s-stable vector bundle with } c_2 = n, \mathrm{rk} = r\}/\cong & \ni & \mathcal{F}(\alpha). \end{array}$$

Remark 1.7. Let $\mathcal{F}$ be an s-stable vector bundle with rank r and $c_2(\mathcal{F}) = n$. Then $h^1(\mathcal{F}(-2)) = h^1(\mathcal{F}(-1)) = n$. Moreover $h^2(\mathcal{F}) = 0$, $h^1(\mathcal{F}) = -\chi(\mathcal{F}) = n - r$. Hence $n \geq r$. The inverse of the above correspondence is given by $\mathcal{F} \mapsto$ the canonical k-linear map $H^1(\mathcal{F}(-2)) \otimes V^* \to H^1(\mathcal{F}(-1))$.

Remark 1.8. Let α be a prestable Kronecker module and $\mathcal{F}$ be the corresponding s-stable vector bundle. Let us consider the matrix $x_0\alpha_0 + x_1\alpha_1 + x_2\alpha_2 \in \mathrm{M}_n(k)$ for $(x_0, x_1, x_2) \in k^3 - \{(0,0,0)\}$. Set $\ell = \{x_0z_0 + x_1z_1 + x_2z_2 = 0\} \subset \mathbb{P}^2$. Then

$$x_0\alpha_0 + x_1\alpha_1 + x_2\alpha_2 \in \mathrm{GL}_n(k) \Longleftrightarrow \mathcal{F}|_\ell \cong \mathcal{O}_\ell^{\oplus r}.$$

2. Quiver

In this section, we introduce a special quiver $\mathcal{Q}$ and the associated path algebra $k[\mathcal{Q}]$. For describing s-stable vector bundles, we use representations of the special quiver $\mathcal{Q}$ instead of Kronecker modules. We introduce the notion of θ-stability of $k[\mathcal{Q}]$-modules according to King [7] and give basic results on θ-stability.

We now consider the following quiver:

$$\mathcal{Q} := \quad \begin{tikzcd} \overset{e_0}{\bullet} \arrow[r, bend left, "\alpha_0"] \arrow[r, "\alpha_1"] \arrow[r, bend right, "\alpha_2"'] & \overset{e_1}{\bullet} \end{tikzcd} .$$

We denote by $k[\mathcal{Q}]$ the path algebra over k associated to the quiver $\mathcal{Q}$ (for definitions see [8]). We can explicitly write down a 5-dimensional algebra $k[\mathcal{Q}]$ with unit 1 as follows:

$$k[\mathcal{Q}] := k[e_0, e_1, \alpha_0, \alpha_1, \alpha_2],$$

$$e_0^2 = e_0, \quad e_1^2 = e_1, \quad e_0 e_1 = e_1 e_0 = 0,$$

$$\alpha_j e_0 = \alpha_j, \quad e_1 \alpha_j = \alpha_j, \quad e_0 \alpha_j = \alpha_j e_1 = 0,$$

$$\alpha_i \alpha_j = 0, \quad 1 = e_0 + e_1.$$

For a commutative ring R we define the path algebra $R[\mathcal{Q}]$ over R in the similar way.

Let $\gamma_0, \gamma_1, \gamma_2 \in \mathrm{Hom}_k(k^m, k^n)$. We construct the left $k[\mathcal{Q}]$-module $M := k^m \oplus k^n$ as follows:

$$\begin{array}{cccccc} e_0 \cdot & : & k^m \oplus k^n & \longrightarrow & k^m \oplus k^n \\ & & (v, w) & \mapsto & (v, 0), \\ e_1 \cdot & : & k^m \oplus k^n & \longrightarrow & k^m \oplus k^n \\ & & (v, w) & \mapsto & (0, w), \\ \alpha_j \cdot & : & k^m \oplus k^n & \longrightarrow & k^m \oplus k^n \\ & & (v, w) & \mapsto & (0, \alpha_j(v)). \end{array}$$

We call (m, n) the *dimension vector* of M.

We remark that any finite-dimensional left $k[\mathcal{Q}]$-module can be obtained as above for suitable $\gamma_0, \gamma_1, \gamma_2 \in \mathrm{Hom}_k(k^m, k^n)$. Let M_γ and M_δ be the $k[\mathcal{Q}]$-modules corresponding to $\gamma_0, \gamma_1, \gamma_2 \in \mathrm{Hom}_k(k^m, k^n)$ and $\delta_0, \delta_1, \delta_2 \in \mathrm{Hom}_k(k^{m'}, k^{n'})$, respectively. Then we have

$$\mathrm{Hom}_{k[\mathcal{Q}]}(M_\gamma, M_\delta)$$
$$= \{(f_0, f_1) \in \underset{k}{\mathrm{Hom}}(k^m, k^{m'}) \oplus \underset{k}{\mathrm{Hom}}(k^n, k^{n'}) \mid \delta_j f_0 = f_1 \gamma_j (j = 0, 1, 2)\}.$$

For a Kronecker module $\alpha = (\alpha_0, \alpha_1, \alpha_2) : k^n \otimes V^* \to k^n$, we denote by M_α the $k[\mathcal{Q}]$-module corresponding to $\alpha_0, \alpha_1, \alpha_2 \in \mathrm{Hom}_k(k^n, k^n)$.

Remark 2.1. The correspondence $\alpha = (\alpha_0, \alpha_1, \alpha_2) \mapsto M_\alpha$ is a bijection from the set of equivalence classes of Kronecker modules to the set of isomorphism classes of $k[\mathcal{Q}]$-modules of dimension vector (n, n).

$$\left\{ \begin{array}{ll} \alpha : & k^n \otimes V^* \to k^n \\ & \text{a Kronecker module} \end{array} \right\} \Big/ \sim$$

$$\xrightarrow{\simeq} \{\ M \mid M:\ \text{a } k[\mathcal{Q}]\text{-module of dimension vector } (n,n)\ \}/_{\cong}\ .$$

We now prepare some terminology on $k[\mathcal{Q}]$-modules.

Definition 2.2. Let M be a $k[\mathcal{Q}]$-module of dimension vector (n,n) with $n \geq 2$. We say that M is *prestable* if M has no submodule of dimension vector $(1,1)$ or $(n-1,n-1)$. Note that if a Kronecker module α corresponds to the $k[\mathcal{Q}]$-module M_α, then α is prestable if and only if M_α is a prestable $k[\mathcal{Q}]$-module.

Definition 2.3. Let M be a $k[\mathcal{Q}]$-module of dimension vector (n,n). Let $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ be the Kronecker module associated to M. Then we define $r(M)$ by

$$r(M) := \operatorname{rk}\begin{pmatrix} 0 & \alpha_2 & -\alpha_1 \\ -\alpha_2 & 0 & \alpha_0 \\ \alpha_1 & -\alpha_0 & 0 \end{pmatrix} - 2n.$$

Note that $r(M)$ is independent of choices of Kronecker modules associated to M.

Definition 2.4. We define two k-categories as follows:

$$\mathcal{M}(r;0,n)^{\text{s-stable}} := \left\{ \mathcal{F} \;\middle|\; \begin{array}{c} \mathcal{F}:\ \text{an s-stable vector bundle over } \mathbb{P}^2 \\ c_1 = 0, c_2 = n, \operatorname{rk} = r \end{array} \right\}$$

$$\mathcal{M}od\text{-}k[\mathcal{Q}](n;r)^{\text{prestable}} := \left\{ M \;\middle|\; \begin{array}{l} \text{a prestable } k[\mathcal{Q}]\text{-module} \\ \text{of dimension vector} \\ (n,n) \text{ with } r(M) = r \end{array} \right\}.$$

In the similar way we define k-categories $\bigcup_{2\leq r\leq n} \mathcal{M}(r;0,n)^{\text{s-stable}}$ and $\bigcup_{2\leq r\leq n} \mathcal{M}od\text{-}k[\mathcal{Q}](n;r)^{\text{prestable}}$.

We can translate Theorem 1.6 into the following proposition under the preparation above.

Proposition 2.5. *There exists an equivalence of k-categories*

$$\bigcup_{2\leq r\leq n} \mathcal{M}(r;0,n)^{s\text{-}stable} \xrightarrow{\simeq} \bigcup_{2\leq r\leq n} \mathcal{M}od\text{-}k[\mathcal{Q}](n;r)^{prestable},$$

which induces an equivalence between k-categories $\mathcal{M}(r;0,n)^{s\text{-}stable}$ and $\mathcal{M}od\text{-}k[\mathcal{Q}](n;r)^{prestable}$ for $2 \leq r \leq n$.

Let us define the k-anti algebra homomorphism S by

$$\begin{array}{rcccc} S & : & k[\mathcal{Q}] & \longrightarrow & k[\mathcal{Q}] \\ & & e_0 & \mapsto & e_1, \\ & & e_1 & \mapsto & e_0, \\ & & \alpha_j & \mapsto & \alpha_j. \end{array}$$

Let M be a $k[\mathcal{Q}]$-module of dimension vector (n,n). Let $\rho : k[\mathcal{Q}] \to \mathrm{End}_k(M)$ be the k-algebra homomorphism associated to the $k[\mathcal{Q}]$-module structure of M. We define the $k[\mathcal{Q}]$-module structure on $M^\vee := \mathrm{Hom}_k(M,k)$ by $k[\mathcal{Q}] \xrightarrow{S} k[\mathcal{Q}] \xrightarrow{{}^t\rho} \mathrm{End}_k(M^\vee)$. If M is prestable, then $M^\vee$ is also prestable. If an s-stable vector bundle $\mathcal{F}$ corresponds to M, then the dual $\mathcal{F}^\vee$ corresponds to $M^\vee$.

Here we introduce θ-stability of $k[\mathcal{Q}]$-modules and vector bundles. Let us define the homomorphism θ by

$$\begin{array}{rcccc} \theta & : & K_0(mod\text{-}k[\mathcal{Q}]) & \to & \mathbb{Z} \\ & & M & \mapsto & n-m, \end{array}$$

where M is a $k[\mathcal{Q}]$-module of dimension vector (m,n). Here we denote by mod-$k[\mathcal{Q}]$ the category of finitely generated (or finite-dimensional) $k[\mathcal{Q}]$-modules.

Definition 2.6 (A.D. King [7])**.** Let M be a finite-dimensional $k[\mathcal{Q}]$-module.

(i) M is *θ-semistable* $\overset{\text{def}}{\Longleftrightarrow}$ $\theta(N) \geq 0$ for each $k[\mathcal{Q}]$-submodule N of M.

(ii) M is *θ-stable* $\overset{\text{def}}{\Longleftrightarrow}$ M is θ-semistable and $\theta(N) > 0$ holds for each $k[\mathcal{Q}]$-submodule $N \neq 0, M$.

Remark 2.7. Suppose that M is a $k[\mathcal{Q}]$-module of dimension vector (n,n) with $n \geq 2$.

(i) M is θ-semistable $\Longleftrightarrow$ M has no $k[\mathcal{Q}]$-submodule of dimension vector (s,t) with $s > t$.

(ii) M is θ-stable $\Longleftrightarrow$ M has no $k[\mathcal{Q}]$-submodule of dimension vector (m,m) with $0 < m < n$.

In the sequel, we assume that any $k[\mathcal{Q}]$-module M has the dimension vector (n,n) with $n \geq 2$.

For $k[\mathcal{Q}]$-modules we easily see that

$$\begin{array}{ccc} (\ \theta\text{-stable}\ ) & \Longrightarrow & (\ \text{prestable}\ ) \\ \Downarrow & & \\ (\ \theta\text{-semistable}\ ). & & \end{array}$$

However prestability and θ-semistability do not imply each other.

Definition 2.8. Let $\mathcal{F}$ be an s-stable vector bundle. We say that $\mathcal{F}$ is a *θ-stable vector bundle* if the associated prestable $k[\mathcal{Q}]$-module is θ-stable.

Lemma 2.9. *If $\mathcal{F}$ is a θ-stable vector bundle, then $\mathcal{F}^\vee$ is also θ-stable.*

Proof. Let M be the θ-stable $k[\mathcal{Q}]$-module associated to $\mathcal{F}$. It is easy to check that $M^\vee$ is θ-stable. Hence $\mathcal{F}^\vee$ is θ-stable since it is the s-stable vector bundle associated to $M^\vee$. □

Lemma 2.10. *Let M and N be $k[\mathcal{Q}]$-modules of dimension vectors (m, m) and (n, n) respectively. Let $f : M \to N$ be a non-zero $k[\mathcal{Q}]$-module homomorphism.*

(i) *M : θ-stable, N : θ-semistable $\Longrightarrow f$: injective*
(ii) *M : θ-semistable, N : θ-stable $\Longrightarrow f$: surjective*

Proof. It is an easy lemma, so here we prove only (i). Suppose that f is not injective. Let (s, t) be the dimension vector of $\mathrm{Ker} f$. Since M is θ-stable and $\mathrm{Ker} f \neq 0, M$, we get $s < t$. The image $\mathrm{Im} f$ is a $k[\mathcal{Q}]$-submodule of N with dimension vector $(m - s, m - t)$. N is θ-semistable, while $m - s > m - t$. It is a contradiction. Therefore $\mathrm{Ker} f = 0$. □

Corollary 2.11. *Let M and N be as above. Suppose that M, N are θ-stable. If $f : M \to N$ is a non-zero $k[\mathcal{Q}]$-module homomorphism, then f is an isomorphism.*

The above corollary can be described in terms of vector bundles as follows.

Corollary 2.12. *Let $\mathcal{F}$ and $\mathcal{G}$ be θ-stable vector bundles. Let $f : \mathcal{F} \to \mathcal{G}$ be a non-zero morphism. Then f is an isomorphism.*

Corollary 2.13. *Let M be a θ-stable $k[\mathcal{Q}]$-module of dimension vector (n, n) with $n \geq 2$. Then $\mathrm{End}_{k[\mathcal{Q}]}(M) \cong k$. In particular if $\mathcal{F}$ is a θ-stable vector bundle, then it is simple.*

Proof. Let $f : M \to M$ be a non-zero $k[\mathcal{Q}]$-module homomorphism. Let γ be the Kronecker module associated to M. Then f can be described as $f_i : k^n \to k^n (i = 0, 1)$ such that $f_1 \gamma_j = \gamma_j f_0$. Take one of eigenvalues of f_0, say it λ. Then $f - \lambda : M \to M$ is a $k[\mathcal{Q}]$-module homomorphism which is not injective. By Corollary 2.11 we get $f - \lambda = 0$. Hence we have $f = \lambda$. □

3. Lemmas in Linear Algebra

In this section we introduce two lemmas in linear algebra. These lemmas can be described in terms of $k[\mathcal{Q}]$-modules (Corollary 3.3). From the corollary we see that $r(M) \geq 2$ holds for each θ-stable $k[\mathcal{Q}]$-module. We shall use this result in the next section. The following elegant proofs of two lemmas are due to A. Ishii.

Lemma 3.1. *Let k be an algebraically closed field. Let $A, B \in \mathrm{M}_n(k)$ with $n \geq 2$. Suppose that $\mathrm{rk}[A, B] \leq 1$, where $[A, B] := AB - BA$. Then there exists a non-trivial subspace $0 \subset W \subset k^n$ such that W is A, B-invariant.*

Proof. (The statement was proved for the case $\mathrm{ch} k = 0$ in [13].)

Let λ be one of eigenvalues of A. Considering $A - \lambda I_n$ instead of A, we only have to prove the statement for the case $\det A = 0$. We may assume that $A \neq 0$. Choose $v \in k^n$ such that $\mathrm{Im}[A, B] = k \cdot v$. We divide the following two cases.

(Case 1) $v \in \mathrm{Im} A$.
For each $x \in k^n$,

$$BAx = -[A, B]x + ABx \in \mathrm{Im}[A, B] + \mathrm{Im} A \subseteq \mathrm{Im} A.$$

Hence $B(\mathrm{Im} A) \subseteq \mathrm{Im} A$. Therefore we can take $\mathrm{Im} A$ as an A, B-invariant subspace.

(Case 2) $v \notin \mathrm{Im} A$.
Then $\mathrm{Im} A \cap \mathrm{Im}[A, B] = 0$. For each $x \in \mathrm{Ker} A$,

$$\begin{aligned} ABx &= [A, B]x + BAx \\ &= [A, B]x \quad (\in \mathrm{Im} A \cap \mathrm{Im}[A, B]) \\ &= 0. \end{aligned}$$

Hence $B(\mathrm{Ker} A) \subseteq \mathrm{Ker} A$. Therefore we can take $\mathrm{Ker} A$ as an A, B-invariant subspace. $\square$

Lemma 3.2. *Let k be an algebraically closed field and let $n \geq 2$. For $A_0, A_1, A_2 \in \mathrm{M}_n(k)$, we assume that*

$$\mathrm{rk} \begin{pmatrix} 0 & A_2 & -A_1 \\ -A_2 & 0 & A_0 \\ A_1 & -A_0 & 0 \end{pmatrix} \leq 2n + 1.$$

Then there exist subspaces $W_1, W_2 \subset k^n$ such that

$$0 < \dim W_1 = \dim W_2 < n$$

and

$$A_i(W_1) \subseteq W_2 \ \ \text{for} \ \ i = 0, 1, 2.$$

Proof. We put

$$\Phi := \begin{pmatrix} 0 & A_2 & -A_1 \\ -A_2 & 0 & A_0 \\ A_1 & -A_0 & 0 \end{pmatrix}.$$

First we assume that A_2 is invertible. To prove that there exist W_1 and W_2 for a triple (A_0, A_1, A_2), we only have to find suitable subspaces W_1', W_2' for the triple $(A_0A_2^{-1}, A_1A_2^{-1}, I_n)$. Indeed if W_1' and W_2' satisfy the condition that $A_0A_2^{-1}(W_1') \subseteq W_2'$, $A_1A_2^{-1}(W_1') \subseteq W_2'$, and that $I_n(W_1') \subseteq W_2'$, then we can take W_1 and W_2 as $A_2^{-1}W_1'$ and W_2' respectively. Note that the rank of Φ for the triple (A_0, A_1, A_2) is equal to the one for $(A_0A_2^{-1}, A_1A_2^{-1}, I_n)$. Hence we may assume that A_2 is the identity matrix I_n. For the triple (A_0, A_1, I_n) we can change Φ into the form

$$\begin{pmatrix} 0 & I_n & -A_1 \\ -I_n & 0 & A_0 \\ 0 & 0 & [A_1, A_0] \end{pmatrix}$$

by elementary transformations. We see that $\mathrm{rk}\Phi = 2n + \mathrm{rk}[A_1, A_0]$. From the assumption we have $\mathrm{rk}[A_1, A_0] \leq 1$. Lemma 3.1 allows us to have a nontrivial A_0, A_1-invariant subspace. This implies that there exist W_1, W_2 for the triple (A_0, A_1, I_n).

Next we assume that A_2 is not invertible. Let us define the linear map ϕ by

$$\begin{array}{rcccl} \phi & : & k^n \oplus k^n \oplus k^n & \to & k^n \oplus k^n \oplus k^n \\ & & (x_0, x_1, x_2) & \mapsto & (A_2x_1 - A_1x_2, A_0x_2 - A_2x_0, A_1x_0 - A_0x_1). \end{array}$$

We remark that Φ is the representation matrix of ϕ with respect to the canonical basis $\{(e_1, 0, 0), \dots, (e_n, 0, 0), (0, e_1, 0), \dots\dots, (0, 0, e_n)\}$. Let $p_j : k^n \oplus k^n \oplus k^n \to k^n$ be the j-th projection $(j = 0, 1, 2)$. Set $\overline{p}_j := p_j \mid_{\mathrm{Im}\phi}: \mathrm{Im}\phi \to k^n$.

First we assume that $\mathrm{Im}A_i + \mathrm{Im}A_j \neq k^n$ for some distinct i, j. For example, suppose that $W := \mathrm{Im}A_0 + \mathrm{Im}A_1 \neq k^n$. If $W = 0$, then we can take an eigenspace of A_2 as W_1 and W_2. If $W \neq 0$, then take a subspace $W_1 \subseteq A_2^{-1}(W)$ such that $\dim W_1 = \dim W$. Put $W_2 := W$. We see that $A_i(W_1) \subseteq W_2$ for $i = 0, 1, 2$. In such a case the claim is true. Hence we may assume that $\mathrm{Im}A_i + \mathrm{Im}A_j = k^n$ for any distict i, j.

Under the assumption above, we have $p_j(\mathrm{Im}\phi) = k^n$ for $j = 0, 1, 2$. Since $\dim \mathrm{Im}\phi \leq 2n + 1$, we get

$$(1) \qquad \dim \mathrm{Ker}\,(\overline{p}_j) \leq n + 1.$$

Note that

$$
\text{(2)}\quad \begin{aligned} \overline{p}_1(\mathrm{Ker}(\overline{p}_0)) &= \{A_0x_2 - A_2x_0 \mid \exists x_1 \text{ such that } A_2x_1 = A_1x_2\} \\ &= \mathrm{Im}A_2 + A_0(A_1^{-1}(\mathrm{Im}A_2)). \end{aligned}
$$

We divide the following two cases.

(i) $A_0(A_1^{-1}(\mathrm{Im}A_2)) \subseteq \mathrm{Im}A_2$.

Then $A_j(A_1^{-1}(\mathrm{Im}A_2)) \subseteq \mathrm{Im}A_2$ holds for $j = 0, 1, 2$. Since A_2 is not invertible, we have $\mathrm{Im}A_2 \neq k^n$. If $\mathrm{Im}A_2 \neq 0$, then put $W_2 := \mathrm{Im}A_2$ and take a subspace of $A_1^{-1}(\mathrm{Im}A_2)$ of dimension $\dim W_2$ as W_1.

If $\mathrm{Im}A_2 = 0$, then A_0, A_1 are invertible since $\mathrm{Im}A_i + \mathrm{Im}A_2 = k^n$ for $i = 0, 1$. Let us choose an eigenspace of $A_1A_0^{-1}$ (if $A_1A_0^{-1}$ is a scalar matrix, then choose a nontrivial subspace). Call it W. Set $W_1 := A_0^{-1}W$ and $W_2 := W$. Then $A_i(W_1) \subseteq W_2$ for $i = 0, 1, 2$.

(ii) $A_0(A_1^{-1}(\mathrm{Im}A_2)) \not\subseteq \mathrm{Im}A_2$.

Note that $\dim \overline{p}_1(\mathrm{Ker}\overline{p}_0) \geq \mathrm{rank}A_2 + 1$ by (2). We see that

$$
\begin{aligned} \dim(\mathrm{Ker}\overline{p}_0 \cap \mathrm{Ker}\overline{p}_1) &\leq \dim \mathrm{Ker}\overline{p}_0 - (\mathrm{rank}A_2 + 1) \\ &\leq n + 1 - (\mathrm{rank}A_2 + 1) \quad (\text{ by (1) }) \\ &= n - \mathrm{rank}A_2 \\ &= \dim \mathrm{Ker}A_2. \end{aligned}
$$

Hence $\dim \overline{p}_2(\mathrm{Ker}\overline{p}_0 \cap \mathrm{Ker}\overline{p}_1) \leq \dim \mathrm{Ker}A_2$. Set $W_1 := \mathrm{Ker}A_2$. Take a subspace of $\dim = \dim \mathrm{Ker}A_2$ which contains $\overline{p}_2(\mathrm{Ker}\overline{p}_0 \cap \mathrm{Ker}\overline{p}_1)$, say it W_2. Then

$$
\begin{aligned} \overline{p}_2(\mathrm{Ker}\overline{p}_0 \cap \mathrm{Ker}\overline{p}_1) &= \{A_1x_0 - A_0x_1 \mid \exists\ x_2 \text{ such that} \\ &\qquad\qquad A_2x_1 = A_1x_2, A_0x_2 = A_2x_0\} \\ &\supseteq A_1(\mathrm{Ker}A_2) + A_0(\mathrm{Ker}A_2). \end{aligned}
$$

Therefore $A_i(W_1) \subseteq W_2$ for $i = 0, 1, 2$. This completes the proof. □

From the lemmas above we obtain the following corollary.

Corollary 3.3. *Let M be a $k[\mathcal{Q}]$-module of dimension vector (m, m) with $m \geq 2$. If M is θ-stable then $r(M) \geq 2$.*

4. μ-stability and θ-stability

In this section we show that for s-stable vector bundles, μ-stability and θ-stability are equivalent. This result follows that there exists a one-to-one correspondence between μ-stable vector bundles with $c_1 = 0$ and θ-stable $k[\mathcal{Q}]$-modules.

First we prove the following useful lemma.

Lemma 4.1. [1] *Let $\mathcal{F}$ be a vector bundle over $\mathbb{P}^2$ with $c_1(\mathcal{F}) \geq 0$. Then there exists an exact sequence*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}$$

such that $\mathcal{E}$ is a μ-stable vector bundle with $c_1(\mathcal{E}) = 0$.

Proof. We proceed with several steps.

(Step 1) Let $\mathcal{F}$ be a vector bundle with $c_1(\mathcal{F}) \geq 0$. Then there exists an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}$$

such that $\mathcal{E}$ is a μ-stable vector bundle with $0 \leq c_1(\mathcal{E}) < \mathrm{rk}\mathcal{E}$.

(Proof) Considering the Harder-Narasimhan filtration and the Jordan-Hölder filtration, we get an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}$$

such that $\mathcal{E}$ is a μ-stable torsion free sheaf with $\mu(\mathcal{E}) \geq \mu(\mathcal{F}) \geq 0$. Taking the double dual, we get $0 \longrightarrow \mathcal{E}^{\vee\vee} \longrightarrow \mathcal{F}$. Then $\mathcal{E}^{\vee\vee}$ is a μ-stable vector bundle. Twisting $\mathcal{E}^{\vee\vee}$ by $\mathcal{O}(-m)$ for a suitable integer m, if necessary, we obtain the desired μ-stable vector bundle with $0 < c_1 <$ rank.

(Step 2) Let $\mathcal{F}$ be a vector bundle with $c_1(\mathcal{F}) > 0$. Then there exists an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F}$$

such that $\mathcal{E}$ is a vector bundle with $c_1(\mathcal{E}) = 0$.

(Proof) By twisting $\mathcal{F}$ by $\mathcal{O}(-m)$ for a suitable integer m, we may assume that $0 < c_1(\mathcal{F}) < \mathrm{rk}\mathcal{F}$. Pick a line $\ell \subset \mathbb{P}^2$. Let us consider the following exact sequence:

$$0 \to \mathcal{F}(-1) \to \mathcal{F} \to \mathcal{F}|_\ell \to 0.$$

Set $\mathcal{F}|_\ell \cong \mathcal{O}_\ell(a_1) \oplus \cdots \oplus \mathcal{O}_\ell(a_r)$, where $r := \mathrm{rk}\mathcal{F}$. Let $i := c_1(\mathcal{F})$. Then $0 < i < r$. We denote by φ the surjective morphism $\mathcal{F} \to \mathcal{F}|_\ell \to \mathcal{O}_\ell(a_1) \oplus \cdots \oplus \mathcal{O}_\ell(a_{r-i})$. Put $\mathcal{E} := \mathrm{Ker}(\varphi)$. We see that $\mathcal{E}$ is a vector bundle with $c_1(\mathcal{E}) = 0$. Hence we get what we want.

We shall prove Lemma 4.1 by induction on $\mathrm{rk}\mathcal{F}$. It is trivial for the case $\mathrm{rk}\mathcal{F} = 1$. We may assume that $\mathrm{rk}\mathcal{F} > 1$. By Step 1 there

[1] M. Inaba and Y. Akahori, the author's colleagues, gave him an idea of the proof of this lemma.

exists an exact sequence $0 \to \mathcal{E} \to \mathcal{F}$ such that $\mathcal{E}$ is a μ-stable vector bundle with $0 \leq c_1(\mathcal{E}) < \mathrm{rk}\mathcal{E}$. If $c_1(\mathcal{E}) = 0$, then the claim is true. If $\mathrm{rk}(\mathcal{E}) < \mathrm{rk}(\mathcal{F})$, then the proof is completed by induction. Hence suppose that $\mathrm{rk}\mathcal{E} = \mathrm{rk}\mathcal{F}$ and $c_1(\mathcal{E}) > 0$. By Step 2 there exists an exact sequence $0 \to \mathcal{E}' \to \mathcal{E}$ such that $\mathcal{E}'$ is a vector bundle with $c_1(\mathcal{E}') = 0$. If $\mathrm{rk}\mathcal{E}' < \mathrm{rk}\mathcal{E}$, then the claim is true by induction. So we may assume that $\mathrm{rk}\mathcal{E}' = \mathrm{rk}\mathcal{E}$. By using Step 1 again, we see that there exists an exact sequence $0 \to \mathcal{E}'' \to \mathcal{E}'$ such that $\mathcal{E}''$ is a μ-stable vector bundle with $c_1(\mathcal{E}'') \geq 0$. As above, we may assume that $\mathrm{rk}\mathcal{E}' = \mathrm{rk}\mathcal{E}''$. Then we shall show that $\mathcal{E}'$ and $\mathcal{E}''$ are isomorphic. If it is true, then $\mathcal{F}$ has a μ-stable vector bundle $\mathcal{E}' \cong \mathcal{E}''$ with $c_1 = 0$ as a subsheaf, so we complete the proof. For proving $\mathcal{E}' \cong \mathcal{E}''$, take the determinant bundles of $0 \to \mathcal{E}'' \to \mathcal{E}'$. We get $0 \to \det \mathcal{E}'' \to \det \mathcal{E}'$. Since $c_1(\mathcal{E}'') \geq c_1(\mathcal{E}') = 0$, $\det \mathcal{E}'' \cong \det \mathcal{E}'$. In particular $\mathcal{E}'' \cong \mathcal{E}'$. $\square$

Now we show that μ-stability and θ-stability are equivalent.

Theorem 4.2 (The main theorem). *For each s-stable vector bundle, it is μ-stable if and only if it is θ-stable.*

Proof. First we consider the following proposition (Q_n).

(Q_n) : If $\mathcal{F}$ is a μ-stable vector bundle with $2 \leq \mathrm{rk}\mathcal{F} \leq n$, then there exist a θ-stable vector bundle $\mathcal{E}$ and a morphism $\mathcal{E} \to \mathcal{F}$ which is generically surjective.

Recall that a morphism $\mathcal{E} \to \mathcal{F}$ is said to be generically surjective if it is surjective at the generic point of $\mathbb{P}^2$. We shall prove that (Q_n) holds for $n \geq 2$. For proving them, it suffices to prove that $(Q_n) \Longrightarrow (Q_{n+1})$. Note that we see that (Q_2) is true from the following discussion.

Suppose that (Q_n) is true. Let $\mathcal{M}$ be a μ-stable vector bundle of $\mathrm{rk} = n + 1$. If $\mathcal{M}$ is θ-stable, then (Q_{n+1}) holds for $\mathcal{M}$. Hence we may assume that $\mathcal{M}$ is not θ-stable. Let M be the prestable $k[\mathcal{Q}]$-module associated to $\mathcal{M}$. We consider $k[\mathcal{Q}]$-submodules of M whose dimension vector has the form (p, p). Let N be a $k[\mathcal{Q}]$-submodule whose p is the smallest among such submodules. Then N is a θ-stable $k[\mathcal{Q}]$-module (note that $p \geq 2$ since M is prestable). By Corollary 3.3, we have $r(N) \geq 2$. Let $\mathcal{T}$ be the associated θ-stable vector bundle to N. Let $\varphi : \mathcal{T} \to \mathcal{M}$ be the non-zero morphism induced by the inclusion $N \to M$. Then we claim that φ is generically surjective. We shall prove this claim by contradiction. Suppose that φ is not generically surjective. Then $\mathrm{rk}(\mathrm{Im}(\varphi)) < \mathrm{rk}\mathcal{M}$. If φ is injective, then the fact that $\mu(\mathcal{T}) = c_1(\mathcal{T})/\mathrm{rk}(\mathcal{T}) = 0$ contradicts μ-stability of $\mathcal{M}$. Therefore $\mathrm{Ker}\,\varphi \neq 0$.

Now we consider $0 \to \mathcal{H} := (\mathrm{Im}\varphi)^{\vee\vee} \to \mathcal{M}$. Then $\mathrm{rk}\mathcal{H} < \mathrm{rk}\mathcal{M}$ and $\mathrm{rk}\mathcal{H} < \mathrm{rk}\mathcal{T}$. By μ-stability of $\mathcal{M}$, we have $c_1(\mathcal{H}) < 0$. Since $\mathcal{T} \to \mathcal{H}$ is generically surjective, $0 \to \mathcal{H}^\vee \to \mathcal{T}^\vee$ is exact. (Note. In this situation we can see that (Q_2) is true. Indeed if $\mathcal{M}$ has rank 2, then $\mathcal{H}^\vee$ is a line bundle with $c_1(\mathcal{H}^\vee) > 0$. Hence we have $h^0(\mathcal{T}^\vee) \neq 0$, which contradicts s-stability of $\mathcal{T}$.)

Since $c_1(\mathcal{H}^\vee) > 0$, Lemma 4.1 follows that there exists an exact sequence $0 \to \mathcal{M}_1 \to \mathcal{H}^\vee$ such that $\mathcal{M}_1$ is a μ-stable vector bundle with $c_1(\mathcal{M}_1) = 0$. The composition $\mathcal{M}_1 \to \mathcal{H}^\vee \to \mathcal{T}^\vee$ is injective and $h^0(\mathcal{T}^\vee) = 0$. Hence we have $\mathcal{M}_1 \ncong \mathcal{O}$ and $\mathrm{rk}\mathcal{M}_1 \geq 2$. Since $\mathrm{rk}\mathcal{M}_1 \leq \mathrm{rk}\mathcal{H}^\vee \leq n$, the induction follows that there exists a generically surjective morphism $\mathcal{T}_1 \to \mathcal{M}_1$ where $\mathcal{T}_1$ is a θ-stable vector bundle. Therefore there exists a nonzero morphism $\psi : \mathcal{T}_1 \to \mathcal{M}_1 \to \mathcal{T}^\vee$ between θ-stable vector bundles. By Corollary 2.12 ψ is an isomorphism while ψ factors through $\mathcal{M}_1$ and $\mathrm{rk}\mathcal{T}^\vee > \mathrm{rk}\mathcal{M}_1$. This is a contradiction. We have thus proved that $(Q_n) \Longrightarrow (Q_{n+1})$.

By the above discussion, we have proved that if $\mathcal{F}$ is μ-stable of $\mathrm{rk} \geq 2$, then there exists a generically surjective morphism $\mathcal{E} \to \mathcal{F}$ such that $\mathcal{E}$ is a θ-stable vector bundle.

Next we shall show that "$\mathcal{F}$: θ-stable $\Longrightarrow$ μ-stable". Let $\mathcal{F}$ be a θ-stable vector bundle. Suppose that $\mathcal{F}$ is not μ-stable. Then there exists an exact sequence $0 \to \mathcal{E} \to \mathcal{F}$ such that $\mathcal{E}$ is a vector bundle with $c_1(\mathcal{E}) \geq 0$ and $0 < \mathrm{rk}\mathcal{E} < \mathrm{rk}\mathcal{F}$. Using lemma 4.1, we see that there exists an exact sequence $0 \to \mathcal{M} \to \mathcal{E}$ such that $\mathcal{M}$ is a μ-stable vector bundle with $c_1(\mathcal{M}) = 0$. Since $h^0(\mathcal{F}) = h^0(\mathcal{M}) = 0$, we have $\mathcal{M} \ncong \mathcal{O}$ and $\mathrm{rk}\mathcal{M} \geq 2$. By the above result, there exists a generically surjective morphism $\mathcal{T} \to \mathcal{M}$ such that $\mathcal{T}$ is a θ-stable vector bundle. Composite two morphisms $\mathcal{T} \to \mathcal{M}$ and $\mathcal{M} \to \mathcal{E} \to \mathcal{F}$. Then the morphism $\mathcal{T} \to \mathcal{F}$ is an isomorphism because of Corollary 2.12. However since the morphism $\mathcal{T} \to \mathcal{F}$ factors through $\mathcal{E}$ and $\mathrm{rk}\mathcal{E} < \mathrm{rk}\mathcal{F}$, it is not an isomorphism. This is a contradiction. Therefore if $\mathcal{F}$ is θ-stable then it is μ-stable.

Finally we shall show that "$\mathcal{F}$: μ-stable $\Longrightarrow$ θ-stable". Let $\mathcal{F}$ be μ-stable. There exists a generically surjective morphism $\psi : \mathcal{T} \to \mathcal{F}$ such that $\mathcal{T}$ is θ-stable. The above result follows that $\mathcal{T}$ is μ-stable. Therefore ψ is an isomorphism. In particular $\mathcal{F}$ is θ-stable. $\square$

5. Appendix

In this section, we construct the moduli space of μ-stable vector bundles with $c_1 = 0, c_2 = n, \mathrm{rk} = r$ by using the results in the previous sections.

Notation 5.1. Let $\mathrm{Rep}_n(3) := \{(A, B, C) \in \mathrm{M}_n \times \mathrm{M}_n \times \mathrm{M}_n\}$ denote the affine scheme over $\mathbb{Z}$ representing three matrices of size $n \times n$. In particular, we have $\mathrm{Rep}_n(3) \cong \mathbb{A}_{\mathbb{Z}}^{3n^2}$.

We consider the group scheme $G := (\mathrm{GL}_n \times \mathrm{GL}_n)/\{(\lambda I_n, \lambda I_n)\}$ over $\mathbb{Z}$. The group scheme G can be described as $\{(A, B) \in \mathbb{P}(\mathrm{M}_n \times \mathrm{M}_n) \mid \det A \cdot \det B \neq 0\} \subseteq \mathbb{P}(\mathrm{M}_n \times \mathrm{M}_n) = \mathrm{Proj} S((\mathrm{M}_n \times \mathrm{M}_n)^\vee)$. We define the action of G on $\mathrm{Rep}_n(3)$ by $(A, B, C) \mapsto (PAQ^{-1}, PBQ^{-1}, PCQ^{-1})$, where $(A, B, C) \in \mathrm{Rep}_n(3)$ and $(P, Q) \in G$. Let $\chi : G \to \mathbb{G}_m$ be the character defined by $(P, Q) \mapsto \det(P)^{-1} \det(Q)$. Let L be the trivial line bundle on $\mathrm{Rep}_n(3)$. Then we define the linearization on L by $g \cdot (x, z) = (g \cdot x, \chi^{-1}(g)z)$ on the total space L^{-1}. We denote by $\mathrm{Rep}_n(3)^\theta$ the open subset of $\mathrm{Rep}_n(3)$ consisting of stable points with respect to the above linearization.

Here let us explain a relation between stable points under the action of G and θ-stable modules over the path algebra of the quiver $\mathcal{Q}$. For a commutative ring R we denote by $R[\mathcal{Q}]$ the path algebra over R. We denote by M_α or $M_{\alpha_0,\alpha_1,\alpha_2}$ the left $R[\mathcal{Q}]$-module $R^n \oplus R^n$ induced by $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathrm{M}_n(R)^{\oplus 3}$ as in §2. Let $\beta_i \in \mathrm{M}_n(R)$ for $i = 0, 1, 2$. Suppose that $f_0, f_1 : R^n \to R^n$ are R-homomorphisms such that $\beta_i f_0 = f_1 \alpha_i$ for $i = 0, 1, 2$. Then we can define the $R[\mathcal{Q}]$-homomorphism $(f_0, f_1) : M_{\alpha_0,\alpha_1,\alpha_2} \to M_{\beta_0,\beta_1,\beta_2}$.

We say that an $R[\mathcal{Q}]$-module $M_{\alpha_0,\alpha_1,\alpha_2}$ is *θ-stable* if the $\overline{k(\wp)}[\mathcal{Q}]$-module $M_{\alpha_0,\alpha_1,\alpha_2} \otimes_R \overline{k(\wp)}$ is θ-stable in the sense of §2 for each $\wp \in \mathrm{Spec}\, R$, where $\overline{k(\wp)}$ is an algebraic closure of the residue field $k(\wp) = R_\wp/\wp R_\wp$ of $\wp$. Note that θ-stability is independent of choices of algebraically closed fields $\overline{k(\wp)}$. For $(A, B, C) \in \mathrm{Rep}_n(3)(R)$, the $R[\mathcal{Q}]$-module $M_{A,B,C}$ is θ-stable if and only if $(A, B, C) \in \mathrm{Rep}_n(3)^\theta(R)$. (See [7].)

Definition 5.2. Let $(A, B, C) \in \mathrm{Rep}_n(3)^\theta$ be the universal three matrices. Let us define the $3n \times 3n$ matrix

$$\mathcal{D}(A, B, C) := \begin{pmatrix} 0 & C & -B \\ -C & 0 & A \\ B & -A & 0 \end{pmatrix}.$$

We define the locally closed subscheme $X_{n,r}$ of $\mathrm{Rep}_n(3)^\theta$ as follows:

$$\begin{array}{rcl} X_{n,r} & := & F \cap U \\ & (= & \{(A,B,C) \in \mathrm{Rep}_n(3)^\theta \mid \mathrm{rk}\mathcal{D}(A,B,C) = 2n+r\}), \end{array}$$

where F is the closed subscheme of $\mathrm{Rep}_n(3)^\theta$ defined by the ideal generated by all minor determinants of $\mathcal{D}(A,B,C)$ of size $(2n+r+1) \times (2n+r+1)$, and U is the open subscheme of $\mathrm{Rep}_n(3)^\theta$ consisting of points where at least one minor determinant of $\mathcal{D}(A,B,C)$ of size $(2n+r) \times (2n+r)$ is a unit.

We easily see that the action of G on $\mathrm{Rep}_n(3)$ induces the one on $X_{n,r}$. The main result in this section is the following:

Theorem 5.3. *Let $n \geq r \geq 2$. Let G and $X_{n,r}$ be as above. Then there exists a geometric quotient $X_{n,r}/G$ over $\mathbb{Z}$. Moreover the quotient is isomorphic to $\mathrm{M}_{\mathbb{P}^2}(r;0,n)^{\mu\text{-}stable}$, that is, the moduli scheme of μ-stable vector bundles with $\mathrm{rk} = r, c_1 = 0, c_2 = n$ over $\mathbb{P}^2$.*

Before proving the theorem, we shall make several preparations.

Definition 5.4. We define the contravariant functor $\Sigma_{\mathbb{P}^2}(r;0,n)$ from the category of locally noetherian schemes to the category of sets as follows:

$$\begin{array}{rcl} \Sigma_{\mathbb{P}^2}(r;0,n): \ \textbf{(LN-Sch)} & \to & \textbf{(Sets)} \\ T & \mapsto & \{\mathcal{F} \mid \mathcal{F} \text{ is a vector bundle on } T \times \mathbb{P}^2_{\mathbb{Z}} \\ & & \quad \text{flat over } T \text{ such that for each} \\ & & \quad t \in T, \mathcal{F}|_{\{t\}\times\mathbb{P}^2_{\mathbb{Z}}} \text{ is } \mu\text{-stable with} \\ & & \quad c_1 = 0, c_2 = n, \text{and } \mathrm{rk} = r\}/\sim, \end{array}$$

where $\mathbb{P}^2_{\mathbb{Z}}$ is the projective plane over $\mathbb{Z}$. Here we say that $\mathcal{F} \sim \mathcal{F}'$ if $\mathcal{F} \cong \mathcal{F}' \otimes L$ for some line bundle L on T.

There exists a natural morphism $\Sigma_{\mathbb{P}^2}(r;0,n) \to \mathrm{M}_{\mathbb{P}^2}(r;0,n)^{\mu\text{-stable}}$. The scheme $\mathrm{M}_{\mathbb{P}^2}(r;0,n)^{\mu\text{-stable}}$ is the coarse moduli scheme associated with the functor $\Sigma_{\mathbb{P}^2}(r;0,n)$.

From the universal three matrices $(A,B,C) \in X_{n,r}$, we get the universal θ-stable representation of the quiver $\mathcal{Q}$ and the associated Kronecker module. Using the following lemma, we obtain the canonical monad on $X_{n,r} \times \mathbb{P}^2_{\mathbb{Z}}$.

Lemma 5.5. *Let R be a local ring, and let $A \in \mathrm{M}_n(R)$. Suppose that any minor determinant of A of size $(r+1) \times (r+1)$ vanishes and that* *some minor determinant of size $r \times r$ is a unit of R. Then there exist $P, Q \in \mathrm{GL}_n(R)$ such that*

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the identity matrix of size $r \times r$.

Proof. It is an easy lemma, so we omit the proof. □

Taking the cohomology bundle of the canonical monad on $X_{n,r} \times \mathbb{P}^2$, we get an element of $\Sigma_{\mathbb{P}^2}(r; 0, n)(X_{n,r})$. Indeed we see that the cohomology bundle is flat over $X_{n,r}$ by Lemma 6.5 in [9]. Hence we obtain the canonical morphism $X_{n,r} \to \Sigma_{\mathbb{P}^2}(r; 0, n)$.

For the proof of Theorem 5.3, we proceed with several lemmas.

Lemma 5.6. *Let R be a noetherian local ring or an artinian ring. Then $X_{n,r}(R)/G(R) \xrightarrow{\sim} \Sigma_{\mathbb{P}^2}(r; 0, n)(R)$.*

Proof. Since any artinian ring is a product of artinian local rings, we only need to consider the case R is a noetherian local ring. Using base change theorem and the relative version of Beilinson's spectral sequences, we see that $X_{n,r}(R) \to \Sigma_{\mathbb{P}^2}(r; 0, n)(R)$ is surjective for each noetherian local ring R. For proving the injectivity we can use Proposition 4 in [2] or Lemma 4.1.3 in [3]. □

Lemma 5.7. *Let R be a commutative ring. Then the map*

$$\begin{array}{ccc} G(R) \times X_{n,r}(R) & \to & X_{n,r}(R) \times X_{n,r}(R) \\ (g, \alpha) & \mapsto & (\alpha,\ g \cdot \alpha) \end{array}$$

is injective.

Proof. It suffices to prove that if $g \cdot \alpha = \alpha$ then $g = e \in G(R)$. Suppose that $(PAQ^{-1}, PBQ^{-1}, PCQ^{-1}) = (A, B, C)$ for $(P, Q) \in G(R)$ and $(A, B, C) \in X_{n,r}(R)$. Since we only have to prove that $(P, Q) = e$ holds locally, we may assume that $(A, B, C) \in \mathrm{M}_n(R)^{\oplus 3}$. By θ-stability we see that $\{A, B, C\}$ generates $\mathrm{M}_n(R)$ as an R-algebra. From the assumption we have $PX = XQ$ for each $X \in \mathrm{M}_n(R)$. Hence we get $P = Q = \lambda I_n$ for some $\lambda \in R^{\times}$. This completes the proof. □

Lemma 5.8. *The morphism*

$$\begin{array}{ccc} G \times X_{n,r} & \to & X_{n,r} \times X_{n,r} \\ (g, \alpha) & \mapsto & (\alpha,\ g \cdot \alpha) \end{array}$$

is proper.

Proof. Note that the morphism is separated since $X_{n,r}$ is a subscheme of the affine scheme $\mathrm{Rep}_n(3)$. We shall prove that the morphism is proper by valuative criterion. Let R be a discrete valuation ring and K the quotient field of R. Let $(\alpha, \beta) \in X_{n,r}(R) \times X_{n,r}(R)$. Suppose that $g \cdot \alpha = \beta$ in $X_{n,r}(K)$ for some $g = (P, Q) \in G(K)$. Then it suffices to prove that there exists $g' \in G(R)$ such that $g' \cdot \alpha = \beta$. Take the smallest integer d such that $\pi^d P, \pi^d Q \in \mathrm{M}_n(R)$, where π is a uniformizer of R. Then $f := (\pi^d Q, \pi^d P) : M_\alpha \to M_\beta$ is an $R[\mathcal{Q}]$-module homomorphism. Taking the reduction of f to the residue field k of R, we get a non-zero homomorphism between θ-stable $k[\mathcal{Q}]$-modules. By Corollary 2.11 the reduction of f is an isomorphism, and so is f. Putting $g' := (\pi^d P, \pi^d Q) \in G(R)$, we have $g' \cdot \alpha = \beta$. □

Corollary 5.9. *The action of G on $X_{n,r}$ is free, that is, the morphism*

$$G \times X_{n,r} \to X_{n,r} \times X_{n,r}$$

is a closed immersion.

Proof. Since the above morphism is proper and quasi-finite by Lemma 5.7 and Lemma 5.8, it is a finite morphism. The statement follows from the fact that any finite morphism which is a monomorphism is a closed immersion. □

By several lemmas above, we can prove Theorem 5.3.

(*Proof of Theorem 5.3.*) From the discussion above and general theory we see that there exists a geometric quotient $X_{n,r}/G$. Let us define the natural transformation $\Phi : \Sigma_{\mathbb{P}^2}(r; 0, n) \to X_{n,r}/G$ as follows. For a locally noetherian scheme T and $\mathcal{F} \in \Sigma_{\mathbb{P}^2}(r; 0, n)(T)$, there exists an open covering $\{U_i\}_{i \in I}$ such that $\mathcal{F}|_{U_i} \in \Sigma_{\mathbb{P}^2}(r; 0, n)(U_i)$ is induced by some $\alpha_i \in X_{n,r}(U_i)$ for each $i \in I$ (*cf.* Lemma 5.6). Let $f_i : U_i \to X_{n,r} \to X_{n,r}/G$ be the morphism induced by α_i. We easily see that $\{f_i\}$ are glued together, and hence we get the morphism $f : T \to X_{n,r}/G$. Therefore from the correspondence $\mathcal{F} \mapsto f$ we obtain the natural transformation $\Phi : \Sigma_{\mathbb{P}^2}(r; 0, n) \to X_{n,r}/G$.

For an algebraically closed field k, the map $\Phi : \Sigma_{\mathbb{P}^2}(r; 0, n)(k) \to (X_{n,r}/G)(k)$ is bijective from Lemma 5.6. Remark that the composition of morphisms $X_{n,r} \to \Sigma_{\mathbb{P}^2}(r; 0, n) \xrightarrow{\Phi} X_{n,r}/G$ is equal to the quotient morphism. For a scheme Z and a morphism $\Sigma_{\mathbb{P}^2}(r; 0, n) \to Z$, there exists a unique morphism $X_{n,r}/G \to Z$ such that the composition $\Sigma_{\mathbb{P}^2}(r; 0, n) \to X_{n,r}/G \to Z$ is equal to a given morphism, since $X_{n,r}/G$ is a categorical quotient. Therefore $X_{n,r}/G \cong \mathrm{M}_{\mathbb{P}^2}(r; 0, n)^{\mu\text{-stable}}$. □

As an appendix, here we mention smoothness of $X_{n,r}$.

Lemma 5.10. *The morphism $X_{n,r} \to \Sigma_{\mathbb{P}^2}(r;0,n)$ is smooth. Moreover $X_{n,r}$ is smooth over $\mathbb{Z}$.*

Proof. Let A be an artinian ring and let I be an ideal of A with $I^2 = 0$. Let us consider the following diagram:

$$\begin{array}{ccc} X_{n,r}(A) & \to & \Sigma_{\mathbb{P}^2}(r;0,n)(A) \\ \downarrow & & \downarrow \\ X_{n,r}(A/I) & \to & \Sigma_{\mathbb{P}^2}(r;0,n)(A/I). \end{array}$$

For proving the statement, we shall show that $X_{n,r}(A) \to X_{n,r}(A/I) \times_{\Sigma_{\mathbb{P}^2}(r;0,n)(A/I)} \Sigma_{\mathbb{P}^2}(r;0,n)(A)$ is surjective. Let us consider the following situation:

$$\begin{array}{ccccc} X_{n,r}(A/I) & \to & \Sigma_{\mathbb{P}^2}(r;0,n)(A/I) & \leftarrow & \Sigma_{\mathbb{P}^2}(r;0,n)(A) \\ \overline{\alpha} & \mapsto & \overline{\mathcal{F}} & \leftarrow\!\shortmid & \mathcal{F} \quad . \end{array}$$

By Lemma 5.6 we can take $\alpha' \in X_{n,r}(A)$ such that $\alpha' \mapsto \mathcal{F}$. Suppose that $\alpha' \mapsto \overline{\alpha}' \in X_{n,r}(A/I)$. Since $\overline{\alpha}$ and $\overline{\alpha}'$ induce the same vector bundle $\overline{\mathcal{F}}$, we have $\overline{g} \in G(A/I)$ such that $\overline{g}(\overline{\alpha}') = \overline{\alpha}$ by using Lemma 5.6 again. It is easy to see that $G(A) \to G(A/I)$ is surjective. Take $g \in G(A)$ such that $g \mapsto \overline{g}$. Set $\alpha := g(\alpha')$. Then we have $\alpha \mapsto \mathcal{F}$ and $\alpha \mapsto \overline{\alpha}$. This implies the surjectivity of the above map. Hence $X_{n,r} \to \Sigma_{\mathbb{P}^2}(r;0,n)$ is smooth. The functor $\Sigma_{\mathbb{P}^2}(r;0,n)$ is smooth over $\mathbb{Z}$ because $\mathrm{Ext}^2(\mathcal{E},\mathcal{E}) = 0$ for $\mathcal{E} \in \Sigma_{\mathbb{P}^2}(r;0,n)(k)$, and so is $X_{n,r}$. □

Remark 5.11. The quotient morphism $X_{n,r} \to X_{n,r}/G$ is a G-principal fiber bundle. Indeed, the schemes $X_{n,r}$ and $X_{n,r}/G$ are smooth over $\mathbb{Z}$ (*cf.* Lemma 5.10). The morphism $X_{n,r} \to X_{n,r}/G$ is flat since the dimensions of fibers are constant. We see that the morphism $G \times X_{n,r} \to X_{n,r} \times_{X_{n,r}/G} X_{n,r}$ is an isomorphism by using Zariski Main Theorem. Using the discussion in Proposition 0.9 in [4], we also see that $X_{n,r} \to X_{n,r}/G$ is a G-principal fiber bundle.

Let us consider μ-stable vector bundles with $c_1 = 0, c_2 = 0, \mathrm{rk} = r$ whose restriction on a fixed line ℓ_0 is a trivial. We end this paper by describing the moduli of such vector bundles on the projective plane.

Notation 5.12. We denote by $\mathrm{Rep}_n(2)$ the affine scheme representing two matrices of size $n \times n$. In other words, the scheme $\mathrm{Rep}_n(2)$ over $\mathbb{Z}$ is the representation variety of degree n for the free monoid of rank 2. We denote by $\mathrm{Rep}_n(2)_{\mathrm{a.i.r.}}$ the absolutely irreducible representation part. (For details see [11].)

The locally closed subscheme $Y_{n,r}$ of $\mathrm{Rep}_n(2)_{\text{a.i.r.}}$ is defined by

$$\begin{array}{rcl} Y_{n,r} & := & F' \cap U' \\ & (= & \{(A,B) \in \mathrm{Rep}_n(2)_{\text{a.i.r.}} \mid \mathrm{rk}[A,B] = r\}), \end{array}$$

where $[A,B] := AB - BA$ and F' is the closed subscheme of $Y_{n,r}$ defined by the ideal generated by all $(r+1)\times(r+1)$ minor determinants of $[A,B]$, and U' is the open subscheme consisting of points where at least one $r \times r$ minor determinant of $[A,B]$ is a unit. Here we denote by (A,B) the universal matrices.

The group scheme PGL_n over $\mathbb{Z}$ acts on $\mathrm{Rep}_n(2)_{\text{a.i.r.}}$ and $Y_{n,r}$ by $(A,B) \mapsto (PAP^{-1}, PBP^{-1})$.

Let us consider the closed immersion

$$\begin{array}{ccc} \mathrm{Rep}_n(2) & \hookrightarrow & \mathrm{Rep}_n(3) \\ (A,B) & \mapsto & (I_n, A, B). \end{array}$$

Under the above immersion we see that $\mathrm{Rep}_n(2)_{\text{a.i.r.}} = \mathrm{Rep}_n(3)^\theta \cap \mathrm{Rep}_n(2)$. If we restrict the immersion to the locally closed subscheme $X_{n,r}$, then we get the following diagram which is a fiber product:

$$\begin{array}{ccc} \mathrm{Rep}_n(2)_{\text{a.i.r.}} & \hookrightarrow & \mathrm{Rep}_n(3)^\theta \\ \uparrow & & \uparrow \\ Y_{n,r} & \hookrightarrow & X_{n,r} \end{array}.$$

Here we define the homomorphism of group schemes $\mathrm{PGL}_n \to G$ by $P \mapsto (P,P)$. Then we get the diagrams

$$\begin{array}{ccc} \mathrm{PGL}_n \times \mathrm{Rep}_n(2)_{\text{a.i.r.}} & \to & G \times \mathrm{Rep}_n(3)^\theta \\ \downarrow & & \downarrow \\ \mathrm{Rep}_n(2)_{\text{a.i.r.}} & \to & \mathrm{Rep}_n(3)^\theta. \end{array}$$

$$\begin{array}{ccc} \mathrm{PGL}_n \times Y_{n,r} & \to & G \times X_{n,r} \\ \downarrow & & \downarrow \\ Y_{n,r} & \to & X_{n,r}. \end{array}$$

We obtain the next theorem in the similar way as Theorem 5.3 (*cf.* Remark 1.8).

Theorem 5.13. *The geometric quotient $Y_{n,r}/\mathrm{PGL}_n$ is isomorphic to the moduli scheme of μ-stable vector bundles with $c_1 = 0, c_2 = n$, and* rk $= r$ *over $\mathbb{P}^2$ whose restrictions on ℓ_0 are trivial, where $\ell_0 := \{z_0 = 0\} \subseteq \mathbb{P}^2$.*

Note that $Y_{n,r}/\mathrm{PGL}_n \hookrightarrow X_{n,r}/G = \mathrm{M}_{\mathbb{P}^2}(r;0,n)^{\mu\text{-stable}}$ is an open immersion. The geometric quotient $Y_{n,r}/\mathrm{PGL}_\mathrm{n}$ can be obtained as a subscheme of the character variety $\mathrm{Ch}_n(3)_{\mathrm{a.i.r.}}$ (*cf.* [11]). The quotient morphism $Y_{n,r} \to Y_{n,r}/\mathrm{PGL}_n$ is a PGL_n-principal fiber bundle.

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