MEMO OF
DIMENSION RESULTS FOR LIMIT SETS OF NON-AUTONOMOUS CONFORMAL ITERATED FUNCTION SYSTEMS

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Abstract. A Non-autonomous Iterated Function System (NIFS) is a sequence of collections of uniformly contracting maps. Unlike ordinary iterated function systems, we allow the contractions applied at each step to vary. In this talk, we give an overview of dimension results for limit sets of NIFSs, focusing mainly on families of parameterized NIFSs by using the transversality method.

1. Overview of the theory of non-autonomous iterated fucntion systems

1.1. Iterated function systems.

Definition. Let I be a finite set. Then a collection $\{\phi_i\}_{i\in I}$ of uniformly contracting maps on $\mathbb{R}^m$ is called an *Iterated Function System* (IFS).

Example.

- A collection

$$\left\{x \mapsto \frac{1}{3}x, x \mapsto \frac{1}{3}x + \frac{2}{3}\right\}$$

is an IFS on $\mathbb{R}$. The limit set of this IFS is the middle third Cantor set.

- For $0 < \lambda < 1$ a collection

$$\{x \mapsto \lambda x, x \mapsto \lambda x + 1 - \lambda\}$$

is an IFS on $\mathbb{R}$. The invariant measure of this IFS (with respect to $(1/2, 1/2)$-probabilty vector) is called the Bernoulli convolution (parameterized by λ).

Dimension results for limit sets of IFS.

- Dimension formula for limit sets of conformal IFS (Mauldin, Urbański [6, 7]).
- Transversality methods for IFS (Pollicott, Simon, Solomyak [11, 15]).
- Additive combinatorics approach (Hochman, Shmerkin [3, 16]).

1.2. Non-autonomous IFS.

Definition. A sequence $(\{\phi_i^{(j)}\}_{i\in I^{(j)}})_{j=1}^{\infty}$ of collections of uniformly contracting maps on a compact set $X \subset \mathbb{R}^m$ is called a Non-autonomous IFS (NIFS).

Example. (1) Set

$$\Phi_1 = \left\{x \mapsto \frac{1}{4}x, x \mapsto \frac{1}{4}x + \frac{3}{4}\right\}, \Phi_2 = \left\{x \mapsto \frac{1}{2}x, x \mapsto \frac{1}{2}x + \frac{1}{2}\right\}.$$

Then the sequence $\Phi = \Phi_1\Phi_2\Phi_1\Phi_1\Phi_2\Phi_1\Phi_1\Phi_1\Phi_2\cdots$ is a NIFS on $[0, 1]$. In general, the sequence $\Phi_\tau = \Phi_{\tau_1}\Phi_{\tau_2}\Phi_{\tau_3}\cdots$ is a NIFS on $[0, 1]$ for $\tau = \tau_1\tau_2\cdots \in \{1, 2\}^{\mathbb{N}}$.

(2) Set
$$\Psi = \left(\left\{ x \mapsto \frac{1}{2}x, x \mapsto \frac{1}{2}x + \frac{1}{j} \right\} \right)_{j=1}^{\infty}.$$
Then the sequence Ψ is a NIFS on $[0, 2]$.

Definition (Limit set of NIFS)**.** Let $\Phi = (\{\phi_i^{(j)}\}_{i \in I^{(j)}})_{j=1}^{\infty}$ be a NIFS on X. For each $n \in \mathbb{N}$, set
$$X_n := \bigcup_{(i_1,\dots,i_n) \in I^{(1)} \times \cdots \times I^{(n)}} \phi_{i_1}^{(1)} \circ \cdots \circ \phi_{i_n}^{(n)}(X),$$
which is a compact set. Then $X_n \supset X_{n+1}$ for all n. Therefore we can define a nonempty compact subset $J(\Phi)$ as $J = J(\Phi) := \bigcap_{n=1}^{\infty} X_n$. The set J is called the limit set of the NIFS Φ.

Remark.

- Geometrical properties of limit sets of NIFS (Sumi et al. [1]).
- Construction of limit sets of NIFS on unbounded spaces (Inui [4]).
- Dimension results for limit sets of NIFS (This talk).

1.3. **Setup and notations.** For each $j \in \mathbb{N}$, let $I^{(j)}$ be a finite set.

Let $X \subset \mathbb{R}^m$ be a compact convex set with non-empty interior $\mathrm{int}(X) \neq \emptyset$ (as a phase space).

For each $j \in \mathbb{N}$, we set
$$\Phi^{(j)} = \{\phi_i^{(j)} : X \to X\}_{i \in I^{(j)}}.$$
Write $\Phi = (\Phi^{(j)})_{j=1}^{\infty}$.

For $n, k \in \mathbb{N}$ with $n \leq k$, we introduce index sets
$$I_n^k = \prod_{j=n}^{k} I^{(j)},\ I_n^{\infty} = \prod_{j=n}^{\infty} I^{(j)},\ I^n = I_1^n \text{ and } I^{\infty} = I_1^{\infty}.$$

For any $\omega = \omega_n \omega_{n+1} \cdots \omega_k \in I_n^k$, we denote
$$\phi_\omega = \phi_{\omega_n}^{(n)} \circ \cdots \circ \phi_{\omega_k}^{(k)}.$$
Moreover, for any $\omega = \omega_n \omega_{n+1} \cdots \in I_n^{\infty}$ and $j \in \mathbb{N}$, write
$$\omega|_j = \omega_n \omega_{n+1} \cdots \omega_{n+j-1} \in I_n^{n+j-1}.$$
Let $W \subset \mathbb{R}^m$ be an open set and let $\phi \colon W \to \phi(W)$ be a diffeomorphism. We say ϕ is *conformal* if for any $x \in W$ the differential $D\phi(x) \colon \mathbb{R}^m \to \mathbb{R}^m$ is a similarity linear map: $D\phi(x) = c_x \cdot M_x$ where $c_x > 0$ is a scaling factor at x and M_x is an $m \times m$ orthogonal matrix. We denote by $|D\phi(x)|$ the scaling factor of ϕ at x.

Definition. We say that $\Phi = (\Phi^{(j)})_{j=1}^{\infty}$ is a Non-autonomous Conformal Iterated Function System (NCIFS) if the following holds:

(i) there exists an open connected set $V \supset X$ such that for all $j \in \mathbb{N}$ and $i \in I^{(j)}$, $\phi_i^{(j)}$ extends to a C^1 conformal map on V such that $\phi_i^{(j)}(V) \subset V$;
(ii) there exists a constant $0 < \gamma < 1$ such that for all $j \in \mathbb{N}$, $i \in I^{(j)}$ and $x \in V$,
$$|D\phi_i^{(j)}(x)| \leq \gamma;$$

(iii) there exists a constant $K > 0$ such that for all $n \in \mathbb{N}$, $\omega \in I_n^\infty$, $j \in \mathbb{N}$ and $x_1, x_2 \in V$,

$$|D\phi_{\omega|_j}(x_1)| \le K|D\phi_{\omega|_j}(x_2)|.$$

For a conformal map $\phi\colon V \to \phi(V)$, write

$$\|D\phi\| = \sup\{|D\phi(x)|\colon x \in X\}.$$

Definition. For any $s \ge 0$, we set

$$\overline{P}(s) = \overline{P}_\Phi(s) := \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in I^n} ||D\phi_\omega||^s, \underline{P}(s) = \underline{P}_\Phi(s) := \liminf_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in I^n} ||D\phi_\omega||^s \in [-\infty, \infty].$$

Then the function $\overline{P}/\underline{P} : [0,\infty) \to [-\infty,\infty]$ is called the upper/lower pressure function of the NCIFS Φ.

Definition. The upper/lower pressure function has the following monotonicity. If $s_1 < s_2$, then either both $\overline{P}(s_1)$ and $\overline{P}(s_2)$ are equal to ∞, both are equal to $-\infty$, or $\overline{P}(s_1) > \overline{P}(s_2)$. Set

$$\overline{s}(\Phi) := \sup\{s \ge 0 \ : \ \overline{P}(s) > 0\} = \inf\{s \ge 0 \ : \ \overline{P}(s) < 0\},$$

where we set $\sup\emptyset = 0$ and $\inf\emptyset = \infty$. We define a value $\underline{s}(\Phi)$ associated with $\underline{P}$ in a similar way.

Definition (Open set condition)**.** A NCIFS Φ satisfies the Open Set Condition (OSC) $\overset{def}{\Longleftrightarrow}$ For any $j \in \mathbb{N}$, and any $a \neq b \in I^{(j)}$, $\phi_a^{(j)}(\mathrm{int}(X)) \cap \phi_b^{(j)}(\mathrm{int}(X)) = \emptyset$.

Example. (1) Set

$$\Phi_1 = \left\{x \mapsto \frac{1}{4}x, x \mapsto \frac{1}{4}x + \frac{3}{4}\right\}, \Phi_2 = \left\{x \mapsto \frac{1}{2}x, x \mapsto \frac{1}{2}x + \frac{1}{2}\right\}.$$

Then the NCIFS $\Phi_\tau = \Phi_{\tau_1}\Phi_{\tau_2}\Phi_{\tau_3}\cdots$ satisfies the OSC for $\tau = \tau_1\tau_2\cdots \in \{1,2\}^{\mathbb{N}}$.

(2) Set

$$\Psi = \left(\left\{x \mapsto \frac{1}{2}x, x \mapsto \frac{1}{2}x + \frac{1}{j}\right\}\right)_{j=1}^\infty.$$

Then the NCIFS Ψ does not satisfy the OSC.

Theorem. [12, 8, (Rempe-Gillen and Urbański + N)] *Let* $\Phi = (\{\phi_i^{(j)}\}_{i\in I^{(j)}})_{j=1}^\infty$ *be a NCIFS satisfying the OSC. If* $\inf\{||D\phi_i^{(j)}|| \ : \ j \in \mathbb{N}, i \in I^{(j)}\} > 0$, *then we have*

$$\overline{\dim}_{\mathrm{B}} J = \dim_{\mathrm{P}} J = \overline{s}(\Phi) \ (N),$$

$$\dim_{\mathrm{H}} J = \underline{s}(\Phi) \ (RU),$$

where $\overline{\dim}_{\mathrm{B}} / \dim_{\mathrm{P}} / \dim_{\mathrm{H}}$: *the upper box/ packing/ Hausdorff dimension.*

Remark.

- Rempe-Gillen and Urbański [12] give a dimension formula under much weaker assumptions.
- Recently Käenmäki and Rutar [5] give a formula for the Assouad dimension under similar assumptions.

Example. (1) Fix $\tau = \tau_1\tau_2\cdots \in \{1,2\}^{\mathbb{N}}$. The NCIFS $\Phi = \Phi_\tau$ satisfies the OSC . Moreover,

$$||D\phi_i^{(j)}|| \geq 1/4 \text{ for all } j \in \mathbb{N}, i \in I^{(j)}.$$

For $s \geq 0$,

$$\sum_{\omega\in I^n} ||D\phi_{\omega_1\omega_2\cdots\omega_n}||^s = \#I^n \left(\prod_{j=1}^n \frac{1}{4}\tau_j\right)^s \text{ since } ||D\phi_{\omega_j}^{(j)}|| = \frac{1}{4}\tau_j.$$

Hence, we have

$$\begin{aligned}\overline{P}_\Phi(s) &= \limsup_{n\to\infty} \frac{1}{n}\log\left[\#I^n\left(\prod_{j=1}^n \frac{1}{4}\tau_j\right)^s\right] \\ &= \log 2 + s\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \log(\tau_j/4),\end{aligned}$$

and

$$\overline{s}(\Phi) = \frac{\log 2}{\log 4 - \limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \log\tau_j}.$$

In a similar way, we obtain

$$\underline{s}(\Phi) = \frac{\log 2}{\log 4 - \liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \log\tau_j}.$$

By using theorem above,

$$\overline{\dim_{\mathrm{B}}} J = \dim_{\mathrm{P}} J = \frac{\log 2}{\log 4 - \limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \log\tau_j},$$

$$\dim_{\mathrm{H}} J = \frac{\log 2}{\log 4 - \liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^n \log\tau_j}.$$

(2) $\Psi \to$ Section 2 (NCIFS with overlaps).

2. Transversality methods for NCIFS

2.1. **Transversality methods.** To obtain typical results from parameterized systems, the transversality method was developed. The transversality condition, which controls the way the parameterized systems depend on parameters, asserts that the graphs of two functions

$$t \mapsto \pi_t(\omega) \text{ and } t \mapsto \pi_t(\tau) \text{ for } \omega \neq \tau \in \Sigma$$

can only intersect at a nonzero angle (transversality). Here, π_t is the natural projection corresponding to the parameter t and ω, τ are elements of the symbolic space Σ.

Selected results on transversality methods.

- 1995: Self-similar sets with overlaps$-\{0,1,3\}$ problems (Pollicot and Simon [11])
- 1995: Absolute continuity of a.e. Bernoulli convolution$-$Erdös problems (Solomyak [15])
- 2001: Limit sets and invariant measures for CIFS with overlaps (Simon, Solomyak, Urbański [13, 14])
- 2013: Expanding rational semigroups with overlaps (Sumi and Urbański [17])
- 2022: Iterated function systems with inverses (Takahashi [18])

- 2024: Limit sets and generalized invariant measures for NCIFS with overlaps (N, N and Takahashi [9, 10])

2.2. **Transversal family of NCIFS.** Let $\Phi = \left(\{\phi_i^{(j)} : X \to X\}_{i\in I^{(j)}}\right)_{j=1}^{\infty}$ be a NCIFS on X.

Definition (Address map for NCIFS)**.** For each $n \in \mathbb{N}$, define *the n-th address map* $\pi_n \colon I_n^{\infty} \to X$ as

$$\{\pi_n(\omega)\} = \bigcap_{j=1}^{\infty} \phi_{\omega|_j}(X) \text{ for } \omega \in I_n^{\infty}.$$

Remark. Let J be the limit set of Φ. Then $\pi_1(I^{\infty}) = J$.

Let $U \subset \mathbb{R}^d$ be an open set (as a parameter space). Now we consider a one-parameter family of NCIFS

$$\Phi_t = (\{\phi_{i,t}^{(j)}\}_{i\in I^{(j)}})_{j=1}^{\infty}, \quad t \in U.$$

From below we always assume the following (we assume that V, γ and K can be taken independently of t):

(iv) for any $n \in \mathbb{N}$, the map

$$U \times I_n^{\infty} \ni (t, \omega) \mapsto \pi_{n,t}(\omega)$$

is continuous;

(v) for any $\eta > 0$ and $t_0 \in U$, there exists $\delta = \delta(\eta, t_0) > 0$ such that for all $t \in U$ with $|t - t_0| \le \delta$, $n, j \in \mathbb{N}$ and $\omega \in I_n^{\infty}$, we have

$$\exp(-j\eta) \le \frac{||D\phi_{\omega|_j,t_0}||}{||D\phi_{\omega|_j,t}||} \le \exp(j\eta);$$

Denote by $\mathcal{L}_d$ the d-dimensional Lebesgue measure on $\mathbb{R}^d$.

Definition (Transversality condition)**.** We say that $\{\Phi_t\}_{t\in U}$ satisfies the *transversality condition on U* if the following holds: for any compact subset $G \subset U$ there exists a sequence $\{C_n\}_{n=1}^{\infty}$ of positive constants such that

- $$\lim_{n\to\infty} \frac{\log C_n}{n} = 0;$$
- for all $\omega, \tau \in I_n^{\infty}$ with $\omega_n \ne \tau_n$ and $r > 0$, we have
$$\mathcal{L}_d\left(\{t \in G : |\pi_{n,t}(\omega) - \pi_{n,t}(\tau)| \le r\}\right) \le C_n r^m.$$

Theorem. [9, Main Theorem A] *Let $\{\Phi_t\}_{t\in U}$ satisfy the transversality condition on U. Then*

(i)
$$\dim_H(J_t) = \min\{m, s(\Phi_t)\} \textit{ for } \mathcal{L}_d - a.e.\ t \in U;$$

(ii)
$$\mathcal{L}_m(J_t) > 0 \textit{ for } \mathcal{L}_d - a.e.\ t \in \{t \in U\ : s(t) > m\}.$$

Example. $U := (0, 2^{-2/3}), 2^{-2/3} \approx 0.6299$. For $t \in U, j \in \mathbb{N}$, $\Psi_t^{(j)} = \{x \mapsto tx, x \mapsto tx + 1/j\}, \Psi_t = (\Psi_t^{(j)})_{j=1}^{\infty}$. Then $\{\Psi_t\}_{t\in U}$ satisfies the transverslity condition. By applyng theorem above, we obtain

Corollary. *Let J_t be the limit set corresponding to t. Then*

$$\dim_H(J_t) = \min\left\{1, \frac{\log 2}{\log t}\right\}$$

for a.e. $t \in \{t \in U \ : \ t \leq 1/2\}$ and

$$\mathcal{L}_1(J_t) > 0$$

for a.e. $t \in \{t \in U \ : \ 1/2 < t\}$.

Proof. Let

$$\mathcal{F} := \left\{ t \mapsto f(t) = \pm 1 + \sum_{j=1}^{\infty} a_j t^j : a_j \in [-1, 1] \right\}.$$

Lemma. [15] *There exists $C > 0$ such that for any $f \in \mathcal{F}$, any $r > 0$,*

$$\mathcal{L}_1(\{t \in U \ : \ |f(t)| \leq r\}) \leq Cr.$$

Lemma. [9, Lemma 5.3] *Let $t \in U$. For any $n \in \mathbb{N}$, $\omega = \omega_n \cdots \omega_{n+j-1} \cdots \in I_n^\infty := \{1, 2\}^{\mathbb{N}}$,*

$$\pi_{n,t}(\omega) = \sum_{i=1}^{\infty} b_{\omega_{n+i-1}}^{(n+i-1)} t^{i-1},$$

where $b_1^{(n+i-1)} = 0, b_2^{(n+i-1)} = 1/(n + i - 1)$ for $i \in \mathbb{N}$.

Then we have for any $t \in U$ and any $\omega, \tau \in I_n^\infty$ with $\omega_n \neq \tau_n$,

$$\begin{aligned}
\pi_{n,t}(\omega) - \pi_{n,t}(\tau) &= \sum_{i=1}^{\infty} b_{\omega_{n+i-1}}^{(n+i-1)} t^{i-1} - \sum_{i=1}^{\infty} b_{\tau_{n+i-1}}^{(n+i-1)} t^{i-1} \\
&= b_{\omega_n}^{(n)} - b_{\tau_n}^{(n)} + \sum_{i=2}^{\infty} \left(b_{\omega_{n+i-1}}^{(n+i-1)} - b_{\tau_{n+i-1}}^{(n+i-1)} \right) t^{i-1} \\
&= \frac{1}{n} \left(\pm 1 + \sum_{i=2}^{\infty} n \left(b_{\omega_{n+i-1}}^{(n+i-1)} - b_{\tau_{n+i-1}}^{(n+i-1)} \right) t^{i-1} \right).
\end{aligned}$$

Then the function $t \mapsto \pm 1 + \sum_{i=2}^{\infty} n(b_{\omega_{n+i-1}}^{(n+i-1)} - b_{\tau_{n+i-1}}^{(n+i-1)}) t^{i-1}$ belongs to $\mathcal{F}$. By Solomyak's lemma, there exists $C > 0$ such that for any $\omega, \tau \in I_n^\infty$ with $\omega_n \neq \tau_n$ and any $r > 0$,

$$\begin{aligned}
&\mathcal{L}_1(\{t \in U \ : \ |\pi_{n,t}(\omega) - \pi_{n,t}(\tau)| \leq r\}) \\
&= \mathcal{L}_1(\{t \in U \ : \ |\pm 1 + \sum_{i=2}^{\infty} n(b_{\omega_{n+i-1}}^{(n+i-1)} - b_{\tau_{n+i-1}}^{(n+i-1)}) t^{i-1}| \leq nr\}) \\
&\leq C(nr).
\end{aligned}$$

If we set $C_n := Cn$ for any $n \in \mathbb{N}$, we have

$$\mathcal{L}_1(\{t \in U \ : \ |\pi_{n,t}(\omega) - \pi_{n,t}(\tau)| \leq r\}) \leq C_n r$$

and

$$\frac{1}{n} \log C_n = \frac{1}{n} \log C + \frac{1}{n} \log n \to 0$$

as $n \to \infty$.

Hence, $\{\Psi_t\}_{t \in U}$ satisfies the transversality condition. □

2.3. **Idea of the proof of the main theorem.**

Lemma. *For any compact subset $G \subset U$ and any α with $0 < \alpha < m$, there exists a sequence $\{\tilde{C}_n\}_{n=1}^{\infty}$ of positive constants such that*

$$\lim_{n\to\infty} \frac{\log \tilde{C}_n}{n} = 0$$

and for any $\omega, \tau \in I_n^{\infty}$ with $\omega_n \neq \tau_n$,

$$\int_G \frac{1}{|\pi_{n,t}(\omega) - \pi_{n,t}(\tau)|^{\alpha}} \, d\mathcal{L}_d(t) \leq \tilde{C}_n.$$

Proof. Let $n \in \mathbb{N}$. By the transversality condition we have that

$$\begin{aligned}
\int_G \frac{1}{|\pi_{n,t}(\omega) - \pi_{n,t}(\tau)|^{\alpha}} \, d\mathcal{L}_d(t) &= \int_0^{\infty} \mathcal{L}_d\left(\left\{t \in G \;:\; \frac{1}{|\pi_{n,t}(\omega) - \pi_{n,t}(\tau)|^{\alpha}} \geq x\right\}\right) \, dx \\
&= \int_0^{\infty} \mathcal{L}_d\left(\left\{t \in G \;:\; |\pi_{n,t}(\omega) - \pi_{n,t}(\tau)| \leq \frac{1}{x^{1/\alpha}}\right\}\right) \, dx \\
&= \int_0^{|X|^{-\alpha}} \mathcal{L}_d(G) \, dx + \int_{|X|^{-\alpha}}^{\infty} C_n \frac{1}{x^{m/\alpha}} \, dx \\
&= |X|^{-\alpha}\mathcal{L}_d(G) + C_n \left[\frac{1}{1 - m/\alpha} x^{1-m/\alpha}\right]_{|X|^{-\alpha}}^{\infty} \\
&= |X|^{-\alpha}\mathcal{L}_d(G) + C_n \frac{1}{m/\alpha - 1} |X|^{m-\alpha} =: \tilde{C}_n.
\end{aligned}$$

Since $\frac{1}{n}\log C_n \to 0$ as $n \to \infty$, we have $\frac{1}{n}\log \tilde{C}_n \to 0$ as $n \to \infty$. □

Lemma. [9, Lemma 4.6 (Existence of a Gibbs-like measure)] *Let $t \in U$ and let $s \geq 0$. Then there exists a Borel probability measure $\mu_{t,s}$ on I^{∞} such that for any $\omega \in I^n$, $n \in \mathbb{N}$,*

$$\mu_{t,s}([\omega]) \leq K^s \frac{||D\phi_{\omega,t}||^s}{Z_{n,t}(s)},$$

where K is the constant coming from the bounded distortion condition and $Z_{n,t}(s) = \sum_{\omega \in I^n} ||D\phi_{\omega,t}||^s$.

The following is a key ingredient.

Proposition. *For any $t_0 \in U$ and any $\epsilon > 0$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that*

$$\dim_H(J_t) \geq \min\{m, s(\Phi_{t_0})\} - \epsilon$$

for $\mathcal{L}_d$- a.e. $t \in B(t_0, \delta)$.

Proof. For any $t_0 \in U$, we set $s := \min\{m, s(\Phi_{t_0})\}$. Let $\mu = \mu_{t_0, s-\epsilon/2}$ be the Gibbs like measure ascribed to t_0 and $s - \epsilon/2$. We set $\mu_2 = \mu \times \mu$ and

$$R(t) := \iint_{I^{\infty} \times I^{\infty}} \frac{1}{|\pi_{1,t}(\omega) - \pi_{1,t}(\tau)|^{s-\epsilon}} \, d\mu_2.$$

Let $n \in \mathbb{N}$. For any $\rho \in I^n \cup \{\emptyset\}$, we set

$$A_{\rho} := \{(\omega, \tau) \in I^{\infty} \times I^{\infty} \;:\; \omega \wedge \tau = \rho\},$$

and take a sufficiently small $\delta > 0$ coming from the condition (v) ascribed to ϵ and t_0. Then

$$\int_{B(t_0,\delta)} R(t)\ d\mathcal{L}_d(t) = \sum_{n\geq 0}\sum_{\rho\in I^n}\iint_{A_\rho}\left(\int_{B(t_0,\delta)} \frac{1}{|\pi_{1,t}(\omega)-\pi_{1,t}(\tau)|^{s-\epsilon}}\ dt\right)\ d\mu_2(\omega,\tau)$$

(by Fubini's Theorem)

$$\leq O_{t_0,\epsilon}(1)\sum_{n\geq 0}\sum_{\rho\in I^n}\iint_{A_\rho}\left(\int_{B(t_0,\delta)} \frac{||D\phi_{\rho,t_0}||^{-s+\epsilon/2}}{|\pi_{n+1,t}(\sigma^n\omega)-\pi_{n+1,t}(\sigma^n\tau)|^{s-\epsilon}}\ dt\right)\ d\mu_2(\omega,\tau)$$

(by the definition of n-th adderss maps and condition (v))

$$\leq O_{t_0,\epsilon}(1)\sum_{n\geq 0}\tilde{C}_{n+1}\sum_{\rho\in I^n}\iint_{A_\rho} ||D\phi_{\rho,t_0}||^{-s+\epsilon/2}\ d\mu_2(\omega,\tau)$$

(by the transversality)

$$\leq O_{t_0,\epsilon}(1)\sum_{n\geq 0}\tilde{C}_{n+1}\sum_{\rho\in I^n}\iint_{A_\rho} \frac{K^{s-\epsilon/2}}{\mu([\rho])Z_{n,t_0}(s-\epsilon/2)}\ d\mu_2(\omega,\tau)$$

(by the Gibbs like property)

$$\leq O_{t_0,\epsilon}(1)\sum_{n\geq 0}\frac{\tilde{C}_{n+1}}{Z_{n,t_0}(s-\epsilon/2)}$$

(since $\mu_2(A_\rho)\leq \mu([\rho])^2$).

Since $\frac{1}{n}\log\tilde{C}_{n+1}\to 0$ as $n\to\infty$, it follows from the fact of exponentially decay property of $1/Z_{n,t_0}(s-\epsilon/2)$ that

$$\int_{B(t_0,\delta)} R(t)\ \mathcal{L}_d(t) < \infty.$$

Hence we have that for $\mathcal{L}_d$-a.e. $t\in B(t_0,\delta)$,

$$R(t) = \iint_{\mathbb{R}^m\times\mathbb{R}^m} \frac{1}{|x-y|^{s-\epsilon}}\ d\left(\pi_{1,t}(\mu)\times\pi_{1,t}(\mu)\right) < \infty,$$

where $\pi_{1,t}(\mu)$ is the push forward measure of μ by $\pi_{1,t}$. Since $\pi_{1,t}(\mu)\,(J_t) = 1$, by [2, Theorem 4.13 (a)] we have

$$\dim_H(J_t) \geq \min\{m, s(t_0)\} - \epsilon$$

for $\mathcal{L}_d$- a.e. $t\in B(t_0,\delta)$.

□

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