Coupled Cell Systems

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1 Coupled Cell Networks

Definition 1.1. A *coupled cell network* Γ consists of the following:

(a) A finite set $C = \{1, 2, \ldots, N\}$ of *cells*.

(b) An equivalence relation $\sim_C$ on cells in C.
The *type* or *cell label* of cell c is the $\sim_C$ −equivalence class $[c]_C$ of c.

(c) A finite set E of *edges* or *arrows*.

(d) An equivalence relation $\sim_E$ on edges in E.
The *type* or *coupling label* of edge e is the $\sim_E$ −equivalence class $[e]_E$ of e.

(e) Two maps $H\colon E \to C$ and $T\colon E \to C$.
For $e \in E$ we call $H(e)$ the *head* of e and $T(e)$ the *tail* of e.

We also require a consistency condition:

(f) Equivalent arrows have equivalent tails and heads. Thus, if $e_1, e_2 \in E$ and $e_1 \sim_E e_2$, then

$$H(e_1) \sim_C H(e_2), \quad T(e_1) \sim_C T(e_2).$$

Here we allow $H(e) = T(e)$ (self-coupling), $H(e_1) = H(e_2)$ and $T(e_1) = T(e_2)$ for $e_1 \neq e_2$ (multiple arrows).

Definition 1.2. If $c \in C$, then the *input set* of c is

$$I(c) = \{e \in E : H(e) = c\}.$$

An element of $I(c)$ is called an *input edge* or *input arrow* of c.

Definition 1.3. The *input equivalence relation* $\sim_I$ on C is defined by:

$$c \sim_I d \iff \text{there exists an arrow-type preserving bijection } \beta\colon I(c) \to I(d).$$

That is, for every input arrow $i \in I(c)$ holds

$$i \sim_E \beta(i).$$

Any such bijection β is called an *input isomorphism* from cell c to d. The set $B(c, d)$ is the collection of all input isomorphisms from cell c to d. The set

$$\mathcal{B}_\Gamma = \bigcup_{c,d \in C} B(c, d)$$

is the *groupoid of the network*, which is an algebraic structure in group, except that the product of two elements is not always defined.

Note: The above union is disjoint and $B(c, c)$ is a permutation group acting on the input set $I(c)$.
By the consistency condition (f) of Definition 1.1, $c \sim_I d$ implies $c \sim_C d$, but the converse fails in general.

Definition 1.4. A *homogeneous network* is a coupled cell network such that $B(c, d) \neq \emptyset$ for every pair of cells c, d.

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2 Admissible vector fields

For each cell in C define a *cell phase space* P_c. This must be a smooth manifold, which for simplicity we assume is a non-zero finite-dimensional real vector space. We require

$$c \sim_C d \Rightarrow P_c = P_d$$

Define the corresponding *total phase space* to be

$$P = \prod_{c \in C} P_c$$

and employ the coordinate system

$$x = (x_c)_{c \in C}$$

on P.

Let $\mathcal{D} = (d_1, d_2, \ldots, d_s)$ be any finite ordered subset of s cells in C. Define

$$P_{\mathcal{D}} = P_{d_1} \times \cdots \times P_{d_s}$$

Further write

$$x_{\mathcal{D}} = (x_{d_1}, \ldots, x_{d_s})$$

where $x_{d_j} \in P_{d_j}$. For a given cell c, the *internal phase space* is P_c and the *coupling phase space* is

$$P_{T(I(c))} = P_{T(i_1)} \times \cdots \times P_{T(i_s)}$$

where $T(I(c))$ denotes the ordered set of cells $(T(i_1), \ldots, T(i_s))$ as the arrows i_k run through $I(c)$.

Suppose $c, d \in C$ and $c \sim_I d$. For any $\beta \in B(c,d)$, define the *pullback map*

$$\beta^* \colon P_{T(I(d))} \to P_{T(I(c))}$$

by

$$(\beta^*(z))_{T(i)} = z_{T(\beta(i))}$$

for all $i \in I(c)$ and $z \in P_{T(I(d))}$.

Definition 2.1. A vector field $f \colon P \to P$ is *$\Gamma-$admissible* if

(a) (domain condition) For all $c \in C$, there exists $\hat{f}_c \colon P_c \times P_{T(I(c))} \to P_c$ such that

$$f_c(x) = \hat{f}_c(x_c, x_{T(I(c))})$$

(b) (pullback condition) For all $c, d \in C$ and $\beta \in B(c,d)$

$$\hat{f}_d(x_d, x_{T(I(d))}) = \hat{f}_c(x_d, \beta^*(x_{T(I(d))}))$$

for all $x \in P$.

Definition 2.2. An ODE defined by admissible vector field on a coupled cell network is called a *coupled cell system.*

3 Balanced equivalence relations

Choose a total phase space P, and let $\bowtie$ be an equivalence relation on C. It defines unique *partition* of C. It follows that *polydiagonal subspace*

$$\Delta_{\bowtie} = \{x \in P : c \bowtie d \Rightarrow x_c = x_d\}$$

is well defined, since x_c and x_d lie in the same space $P_c = P_d$.
The polydiagonal $\Delta_{\bowtie}$ is a linear subspace of P.

Definition 3.1. Let $\bowtie$ be an equivalence relation on C. $\bowtie$ is *robustly polysynchronous* if $\Delta_{\bowtie}$ is invariant under every $f \in \mathcal{F}_{\Gamma}^{P}$. That is,

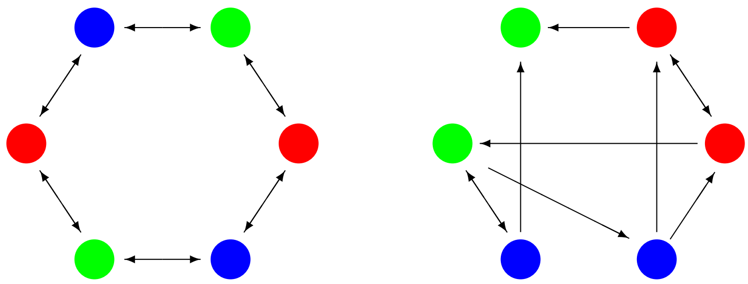
$$f(\Delta_{\bowtie}) \subseteq \Delta_{\bowtie}$$

for all $f \in \mathcal{F}_{\Gamma}^{P}$.

Definition 3.2. An equivalence relation $\bowtie$ is *balanced* if for every $c, d \in C$ with $c \bowtie d$, there exists an input isomorphism $\beta \in B(c, d)$ such that $T(i) \bowtie T(\beta(i))$ for all $i \in I(c)$.

If color the cells in C so that two cells have the same color iff they are $\bowtie$ −equivalent, and the tail of each arrow by the color of the corresponding cell. Then $\bowtie$ is balanced if and only if:
Every pair of identically colored cells admits a *color-preserving* input isomorphism (an isomorphism $\beta \colon I(c) \to I(d)$ such that $T(i)$ and $T(\beta(i))$ have the same color for all $i \in I(c)$).
In particular, $B(c, d) \neq \emptyset$ implies $c \sim_I d$.

Example 3.3.



Theorem 3.4. "$\bowtie$" is robustly polysynchronous if and only if "$\bowtie$" is balanced.

4 Quotient networks

Let $\bowtie$ be a balanced equivalence relation of Γ.

(a) Let $\bar{c}$ be the $\bowtie$ −equivalence class of $c \in C$. Then define

$$C_{\bowtie} = \{\bar{c} : c \in C\}.$$

That is, $C_{\bowtie} = C/\bowtie$.

(b) Define

$$\bar{c} \sim_{C_{\bowtie}} \bar{d} \iff c \sim_C d.$$

(c) Let $S \subset C$ be a set of cells which $\bowtie$ −equivalent to cell c. Then define $I(\bar{c}) = I(c)$, where $c \in S$. An arrow $i \in I(c)$ denoted by $\bar{i}$ in $I(\bar{c})$. Then define

$$E_{\bowtie} = \textstyle\bigcup_{c \in S} I(c).$$

(d) Define

$$\bar{i}_1 \sim_{E_{\bowtie}} \bar{i}_2 \iff i_1 \sim_E i_2.$$

(e) Define

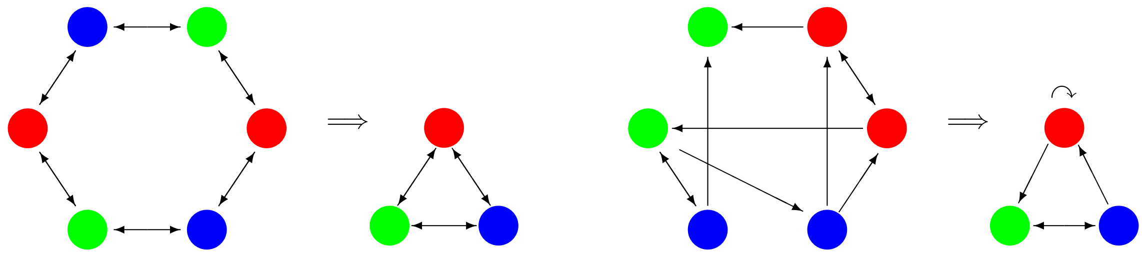
$$H(\bar{i}) = \overline{H(i)}, \quad T(\bar{i}) = \overline{T(i)}.$$

(f) We can easy check the *consistency condition*.

Definition 4.1. A network which defined as above is called the *quotient network* of a coupled cell network Γ. We denote it by $\Gamma_{\bowtie}$.

Remark 4.2. The quotient network is independent of the choice of cells in S because $\bowtie$ is balanced.

Example 4.3.



Theorem 4.4. Let "$\bowtie$" be a balanced equivalence relation on a coupled cell network Γ. Then

(a) The restriction of a $\Gamma-$admissible vector field to $\Delta_{\bowtie}$ is $\Gamma_{\bowtie}-$admissible.

(b) Every $\Gamma_{\bowtie}-$admissible vector field on the quotient network lifts to a $\Gamma-$admissible vector field on the original network.

References

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