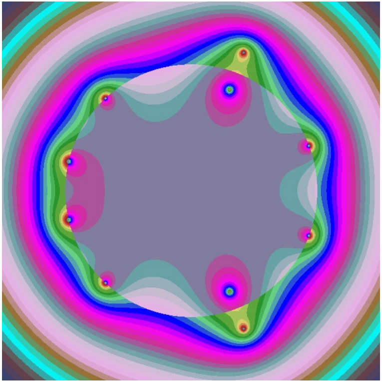
Real slice of complex surface automorphism and complex Salem number



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Abstract

The characteristic polynomials of the linear map of the homologies induced by automorphisms of complex surface and its restriction to the real slice are studied.

Counting the number of periodic points, using Lefschetz formula, we conclude that in the case of the surface automorphism of smallest positive entropy, the entropy of the automorphism and that of its restriction to the real slice are same.

In some cases, complex Salem numbers appear as the leading eigenvalues of homology homomorphisms.

Contents

0. Main results

THEOREM. The characteristic polynomial of the linear map of homology group induced by the real slice of surface automorphisms are :

$$(\Gamma_1) \quad \phi_n(z) = \frac{1}{z+1}\{z^{n+1}(z^3 - z + 1) - (-1)^n(z^3 - z^2 + 1)\},$$

$$(\Gamma_2) \quad \psi_{2k}(z) = \frac{z+1}{z^2+1}\{z^{2k+1}(z^3 + z + 1) + (-1)^k(z^3 + z^2 + 1)\},$$

$$(\Gamma_3) \quad \varphi_{3k}(z) = \frac{z^2+z+1}{z^3+1}\{z^{3k+1}(z^3+z+1) - (-1)^k(z^3+z^2+1)\}.$$

THEOREM. The leading eigenvalues of the homology homomorphisms are :

(Γ_1) case negative Salem,

(Γ_2) and (Γ_3) cases complex Salem.

1. Surface automorphism

Rational automorphism

For parameters $(\alpha, \beta) \in \mathbb{C}^2$, let

$$f_{\alpha,\beta}(x, y) = (y, \frac{y + \alpha}{x + \beta} + \beta)$$

be a birational map.

The indeterminate point, p_*, of $f_{\alpha,\beta}$ and the indeterminate point, q_*, of the inverse map $f_{\alpha,\beta}^{-1}$ are given by

$$p_* = (-\beta, -\alpha), \qquad q_* = (-\alpha, \beta).$$

For $n \in \mathbb{N}$, let $\mathcal{V}_n$ denote the set of parameters $(\alpha, \beta) \in \mathbb{C}^2$ satisfying

$$f_{\alpha,\beta}^k(q_*) \neq p_*, \quad k = 0, 1, \cdots, n-1, \quad \text{and} \quad f_{\alpha,\beta}^n(q_*) = p_*.$$

Surface automorphism

For $(\alpha, \beta) \in \mathcal{V}_n$, let

$$\pi : \mathcal{S} \to \mathbb{CP}^2$$

be the blowup of $\mathbb{CP}^2$ at $n+3$ points

$$q_*, f_{\alpha,\beta}(q_*), \cdots, f^n_{\alpha,\beta}(q_*) = p_*, \;\; p_x = (\infty, 0), \;\; p_y = (0, \infty).$$

THEOREM(Bedford-Kim, 2007) If $(\alpha, \beta) \in \mathcal{V}_n$, then $f_{\alpha,\beta}$ lifts to an automorphism of surface $\mathcal{S}$.

Characteristic polynomial

Let $q_0 = q_*, q_k = f_{\alpha,\beta}(q_{k-1}),\ k = 1, \cdots, n.$

$$Q_k = \pi^{-1}(q_k) \subset \mathcal{S},\ k = 0, \cdots, n,$$

$$E_x = \pi^{-1}(p_x),\ E_y = \pi^{-1}(p_y) \subset \mathcal{S},$$

$$H \subset \mathcal{S}, \qquad \text{generic line.}$$

A basis of $H^2(\mathcal{S}, \mathbb{Z})$ is given by the classes of these lines.

Let $F : \mathcal{S} \to \mathcal{S}$ denote the lift of $f_{\alpha,\beta}$.

We abuse the name of a curve for the corresponding cohomology class.

$$
\begin{aligned}
F^*(H) &= 2H - E_x - E_y - Q_0, \\
F^*(E_x) &= H - E_x - Q_0, \\
F^*(E_y) &= H - E_x - E_y, \\
F^*(Q_n) &= H - E_y - Q_0, \\
F^*(Q_k) &= Q_{k+1}, \quad k = 0, 1, \cdots, n-1.
\end{aligned}
$$

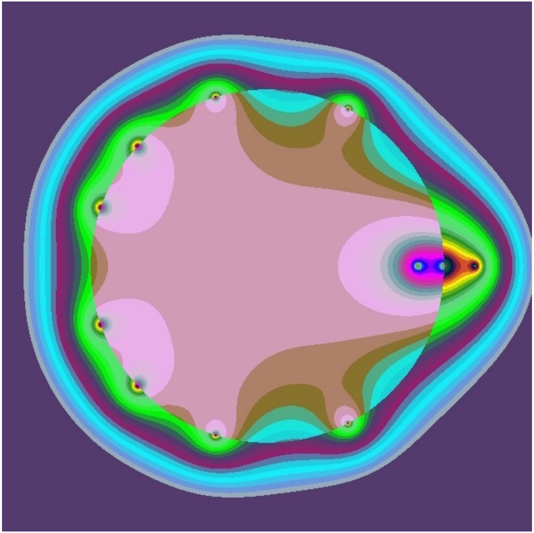
Characteristic polynomial of $F^* : H^2(\mathcal{S}, \mathbb{Z}) \to H^2(\mathcal{S}, \mathbb{Z})$ is the determinant of

$$\begin{array}{c} \\ H \\ E_y \\ E_x \\ Q_n \\ Q_0 \\ Q_1 \\ \vdots \\ Q_{n-2} \\ Q_{n-1} \end{array} \begin{array}{c} \begin{array}{ccccccccc} H & E_y & E_x & Q_n & Q_0 & Q_1 & \cdots & Q_{n-2} & Q_{n-1} \end{array} \\ \begin{pmatrix} 2-z & 1 & 1 & 1 & & & & & \\ -1 & -1-z & & -1 & & & & & \\ -1 & -1 & -1-z & & & & & & \\ & & & -z & & & & & 1 \\ -1 & & -1 & -1 & -z & & & & \\ & & & & 1 & -z & & & \\ & & & & & & \ddots & & \\ & & & & & & & -z & \\ & & & & & & & 1 & -z \end{pmatrix} \end{array},$$

which gives (by multiplying $(-1)^{n+4}$)

$$\chi_n(z) = z^{n+1}(z^3 - z - 1) + z^3 + z^2 - 1.$$

$\chi_7(z)$



2. Invariant cubic curve

Γ family

For $t \in \mathbb{C}$, define parameters α and β by

$$(\Gamma_1) \qquad \alpha_1 = -\frac{t^6 + t^5 - 2t^3 + t + 1}{2t^2(t+1)^2}, \quad \beta_1 = -\frac{t^5 - 1}{2t^2(t+1)},$$

$$(\Gamma_2) \qquad \alpha_2 = \frac{t^4 + t^3 + 2t^2 + t + 1}{2t(t+1)^2}, \quad \beta_2 = \frac{t^3 - 1}{2t(t+1)},$$

$$(\Gamma_3) \qquad \alpha_3 = \frac{(t+1)^2}{2t}, \quad \beta_3 = \frac{t^2 - 1}{2t}.$$

THEOREM(Bedford-Kim 2007)

Let $t \neq 0, \pm 1$ with $t^3 \neq 1$ be given. Then there is an invariant cubic curve of $f_{\alpha,\beta}$ if and only if $(\alpha, \beta) = (\alpha_j(t), \beta_j(t))$ for some $1 \leq j \leq 3$.

Invariant cubic curve

The cubic polynomial $Q(x, y)$ that defines the invariant cubic curve is a solution of equation

$$Q \circ f_{\alpha,\beta} = t \; Q \det Df_{\alpha,\beta}.$$

It defines a meromorphic eigenform $\eta = \frac{du \wedge dv}{Q}$ with

$$f^*_{\alpha,\beta}\eta = t^{-1}\eta.$$

$Q(x, y) = 0$ defines the invariant curve as follows.

(Γ_1) : irreducible cubic with a cusp.
(Γ_2) : line tangent to a quadric.
(Γ_3) : three lines passing through a point.

Surface automorphism with invariant cubic

THEOREM(Bedford-Kim, 2007)

Suppose that $n, 1 \leq j \leq 3$, and t are given, and suppose that $(\alpha_j(t), \beta_j(t)) \notin \mathcal{V}_k$ for any $k < n$. Then the point $(\alpha_j(t), \beta_j(t))$ belongs to $\mathcal{V}_n$ if and only if : j divides n and t is a root of χ_n.

3. Salem number

Salem number

A complex number is an **algebraic integer** if it is the zero of a polynomial with integer coefficients and leading coefficient 1.

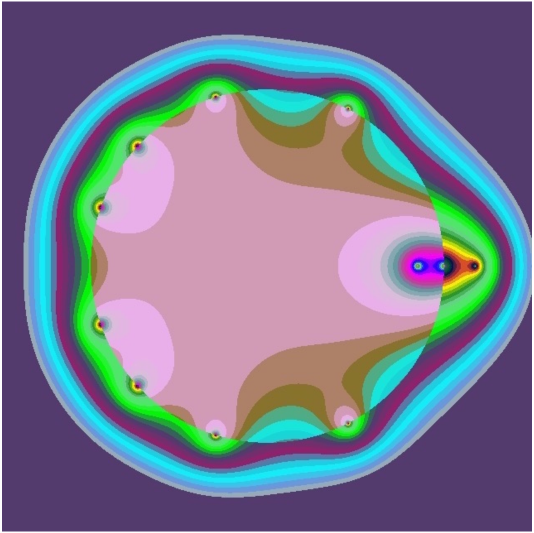
Its **minimal polynomial** is the lowest degree polynomial of that type it satisfies.

Its (Galois)**conjugates** are the zeros of its minimal polynomial.

A **Salem number** is an algebraic integer $\tau > 1$ conjugate to τ^{-1}, all of whose conjugates, excluding τ and τ^{-1} lie on $|z| = 1$.

A **Salem polynomial** is the minimal polynomial of a Salem number.

$\chi_7(z)$



Cyclotomic polynomial

It is known that an algebraic integer lying with all its conjugates on the unit circle must be a root of unity(Kronecker, 1857).

A **cyclotomic polynomial** is the minimal polynomial of a root of unity.

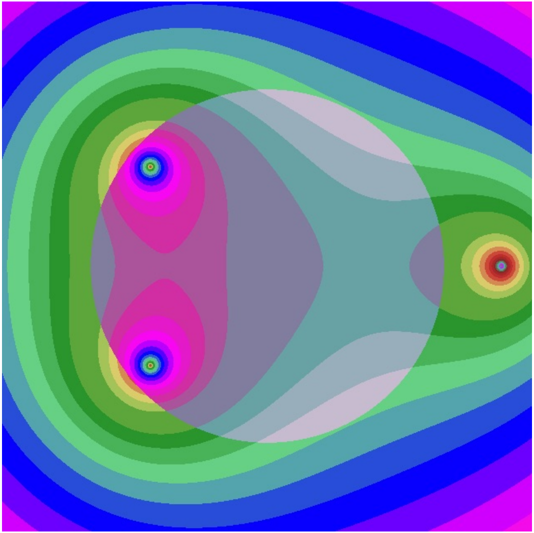
Pisot polynomial

An algebraic integer $\lambda > 1$ is a **Pisot number** if its Galois conjugates satisfy $|\lambda'| < 1$.

A **Pisot polynomial** is the minimal polynomial of a Pisot number.

$P(z) = z^3 - z - 1$ is the minimal polynomial of the smallest Pisot number.

$P(z) = z^3 - z - 1$



Pisot polynomial and Salem number

If $P(z)$ is a Pisot polynomial, then except possibly for small values of n, the polynomials

$$S_{n,P,\pm 1}(z) = z^n P(z) \pm z^{\deg P} P(z^{-1})$$

factor as the minimal polynomial of a Salem number, possibly multiplied by some cyclotomic polynomials.

$P^*(z) = z^{\deg P} P(z^{-1})$ is called the **reciprocal polynomial** of $P(z)$.

Characteristic polynomial

For $n \geq 7$, the characteristic polynomial

$$\chi_n(z) = z^{n+1}(z^3 - z - 1) + z^3 + z^2 - 1$$

factors as

$$\chi_n(z) = C_n(z)S_n(z),$$

where $C_n(z)$ is a product of cyclotomic polynomials and $S_n(z)$ is a Salem polynomial.

Let $\lambda_n > 1$ denote the leading eigenvalue, which is the Salem number given by $\chi_n(z) = 0$.

For $n < 7$, $\chi_n(z)$ is a product of cyclotomic polynomials.

$$
\begin{aligned}
\chi_0(z) &= (z-1)(z+1)(z^2+z+1), \\
\chi_1(z) &= (z-1)(z^4+z^3+z^2+z+1), \\
\chi_2(z) &= (z-1)(z+1)(z^4+1), \\
\chi_3(z) &= (z-1)(z^2+z+1)(z^4-z^2+1), \\
\chi_4(z) &= (z-1)(z^6-z^3+1), \\
\chi_5(z) &= (z-1)(z^8+z^7-z^5-z^4-z^3+z+1), \\
\chi_6(z) &= (z-1)^3(z+1)(z^2+z+1)(z^4+z^3+z^2+z+1).
\end{aligned}
$$

4. Lefschetz formula

Lefschetz number

Let $|K|$ be a finite polyhedra, and let $f : |K| \to |K|$ be a continuous map.

Let $T_i(|K|)$ denote the torsion subgroup of the homology group $H_i(|K|, \mathbb{Z})$.

Let $B_i(|K|) = H_i(|K|, \mathbb{Z}) / T_i(|K|)$.

f induces a homomorphism $f_*|_{B_i(|K|)} : B_i(|K|) \to B_i(|K|)$.

Lefschetz number Λ_f of f is defined by

$$\Lambda_f = \sum_{i=0}^{\dim K} (-1)^i \operatorname{trace}(f_*|_{B_i(|K|)}).$$

Lefschetz formula

Suppose M is a compact smooth manifold without boundary.
And suppose $f : M \to M$ is a differentiable map satisfying $\det(Df - I) \neq 0$ at all fixed points.

The **Lefschetz index** of fixed point p of f is defined as

$$\mathrm{Ind}(f;p) = \mathrm{sign}(\det(Df_p - I)).$$

The **Lefschetz formula** is

$$\sum_{f(p)=p} \mathrm{Ind}(f;p) = \sum_{k=0}^{\dim M} (-1)^k \mathrm{trace}(f^*|_{H^k(M,\mathbb{R})}).$$

Example

As an example, let us consider the case of $F = F_{\alpha,\beta} : \mathcal{S} \to \mathcal{S}$ with $(\alpha, \beta) \in \mathcal{V}_n$, $n \geq 7$.

$$\begin{aligned} H^4(\mathcal{S}, \mathbb{Z}) &\simeq \mathbb{Z}, & \mathrm{trace}(F^*|_{H^4}) &= 1, \\ H^2(\mathcal{S}, \mathbb{Z}) &\simeq \mathbb{Z}^{1,n+3}, & \mathrm{trace}(F^*|_{H^2}) &= 0, \\ H^0(\mathcal{S}, \mathbb{Z}) &\simeq \mathbb{Z}, & \mathrm{trace}(F^*|_{H^0}) &= 1. \end{aligned}$$

By the Lefschetz formula, we conclude that F has two fixed points, because in the complex dynamical system case, Lefschetz index is always 1.

As

$$\chi_n(z) = z^{n+1}(z^3 - z - 1) + z^3 + z^2 - 1$$

is the characteristic polynomail of $F^*|_{H^2}$.
The zeros of $\chi_n(z)$ are the eigenvalues of $F^*|_{H^2}$.

Let $\nu_0, \nu_1, \cdots, \nu_{n+3}$ denote the roots of χ_n, with $\nu_0 = 1$, and $\nu_1 = \lambda_n > 1$.

Let

$$\tau_k = \mathrm{trace}(F^{k*}|_{H^2}) \;=\; \sum_{i=0}^{n+3} \nu^k, \quad k = 1, 2, \cdots .$$

Then

$$\Lambda_{F^k} = 2 + \tau_k.$$

gives the number of fixed points of F^k.

The topological entropy of F is $\log \nu_1 = \log \lambda_n > 0$.

Real surface case

Consider the case with $\alpha, \beta \in \mathbb{R}$.

Let $\mathcal{R}$ denote the real slice of the complex surface $\mathcal{S}$.

$\mathcal{R}$ is a non-orientable real 2-dimensional analytic manifold without boundary.

In this case, $\mathcal{R}$ is invariant under $F : \mathcal{S} \to \mathcal{S}$.

Let $f : \mathcal{R} \to \mathcal{R}$ denote the restriction of F to $\mathcal{R}$.

$$H_2(\mathcal{R}, \mathbb{Z}) \simeq 0,$$

$$H_1(\mathcal{R}, \mathbb{Z}) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}^{n+3}, \qquad B_1(\mathcal{R}) \simeq \mathbb{Z}^{n+3},$$

$$H_0(\mathcal{R}, \mathbb{Z}) \simeq \mathbb{Z}.$$

Local index

In our cases, f has an eigenform $\eta = \frac{dx \wedge dy}{Q(x,y)}$ satisfying $f^*\eta = t^{-1}\eta$, in the complement of the invariant cubic curve C.

$t > 1$ is a Salem number.

If $p \in \mathcal{R} \setminus C$ is an isolated fixed point of f^m, then $\det Df_p^m = t^{-m} > 0$.

If p is a sink or a source of period m, then $\mathrm{Ind}(f^m, p) = 1$, since

$$\mathrm{Ind}(f^m, p) = \mathrm{sign}(\det(Df_p^m - I)).$$

If p is a saddle of period m, with eigenvalues μ_1, μ_2, then

$$\mathrm{Ind}(f^m, p) = 1, \quad \text{if} \quad \mu_1 < -1 < \mu_2 < 0. \quad \text{(bi-flip saddle)}$$

$$\mathrm{Ind}(f^m, p) = -1, \quad \text{if} \quad 0 < \mu_1 < 1 < \mu_2. \quad \text{(non-flip saddle)}$$

Index of saddle

If p is a bi-flip saddle of period m, with $\mathrm{Ind}(f^m, p) = 1$, then

$$\mathrm{Ind}(f^{km}, p) = -(-1)^k, \qquad k = 1, 2, \cdots .$$

If p is a non-flip saddle of period m, with $\mathrm{Ind}(f^m, p) = -1$, then

$$\mathrm{Ind}(f^{km}, p) = -1, \qquad\qquad k = 1, 2, \cdots .$$

5. Irreducible cubic with a cusp

$\mathcal{V}_n\Gamma_1$ case

Let us compute f_* for $\mathcal{V}_n\Gamma_1$ case.

For $n = 7$, $\alpha = -0.2916161663\cdots$, $\beta = -0.207880343\cdots$, with $q_0 = (-\alpha, \beta)$, $p_* = (-\beta, -\alpha)$, and

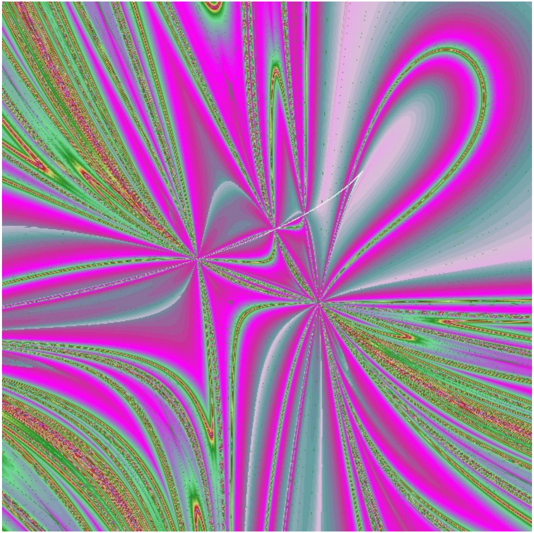
$$f(x,y) = (y, \frac{y+\alpha}{x+\beta} + \beta), \qquad q_k = f^k(q_0), \qquad p_* = q_n.$$

$\mathcal{R}$ is obtained by blowing up the real projective plane $\mathbb{RP}^2$ in $(n+3)$ points :

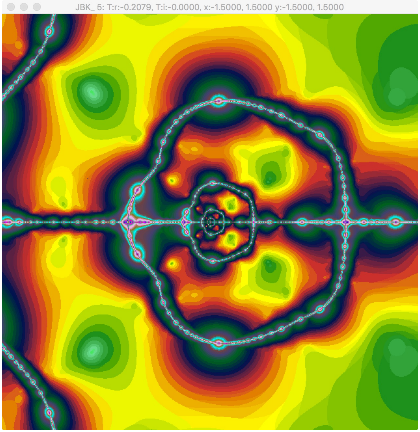
$$(\infty, 0), (0, \infty), q_0, q_1, \cdots, q_n.$$

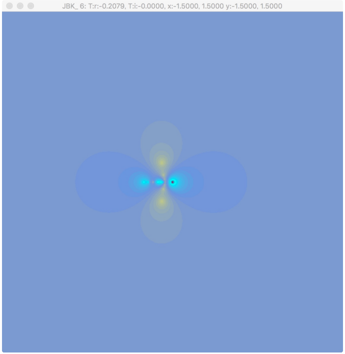
The invariant curve $C = \{Q(x,y) = 0\}$ passes through these ten points.

Real slice, J^-

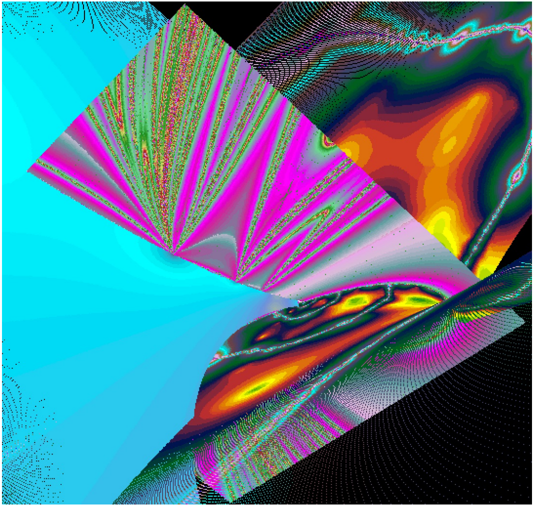


Unstable/stable manifold of fixed saddle

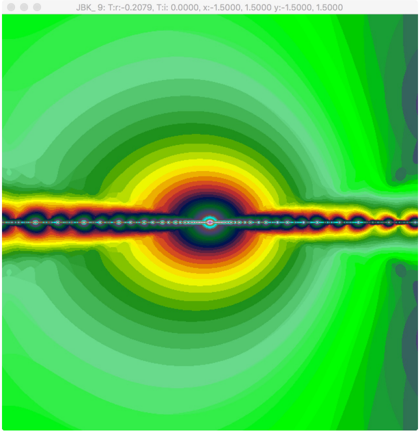


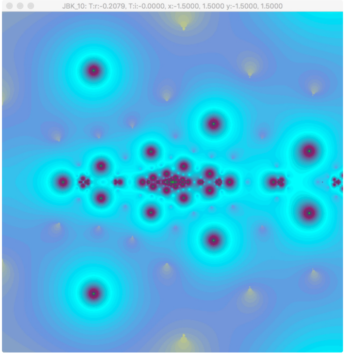


Unstable/stable manifold of fixed saddle

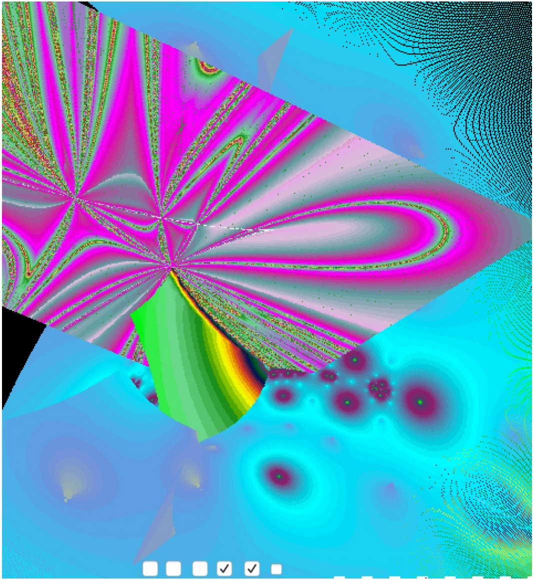


Unstable/stable manifold of period 3 saddle

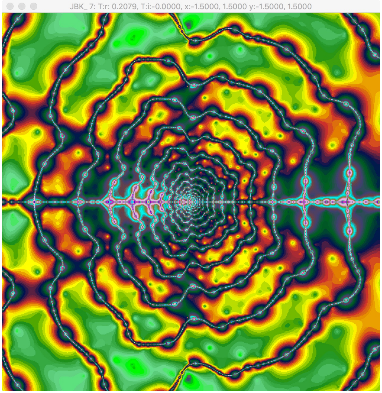


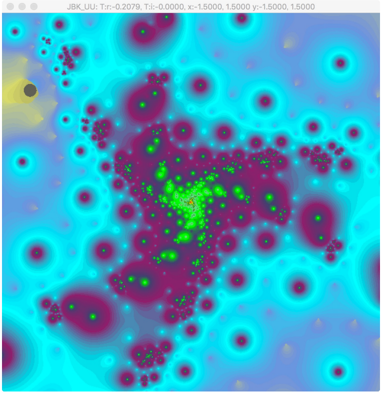


Unstable/stable manifold of period 3 saddle



Fixed repeller and per 2 sink





For $k = 0, 1, 2, \cdots, n$, let e_k denote the homology class representing the blowup fiber $\pi^{-1}(q_k) \subset \mathcal{R}$, with the clockwise orientation in $\mathbb{R}^2 \subset \mathbb{RP}^2$.

The homology class for blowup fibers at $(0, \infty)$ and $(\infty, 0)$ will be denoted as a and b respectively. The orientation is specified by the inner coordinates $x : \infty \to -\infty$ for a, and $y : \infty \to -\infty$ for b.

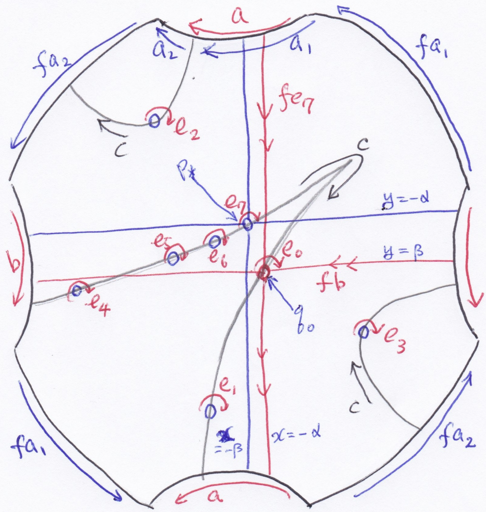
The 1-dimensional homology class of the invariant curve C will be denoted as c, orientation is specified to see the cusp point on the right hand side.

PROPOSITION.

$$H_1(\mathcal{R}, \mathbb{Z}) \text{ is generated by } c, a, b, e_0, e_1, \cdots, e_n,$$

and $2c \sim 0$.

$V_7 \Gamma_1$ $\alpha = -0.2916\cdots$, $\beta = -0.2078\cdots$



PROOF.

Let ℓ_∞ denote the homology class of the line at infinity, with counter-clockwise orientation.

Observing the figure, we have

$$c + e_0 + e_1 + \cdots + e_n + \ell_\infty \sim 0,$$

$$c - e_0 - e_1 - \cdots - e_n - \ell_\infty \sim 0.$$

So, $2c \sim 0$.

$f_* : H_1(\mathcal{R}, \mathbb{Z}) \to H_1(\mathcal{R}, \mathbb{Z})$, $\mathcal{V}_n\Gamma_1$ case

$$f_*(a) \sim \ell_\infty \sim c - e_0 - e_1 - \cdots - e_n,$$

$$f_*(b) \sim a + \ell_\infty + e_0 + 2e_2 + 2e_5 + 2e_6 + \cdots + 2e_n$$

$$\sim c + a - e_1 + e_2 - e_3 - e_4 + e_5 + e_6 + \cdots + e_n.$$

$$f_*(e_n) \sim b + \ell_\infty + e_0 + 2e_1 + 2e_2 + 2e_4 + 2e_5 + 2e_6 + \cdots + 2e_n$$

$$\sim c + b + e_1 + e_2 - e_3 + e_4 + e_5 + e_6 + \cdots + e_n.$$

$$f_*(e_0) = -e_1, \quad f_*(e_1) = -e_2, \quad f_*(e_2) = e_3, \quad f_*(e_3) = -e_4,$$

$$f_*(e_4) = -e_5, \quad f_*(e_5) = -e_6, \cdots, \quad f_*(e_{n-1}) = -e_n,$$

Let $B_1 = H_1(\mathcal{R}, \mathbb{Z})/\mathbb{Z}_2 c$. f_* induces an isomorphism of B_1, which is represented by a matrix below (case of $n = 7$).

$f_*|_{B_1} : B_1 \to B_1$, $\mathcal{V}_7\Gamma_1$ case

	a	b	e_7	e_0	e_1	e_2	e_3	e_4	e_5	e_6
a		1								
b			1							
e_7	-1	1	1							-1
e_0	-1									
e_1	-1	-1	1	-1						
e_2	-1	1	1		-1					
e_3	-1	-1	-1			1				
e_4	-1	-1	1				-1			
e_5	-1	1	1					-1		
e_6	-1	1	1						-1	

Set $t = -z$, and compute the determinant of the following matrix.

$f_*|_{B_1} + tI, \mathcal{V}_7\Gamma_1$ case

	a	b	e_7	e_0	e_1	e_2	e_3	e_4	e_5	e_6
a	t	1								
b		t	1							
e_7	-1	1	$1+t$							-1
e_0	-1			t						
e_1	-1	-1	1	-1	t					
e_2	-1	1	1		-1	t				
e_3	-1	-1	-1			1	t			
e_4	-1	-1	1				-1	t		
e_5	-1	1	1					-1	t	
e_6	-1	1	1						-1	t

Use the first and second lines with columns e_0 to e_6 to simplify the second and third columns.

$$
\begin{array}{c|cccccccccc|}
 & a & b & e_7 & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
\hline
a & t & 1 & & & & & & & & \\
b & & t & 1 & & & & & & & \\
e_7 & -1 & & 1+t & & & & & & & -1 \\
e_0 & -1 & & & t & & & & & & \\
e_1 & -1 & & & -1 & t & & & & & \\
e_2 & & & & & -1 & t & & & & \\
e_3 & -1 & & & & & 1 & t & & & \\
e_4 & -1 & & & & & & -1 & t & & \\
e_5 & -1 & & & & & & & -1 & t & \\
e_6 & -1 & & & & & & & & -1 & t
\end{array}
$$

Use columns e_0 to e_6 to simplify the first column.

$$
\begin{array}{c} \\ a \\ b \\ e_7 \\ e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} a & b & e_7 & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\left(\begin{array}{cccccccccc}
t & 1 & & & & & & & & \\
 & t & 1 & & & & & & & \\
t^3+t^2-1 & & & & & & & & & -1 \\
\rho(t) & & & t & & & & & & \\
 & & & -1 & t & & & & & \\
 & & & & -1 & t & & & & \\
 & & & & & 1 & t & & & \\
 & & & & & & -1 & t & & \\
 & & & & & & & -1 & t & \\
 & & & & & & & & -1 & t
\end{array}\right)
\end{array}
$$

Where $\rho(t) = t^6 + t^5 + t^4 + t^3 - t - 1$.

$$
\det(f_*|_{B_1} + tI) \;=\; t^7(t^3 + t^2 - 1) - \rho(t).
$$

Lehmer's polynomial

$$\det(f_*|_{B_1} + tI) \;=\; t^7(t^3 + t^2 - 1) - \rho(t)$$
$$= t^{10} + t^9 - t^7 - t^6 - t^5 - t^4 - t^3 + t + 1.$$

We obtained the Lehmer's polynomial.

The Lehmer's polynomial is $\chi_7(t)/(t-1)$.

Hence, the characteristic polynomial of $f_*|_{B_1}$ is given by

$$\det(f_*|_{B_1} - zI) \;=\; \chi_7(-z)/(-z-1).$$

Case of $\mathcal{V}_n\Gamma_1$

Similarly, in the case of $\mathcal{V}_n\Gamma_1$ for $n \geq 7$, we get

$$\det(f_*|_{B_1} - zI) \;=\; \chi_n(-z)/(-z-1).$$

Recall that

$$\chi_n(z) \;=\; z^{n+1}(z^3 - z - 1) + z^3 + z^2 - 1.$$

Let

$$\phi_n(z) \;=\; \det(zI - f_*|_{B_1}) \;=\; (-1)^n\chi_n(-z)/(z+1)$$

$$= \frac{1}{z+1}\{z^{n+1}(z^3 - z + 1) - (-1)^n(z^3 - z^2 + 1)\}.$$

Recall that $\nu_0, \nu_1, \cdots, \nu_{n+3}$ denote the roots of $\chi_n(z)$, with $\nu_0 = 1$, and $\nu_1 = \lambda_n > 1$.

$$\tau_k = \mathrm{trace}(F^{k*}|_{H^2}) \ = \ \sum_{i=0}^{n+3} \nu^k, \quad k = 1, 2, \cdots .$$

$$\Lambda_{F^k} = 2 + \tau_k.$$

gives the number of fixed points of F^k.

The topological entropy of F is $\log \lambda_n$.

The roots of $\phi_n(z)$ are given by $-\nu_1, -\nu_2, \cdots, -\nu_{n+3}$. Hence we have

$$\Lambda_{f^k} \ = \ 1 - (-1)^k(\tau_k - 1).$$

The topological entropy of f is same as F.

We got a theorem.

THEOREM 1.

In the case of $\mathcal{V}_n\Gamma_1$, with $n \geq 7$,

$$h_{top}(F) = h_{top}(f).$$

All the saddle periodic points of F are in the real slice $\mathcal{R}$.

REMARK This theorem is stated in papers [BK] and [DK]. But there are gaps in their proofs.

$\mathcal{V}_n\Gamma_1$ map $F : \mathcal{S} \to \mathcal{S}$

By direct calculations, we see that F has two fixed points : a source and a non-flip saddle in $\mathcal{R}$.

F has one cycle of sinks of period two in $\mathcal{S} \setminus \mathcal{R}$.

There are no other sources, since $0 < \det Df^m(z_0) = \lambda_n^{-m} < 1$ for m-periodic point $z_0 \in \mathcal{S} \setminus C$.

From the trace formula, we can say the followings.

There are no other sinks. Because if there exists a periodic sink of period $p \geq 3$, we would have $\Lambda_{f^{2p}} \geq p - \tau_{2p}$, but $\Lambda_{f^{2p}} = 2 - \tau_{2p}$ holds.

All periodic points, except the 2-cycle, are in $\mathcal{R}$. Because $\Lambda_{f^{2k+1}} = \tau_{2k+1} = \Lambda_{F^{2k+1}} - 2$, and $\Lambda_{f^{2k}} = 2 - \tau_k = 4 - \Lambda_{F^{2k}}$.

All periodic points of period $p \geq 3$ are saddles. Because they must have negative index for f^{2p}.

Periodic saddles of odd prime period are bi-flip saddles.

Periodic saddles of even prime period are non-flip saddles.

Periodic orbits of $\mathcal{V}_7$ maps

k	τ_k	Λ_{F^k}	F-cycles	Λ_{f^k}	f-cycles
1	0	2	1, 1	0	
2	2	4	1, 1, 2	0	
3	3	5	1, 1, 3	3	3
4	2	4	1, 1, 2	0	
5	5	7	1, 1, 5	5	5
6	5	7	1, 1, 2, 3	−3	−3
7	7	9	1, 1, 7	7	7
8	2	4	1, 1, 2	0	
9	3	5	1, 1, 3	3	3
10	7	9	1, 1, 2, 5	−5	−5
11	11	13	1, 1, 11	11	11
12	5	7	1, 1, 2, 3	−3	−3
13	13	15	1, 1, 13	13	13
14	9	11	1, 1, 2, 7	−7	−7
15	8	10	1, 1, 3, 5	8	3, 5
16	18	20	1, 1, 2, 16	−16	−16

6. Line tangent to a quadric

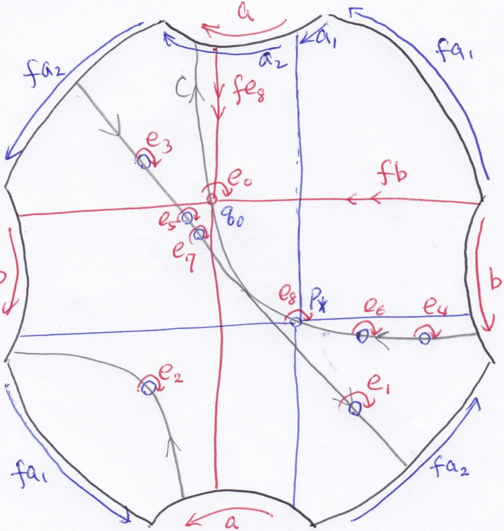
$\mathcal{V}_{2k}\Gamma_2$ case

For $n = 2k$, $k \geq 4$, Γ_2 family gives surface automorphisms with invariant cubic curve, which consists of a line and a quadric tangent to each other at a point. The automorphism $F : \mathcal{S} \to \mathcal{S}$ maps the line and the quadric to each other.

Let $f = F|_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}$ and C denote the invariant cubic curve in $\mathcal{R}$.

Let c be the homology class of C in $H_1(\mathcal{R}, \mathbb{Z})$.
Generators a, b, e_0, e_1, $\cdots$, e_{2k} are defined as in the case of Γ_1.
The class of the line at infinity is denote as ℓ_∞.

$V_8 \Gamma_2 \qquad \alpha = 0.7689\cdots, \quad \beta = 0.1571\cdots$



PROPOSITION.

$$H_1(\mathcal{R}, \mathbb{Z}) \text{ is generated by } c, a, b, e_0, e_1, \cdots, e_{2k},$$

and $2c \sim 0$.

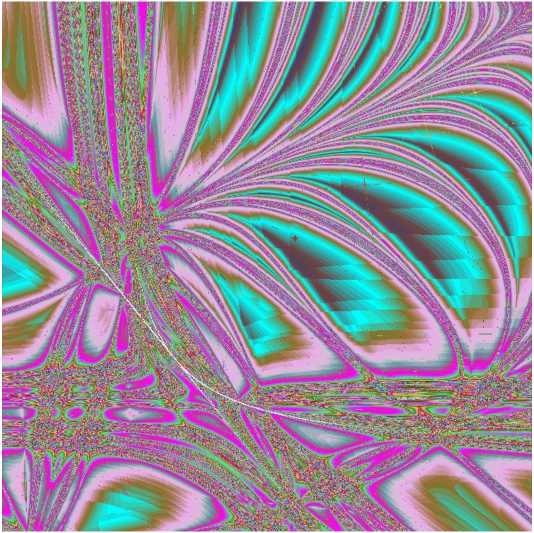
Observing the figure, we have

$$c + e_0 + e_1 + \cdots + e_{2k} + \ell_\infty \sim 0,$$

$$c - e_0 - e_1 - \cdots - e_{2k} - \ell_\infty \sim 0.$$

So, $2c \sim 0$.

Real slice, J^+ and J^-



$\mathcal{V}_{2k}\Gamma_2$ case, $f_* : H_1(\mathcal{R}, \mathbb{Z}) \to H_1(\mathcal{R}, \mathbb{Z})$

$$f_*(a) \sim \ell_\infty \sim c - e_0 - e_1 - \cdots - e_{2k},$$

$$f_*(b) \sim c + a - e_1 - e_2 + e_3 - e_4 - e_5 - e_6 - \cdots - e_{2k}.$$

$$f_*(e_{2k}) \sim c + b - e_1 + e_2 + e_3 - e_4 + e_5 - e_6 + \cdots - e_{2k}.$$

$$f_*(e_0) = e_1, \quad f_*(e_1) = -e_2, \quad f_*(e_2) = -e_3, \quad f_*(e_3) = e_4,$$

$$f_*(e_4) = -e_5, \quad f_*(e_5) = e_6, \quad \cdots, \quad f_*(e_{2k-1}) = e_{2k},$$

Let $B_1 = H_1(\mathcal{R}, \mathbb{Z})/\mathbb{Z}_2 c$. f_* induces an isomorphism of B_1, which is represented by a matrix below (case of $n = 2k = 8$).

$\mathcal{V}_8\Gamma_2$ case, $f_* + tI : B_1 \to B_1$

	a	b	e_8	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
a	t	1									
b		t	1								
e_8	-1	-1	$-1+t$								1
e_0	-1			t							
e_1	-1	-1	-1	1	t						
e_2	-1	-1	1		-1	t					
e_3	-1	1	1			-1	t				
e_4	-1	-1	-1				1	t			
e_5	-1	-1	1					-1	t		
e_6	-1	-1	-1						1	t	
e_7	-1	-1	1							-1	t

Use the first and second lines with columns e_0 to e_6 to simplify the second and third columns.

$\mathcal{V}_8\Gamma_2$ case

	a	b	e_8	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
a	t	1									
b		t	1								
e_8	-1		$-1+t$								1
e_0	-1			t							
e_1	-1			1	t						
e_2					-1	t					
e_3	-1					-1	t				
e_4	-1						1	t			
e_5	-1							-1	t		
e_6	-1								1	t	
e_7	-1									-1	t

Use columns e_0 to e_6 to simplify the first column.

$\mathcal{V}_8\Gamma_2$ case

$$
\begin{array}{c@{\quad}c}
 & \begin{array}{ccccccccccc} a & b & e_8 & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\
\begin{array}{c} a \\ b \\ e_8 \\ e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{array} &
\left(\begin{array}{ccccccccccc}
t & 1 & & & & & & & & & \\
 & t & 1 & & & & & & & & \\
t^3-t^2-1 & & & & & & & & & & 1 \\
\rho(t) & & & t & & & & & & & \\
 & & & 1 & t & & & & & & \\
 & & & & -1 & t & & & & & \\
 & & & & & -1 & t & & & & \\
 & & & & & & 1 & t & & & \\
 & & & & & & & -1 & t & & \\
 & & & & & & & & 1 & t & \\
 & & & & & & & & & -1 & t
\end{array}\right)
\end{array}
$$

Where $\rho(t) = t^7 + t^6 - t^5 - t^4 + t^3 + t - 1$.

$$\det(f_*|_{B_1} + tI) \;=\; t^8(t^3 - t^2 - 1) + \rho(t).$$

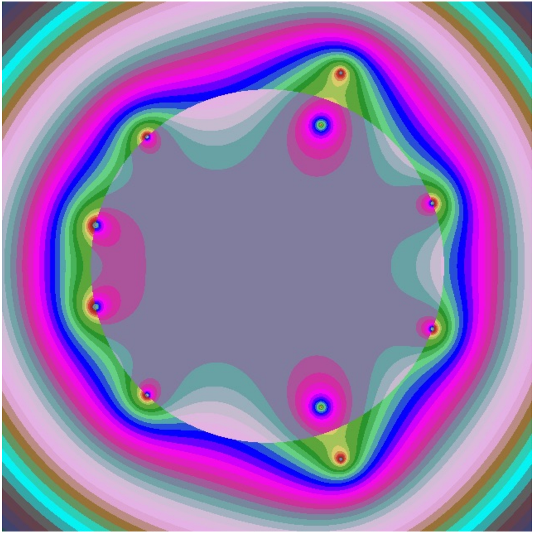
$\mathcal{V}_8\Gamma_2$ case

$$
\begin{aligned}
\det(f_*|_{B_1} + tI) &= t^{11} - t^{10} - t^8 + t^7 + t^6 - t^5 - t^4 + t^3 + t - 1 \\
&= (t-1)(t^{10} - t^7 + t^5 - t^3 + 1).
\end{aligned}
$$

Hence, the characteristic polynomial of $f_*|_{B_1}$ is given by

$$
\det(zI - f_*|_{B_1}) = (z+1)(z^{10} + z^7 - z^5 + z^3 + 1).
$$

$\psi_8(z)/(z+1)$



$\mathcal{V}_8\Gamma_2$ case

The leading eigenvalue, say μ_8, is outside the unit disk with absolute value

$$|\mu_8| \; = \; 1.170042168\cdots$$

which is smaller than the leading eigenvalue of F,

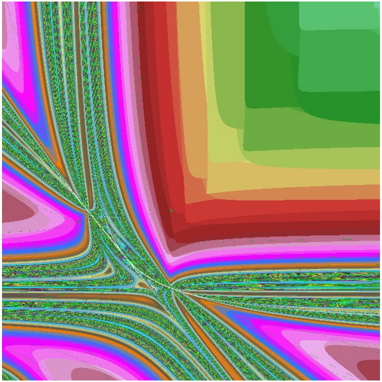
$$\lambda_8 = 1.2303039143\cdots .$$

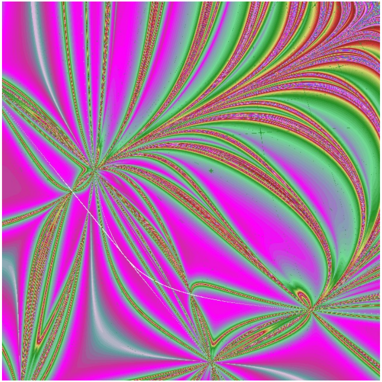
Unfortunately, this does not give a precise estimate of the topological entropy.

$$\log|\mu_8| \leq h_{top}(f) \leq \log \lambda_8 = h_{top}(F).$$

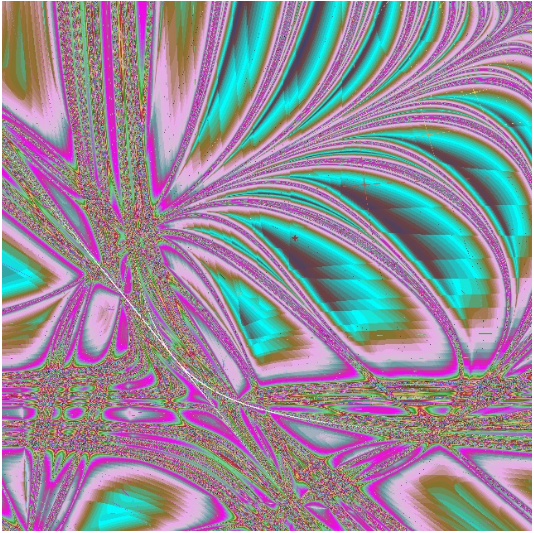
Note that $\lambda_{\mathrm{Lehmer}} = 1.17628081...$

Real slice forward/backward iteration

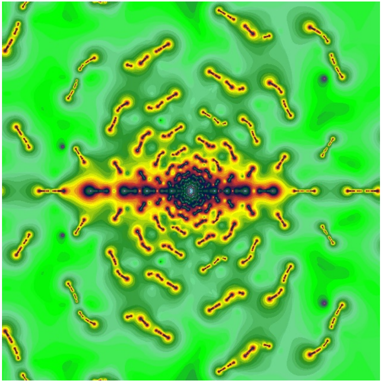


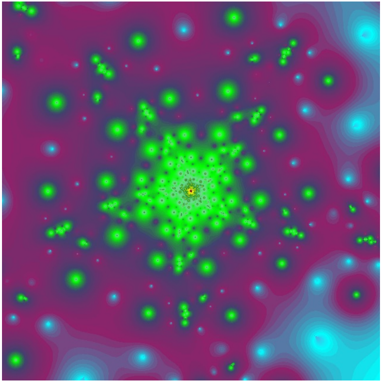


Real slice, J^+ and J^-

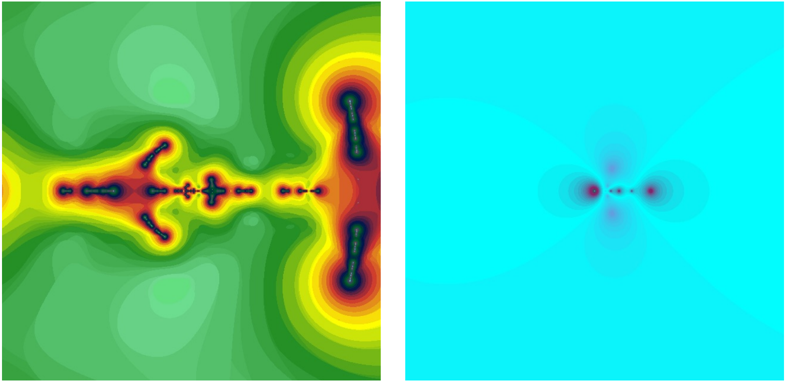


Fixed source and fixed sink

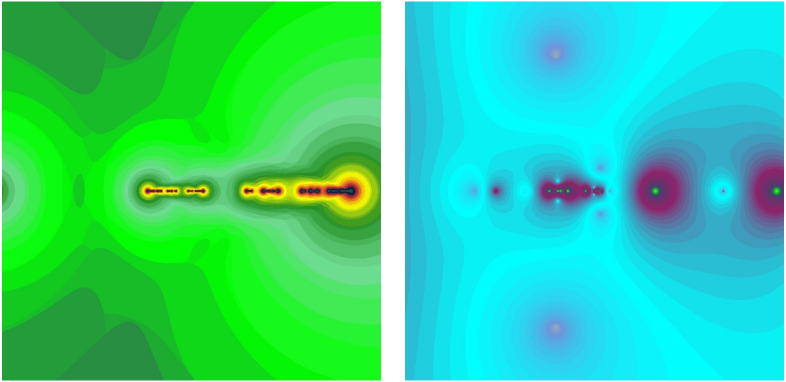




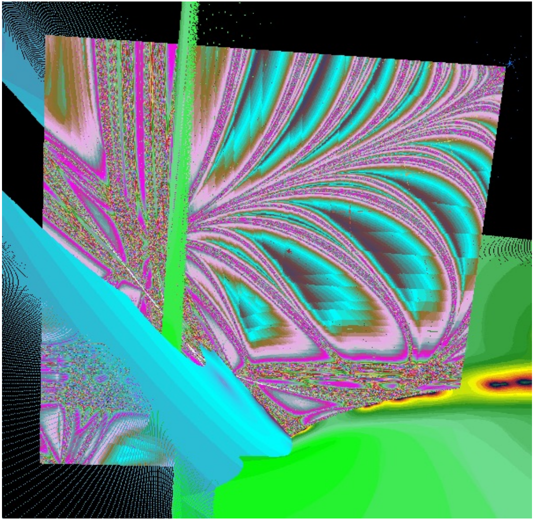
Unstable/stable manifold of per 2 saddle



Unstable/stable manifold of per 3 saddle



Unstable/stable manifold of per 3 saddle



$\mathcal{V}_{2k}\Gamma_2$ case

Similarly, for the case of $\mathcal{V}_{2k}\Gamma_2$, we obtain the characteristic polynomials

$$\psi_{2k}(z) = \det(zI - f_*|_{B_1})$$

as follows.

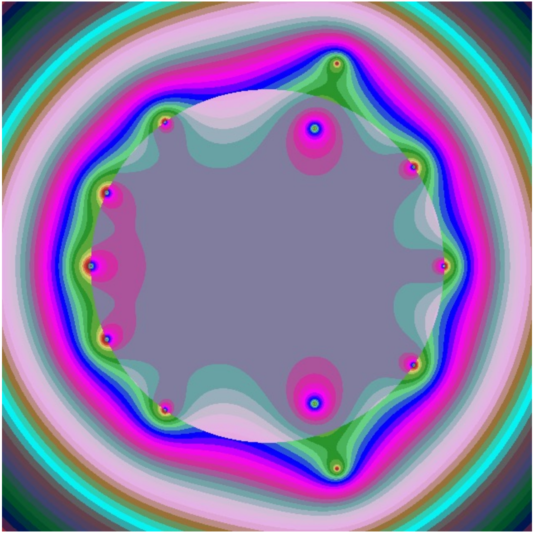
If k is even,

$$\psi_{2k}(z) \;=\; (z+1)(z^{2k+2} + z^{2k-1} - z^{2k-3} + z^{2k-5} - \cdots - z^3 + 1).$$

If k is odd,

$$\psi_{2k}(z) \;=\; (z+1)(z^{2k+2} + z^{2k-1} - z^{2k-3} + z^{2k-5} - \cdots + z^3 - 1).$$

$\psi_{10}(z)/(z+1)$



Periodic orbits of $\mathcal{V}_8$ maps

k	Λ_{F^k}	F-cycles	$\Lambda_{f^k}(\Gamma_1)$	f-cycles(Γ_1)	$\Lambda_{f^k}(\Gamma_2)$	f-cycles(Γ_2)
1	2	$1, 1$	0		2	$1, 1$
2	4	$1, 1, 2$	0		0	
3	5	$1, 1, 3$	3	3	5	$1, 1, 3$
4	4	$1, 1, 2$	0		0	$1, 1, -2$
5	7	$1, 1, 5$	5	5	-3	$1, 1, -5$
6	7	$1, 1, 2, 3$	-3	-3	-3	$1, 1, -2, -3$
7	9	$1, 1, 7$	7	7	9	$1, 1, 7$
8	12	$1, 1, 2, 8$	-8	-8	8	$1, 1, -2, 8$
9	5	$1, 1, 3$	3	3	5	$1, 1, 3$
10	9	$1, 1, 2, 5$	-5	-5	-5	$1, 1, -2, -5$
11	13	$1, 1, 11$	11	11	-9	$1, 1, -11$
12	19	$1, 1, 2, 3, 12$	-15	$-3, -12$	9	$1, 1, -2, -3, 12$
13	15	$1, 1, 13$	13	13	15	$1, 1, 13$
14	25	$1, 1, 2, 7, 14$	-21	$-7, -14$	7	$1, 1, -2, -7, 14$
15	25	$1, 1, 3, 5, 15$	23	$3, 5, 15$	-15	$1, 1, 3, -5, -15$
16	28	$1, 1, 2, 8, 16$	-24	$-8, -16$	-24	$1, 1, -2, -8, -16$
17	36	$1, 1, 17 \times 2$	34	17×2	2	$1, 1, -2, 17, -17$

k	Λ_{F^k}	F-cycles	$\Lambda_{f^k}(\Gamma_2)$	f-cycles(Γ_2)
17	36	$1, 1, 17 \times 2$	2	$1, 1, -2, 17, -17$
34	1160	$1, 1, 2, 17 \times 2,$ 34×33	-407	$1, 1, -2, -17 \times 2,$ $34 \times 23, -34 \times 10$

7. Three lines passing through a point

$\mathcal{V}_{3k}\Gamma_3$ case

For $n = 3k$, $k \geq 3$, Γ_3 family gives surface automorphisms with invariant cubic curve, which consists of three lines passing through a point. The automorphism $F : \mathcal{S} \to \mathcal{S}$ maps each line to another periodically.

Let $f = F|_{\mathcal{R}} : \mathcal{R} \to \mathcal{R}$ and C denote the invariant cubic curve in $\mathcal{R}$.

Let c be the homology class of C in $H_1(\mathcal{R}, \mathbb{Z})$.
Generators a, b, e_0, e_1, $\cdots$, e_{3k} are defined as in the case of Γ_2.
The class of the line at infinity is denote as ℓ_∞.

$H_1(\mathcal{R}, \mathbb{Z})$

PROPOSITION.

$$H_1(\mathcal{R}, \mathbb{Z}) \text{ is generated by } c, a, b, e_0, e_1, \cdots, e_{3k},$$

and $2c \sim 0$.

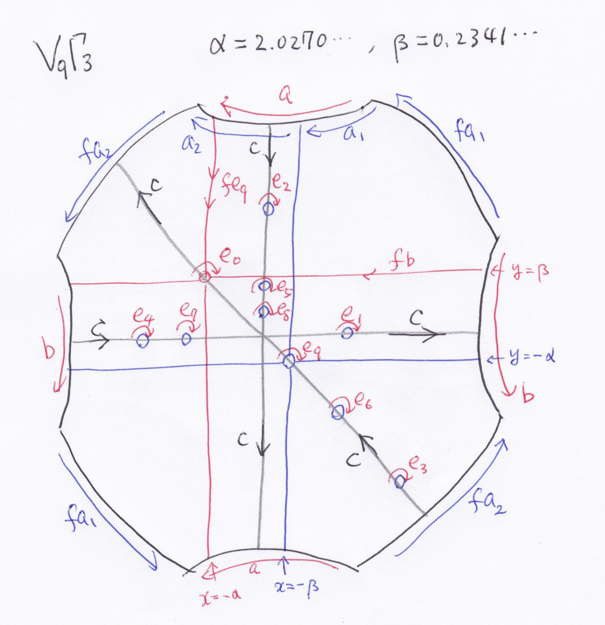
Observing the figure, we have

$$c + e_0 + e_1 + \cdots + e_{3k} + \ell_\infty \sim 0,$$

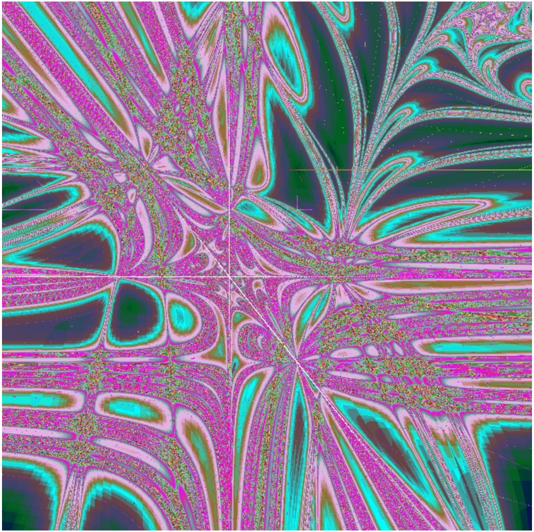
$$c - e_0 - e_1 - \cdots - e_{3k} - \ell_\infty \sim 0.$$

So, $2c \sim 0$.

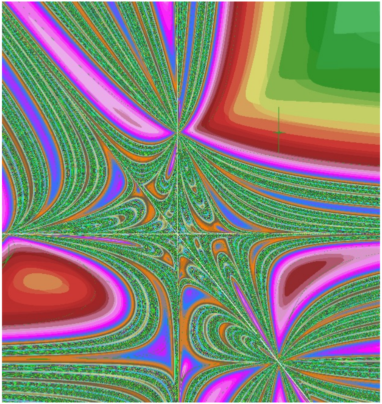
Basis of $H_1(\mathcal{R})$

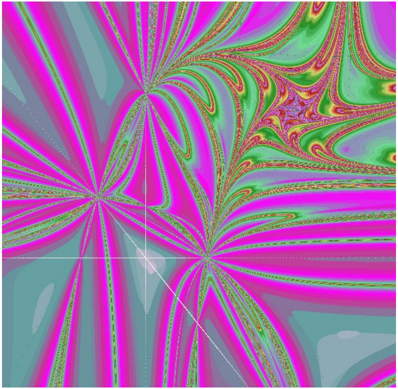


Real slice, J^+ and J^-

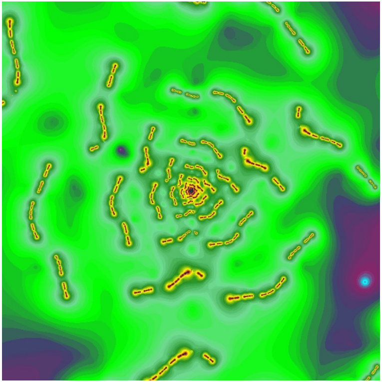


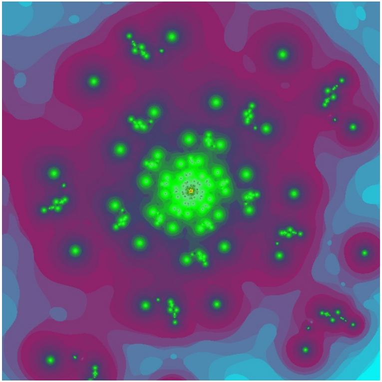
Real slice forward/backward iteration



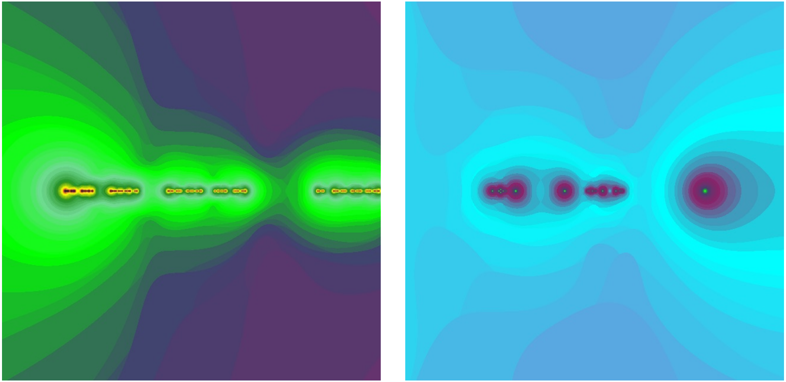


Fixed source and fixed sink

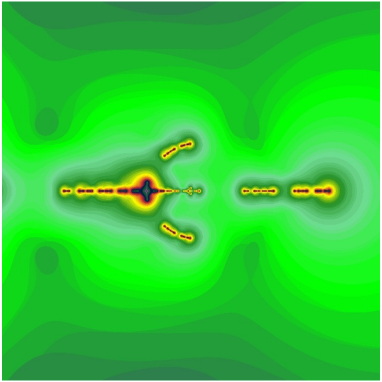


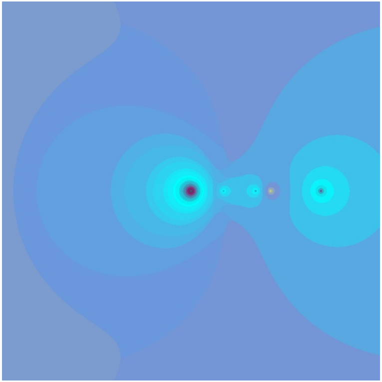


Unstable/stable manifold of per 2 saddle



Unstable/stable manifold of per 3 saddle





$\mathcal{V}_9\Gamma_3$ case, $f_* : H_1(\mathcal{R},\mathbb{Z}) \to H_1(\mathcal{R},\mathbb{Z})$

$$f_*(a) \sim \ell_\infty \sim c - e_0 - e_1 - \cdots - e_{3k},$$

$$f_*(b) \sim -c + a - e_1 + e_2 - e_3 - e_4 - e_5 - e_6 - \cdots - e_{3k}.$$

$$f_*(e_{3k}) \sim -c + b - e_1 - e_2 - e_3 + e_4 - e_5 - e_6 + e_7 - \cdots - e_{3k}.$$

$$f_*(e_0) = e_1, \quad f_*(e_1) = e_2, \quad f_*(e_2) = e_3, \quad f_*(e_3) = -e_4,$$

$$f_*(e_4) = e_5, \quad f_*(e_5) = e_6, \quad f_*(e_6) = -e_7, \quad \cdots \quad , f_*(3k-1) = e_{3k},$$

Let $B_1 = H_1(\mathcal{R},\mathbb{Z})/\mathbb{Z}_2 c$. f_* induces an isomorphism of B_1, which is represented by a matrix below (case of $n = 3k = 9$).

$\mathcal{V}_9\Gamma_3$ case, $f_* + tI : B_1 \to B_1$

	a	b	e_9	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
a	t	1										
b		t	1									
e_9	-1	-1	$-1+t$									1
e_0	-1			t								
e_1	-1	-1	-1	1	t							
e_2	-1	1	-1		1	t						
e_3	-1	-1	-1			1	t					
e_4	-1	-1	1				-1	t				
e_5	-1	-1	-1					1	t			
e_6	-1	-1	-1						1	t		
e_7	-1	-1	1							-1	t	
e_8	-1	-1	-1								1	t

Use the first and second lines with columns e_0 to e_7 to simplify the second and third columns.

$\mathcal{V}_9\Gamma_3$

	a	b	e_9	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
a	t	1										
b		t	1									
e_9	-1	-2	$-1+t$									1
e_0	-1			t								
e_1	-1			1	t							
e_2	-2				1	t						
e_3	-1					1	t					
e_4	1						-1	t				
e_5	-1							1	t			
e_6	-1								1	t		
e_7	1									-1	t	
e_8	-1										1	t

Use columns e_0 to e_7 to simplify the first column.

$\mathcal{V}_9\Gamma_3$

$$
\begin{array}{c|cccccccccccc}
 & a & b & e_9 & e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\
\hline
a & t & 1 & & & & & & & & & & \\
b & & t & 1 & & & & & & & & & \\
e_9 & -1 & -2 & -1+t & & & & & & & & & 1 \\
e_0 & \rho(t) & & & t & & & & & & & & \\
e_1 & & & & 1 & t & & & & & & & \\
e_2 & & & & & 1 & t & & & & & & \\
e_3 & & & & & & 1 & t & & & & & \\
e_4 & & & & & & & -1 & t & & & & \\
e_5 & & & & & & & & 1 & t & & & \\
e_6 & & & & & & & & & 1 & t & & \\
e_7 & & & & & & & & & & -1 & t & \\
e_8 & & & & & & & & & & & 1 & t
\end{array}
$$

where

$$\rho(t) = -t^8 - t^7 + t^6 - t^5 - t^4 + t^3 - 2t^2 + t - 1.$$

$\mathcal{V}_9\Gamma_3$

So, we get

$$\det(f_*|_{B_1} + tI) \;=\; t^9(t^3 - t^2 + 2t - 1) - \rho(t)$$

$$= t^{12} - t^{11} + 2t^{10} - t^9 + t^8 + t^7 - t^6 + t^5 + t^4 - t^3 + 2t^2 - t + 1$$

$$= (t^2 - t + 1)(t^{10} + t^8 + t^5 + t^2 + 1)$$

We prefer the characteristic polynomial of $f_*|_{B_1}$ in the form

$$\varphi_9(z) = (z^2 + z + 1)(z^{10} + z^8 - z^5 + z^2 + 1).$$

Compare this with

$$\chi_9(z) = (z^3 - 1)(z^{10} - z^8 - z^5 - z^2 + 1),$$

$$\phi_9(z) = (z^2 - z + 1)(z^{10} - z^8 + z^5 - z^2 + 1).$$

$\mathcal{V}_{3k}\Gamma_3$ case

Similarly, for the case of $\mathcal{V}_{3k}\Gamma_3$, we obtain the characteristic polynomials

$$\varphi_{3k}(z) = \det(zI - f_*|_{B_1})$$

as follows.

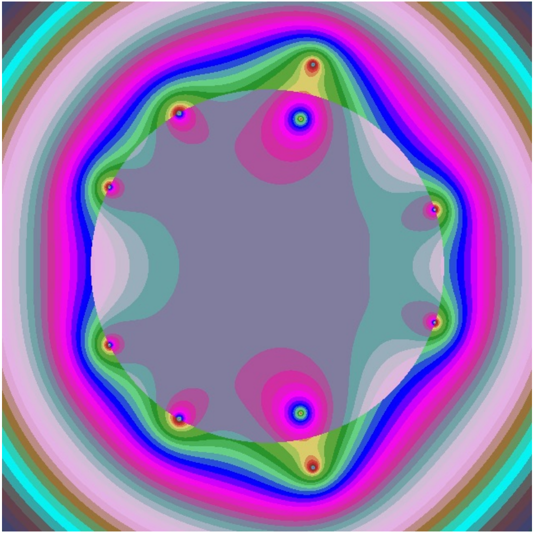
If k is even,

$$\varphi_{3k}(z) = (z^2 + z + 1)(z^{3k+1} + z^{3k-1} - z^{3k-4} + \cdots - z^2 - 1).$$

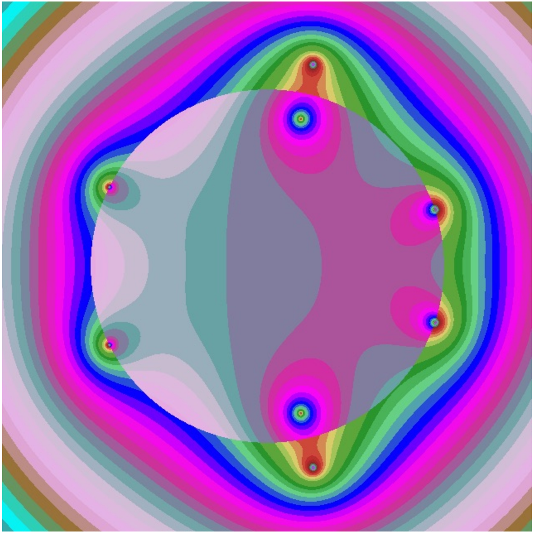
If k is odd,

$$\varphi_{3k}(z) = (z^2 + z + 1)(z^{3k+1} + z^{3k-1} - z^{3k-4} + \cdots + z^2 + 1).$$

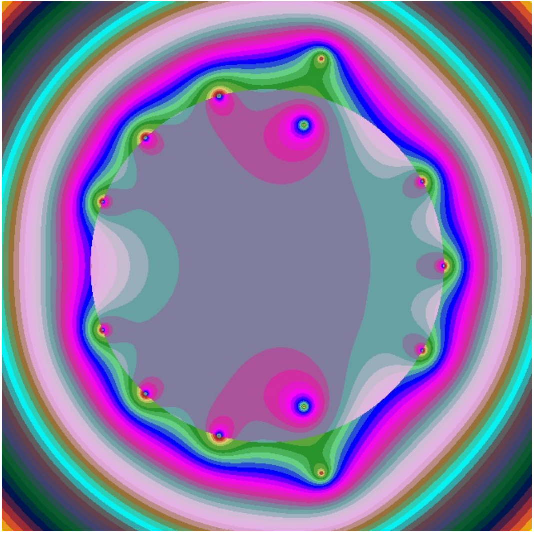
$\mathcal{V}_9\Gamma_3$ case, $\varphi_9(z)/(z^2+z+1)$



$\mathcal{V}_9\Gamma_3$ case, $\varphi_9(z)/(z^2+z+1)^2$



$\mathcal{V}_{12}\Gamma_3$ case, $\varphi_{12}(z)/(z^2+z+1)$



8. Complex Salem number

Complex Salem number

A **complex Salem number** is a non-real algebraic integer τ, $|\tau| > 1$, which is Galois conjugate to τ^{-1}, and all of whose Galois conjugates, excluding τ^{-1}, $\bar{\tau}$ and $\bar{\tau}^{-1}$, lie on $|z| = 1$.

A **complex Salem polynomial** is the minimal polynomial of a complex Salem number.

A **complex Pisot number** is a non-real algebraic integer λ, $|\lambda| > 1$, whose Galois conjugates, excluding λ and $\bar{\lambda}$, satisfy $|\lambda'| < 1$.

A **complex Pisot polynomial** is the minimal polynomial of a complex Pisot number.

Complex Pisot polynomial and complex Salem number

Proposition

If $P(z)$ is a complex Pisot polynomial, then for $n > \deg P$, the polynomial

$$S_{n,P,\pm 1}(z) = z^n P(z) \pm z^{\deg P} P(z^{-1})$$

has at least $(\deg(S_{n,P,\pm 1}) - 4)$ roots on $|z| = 1$.

PROOF

Let $\alpha_0, \bar{\alpha_0}, \alpha_1, \bar{\alpha}_1, \cdots, \alpha_\ell, \bar{\alpha_\ell}, \beta_1, \cdots, \beta_m$ be the roots of $P(z)$, with

$$|\alpha_0| > 1, \ \ |\alpha_i| < 1, \ \ (1 \le i \le \ell), \quad \beta_j \in \mathbb{R}, \ |\beta_j| < 1, \ \ (1 \le j \le m).$$

Then,

$$P(z) = (z - \alpha_0)(z - \bar{\alpha}_0) \prod_{i=1}^{\ell} (z - \alpha_i)(z - \bar{\alpha}_i) \ \prod_{j=1}^{m} (z - \beta_j).$$

The reciprocal polynomial $P^*(z) = z^{\deg P} P(z^{-1})$ is

$$P^*(z) = (1 - \alpha_0 z)(1 - \bar{\alpha}_0 z) \prod_{i=1}^{\ell} (1 - \alpha_i z)(1 - \bar{\alpha}_i z) \ \prod_{j=1}^{m} (1 - \beta_j z).$$

So,

$$z^n \frac{P(z)}{P^*(z)} = z^n \frac{z-\alpha_0}{1-\bar{\alpha}_0 z} \frac{z-\bar{\alpha}_0}{1-\alpha_0 z} \prod_{i=1}^{\ell} \frac{z-\alpha_i}{1-\bar{\alpha}_i z} \frac{z-\bar{\alpha}_i}{1-\alpha_i z} \prod_{j=1}^{m} \frac{z-\beta_j}{1-\beta_j z}$$

is a finite Blaschke product, and maps the unit circle into itself with mapping degree $n + \deg P - 4$.

Hence, $z^n \frac{P(z)}{P^*(z)} = \mp 1$ has at least $\deg(S_{n,P,\pm 1}) - 4$ solutions on $|z| = 1$.

Sufficient condition

PROPOSITION

There exists an integer n_0, such that for any integer $n \geq n_0$, $S_{n,P,\pm 1}(z)$ has a root γ_n with $|\gamma_n| > 1$.

PROOF

Consider equation

$$z^n = \mp \frac{P^*(z)}{P(z)},$$

whose solutions give roots of the polynomial $S_{n,P,\pm 1}(z)$.

The modulus of the derivative of the right-hand side is bounded on the unit circle. Take an integer n_0 satisfying

$$n_0 \geq \sup_{|z|=1} \left| \frac{d}{dz} \left(\frac{P^*(z)}{P(z)} \right) \right|.$$

Then $S_{n,P,\pm 1}(z)$ has exactly $\deg(S_{n,P,\pm 1}) - 4$ roots on $|z| = 1$. Half of the rest of roots are of modulus greater than 1.

Factorization of $\psi_{2k}(z)$

THEOREM 2.

Except possibly for small values of k, the characteristic polynomial (case of $\mathcal{V}_{2k}\Gamma_2$)

$$\psi_{2k}(z) \;=\; \det(zI - f_*|_{B_1})$$

factors as

$$\psi_{2k}(z) \;=\; D_{2k}(z)\,T_{2k}(z),$$

where $D_{2k}(z)$ is a product of cyclotomic polynomials and $T_{2k}(z)$ is a complex Salem polynomial.

REMARK

Numerically, this holds for $k \geq 4$.

Proof

First, we compute the characteristic polynomial $\psi_{2k}(z)$ as follows.

$$\psi_{2k}(z) = \det(zI - f_*|_{B_1})$$

$$= (z+1)(z^{2k+2} + z^{2k-1} - z^{2k-3} + z^{2k-5} - \cdots + (-1)^{k-4}z^3 + (-1)^k)$$

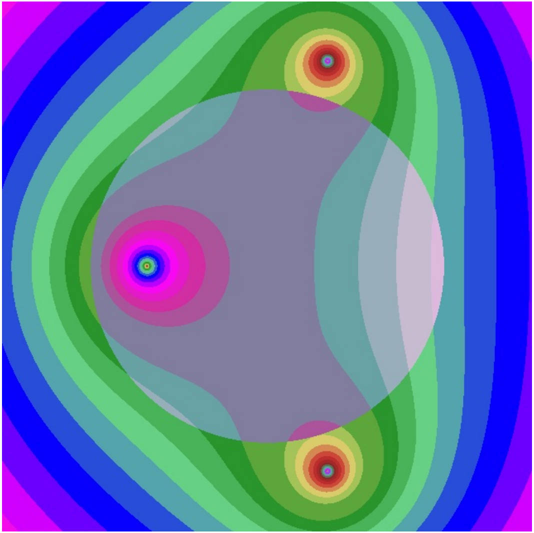
$$= (z+1)(z^{2k+2} + (-1)^k - (-1)^k z \sum_{j=1}^{k-1}(-z^2)^j)$$

$$= (z+1)(z^{2k+2} + (-1)^k - (-1)^k z \frac{-z^2 - (-z^2)^k}{z^2+1})$$

$$= \frac{z+1}{z^2+1}(z^{2k+1}(z^3+z+1) + (-1)^k(z^3+z^2+1)).$$

From the third line, we see that $\psi_{2k}(\pm i) \neq 0$.

$P(z) = z^3 + z + 1$



As $z^3 + z + 1$ has a real root $-1 < \beta_1 < 0$, and a pair of complex conjugate roots α_0 and $\bar{\alpha}_0$ of modulus greater than 1, $P(z) = z^3 + z + 1$ is a complex Pisot polynomial.

There exists an integer n_0, such that for all $2k + 1 \geq n_0$, $S_{2k+1,P,(-1)^k}(z)$ has exactly $\deg(S_{2k+1,P,(-1)^k}) - 4$ roots on $|z| = 1$.

From these roots, we exclude $\pm i$, and together with $z = -1$, ψ_{2k} has exactly $2k - 1$ roots on $|z| = 1$. As $\deg(\psi_{2k}(z)) = 2k + 3$, we conclude that $\psi_{2k}(z)$ has four non-unitary roots.

As $S'_{2k+1,P,(-1)^k}(\pm 1) \neq 0$, $z = \pm 1$ is at most simple root.

To complete the proof, we show that $S_{2k+1,P,(-1)^k}(z)$ has no real roots with $|z| \neq 1$.

If t is a root of $S_{2k+1,P,(-1)^k}(z)$, then t^{-1} is a root, too.

If $t > 1$, then

$$t^{2k+1}(t^3 + t + 1) > t^3 + t^2 + 1,$$

which implies

$$S_{2k+1,P,(-1)^k}(t) > 0.$$

If $t < -1$ then,
if k is even,

$$t^{2k+2} + t^{2k-1} > 0, \ -t^{2k-3} + t^{2k-5} > 0, \cdots, -z^5 + z^3 > 0, \ 1 > 0,$$

and if k is odd,

$$t^{2k+2} + t^{2k-1} > 0, \ -t^{2k-3} + t^{2k-5} > 0, \cdots, \ -z^3 - 1 > 0,$$

which implies $(t^2+1)S_{2k+1,P,1}(t) > 0$. Theorem 2 is proved.

Factorization of $\varphi_{3k}(z)$

THEOREM 3.

Except possibly for small values of k, the characteristic polynomial (case of $\mathcal{V}_{3k}\Gamma_3$)

$$\varphi_{3k}(z) \;=\; \det(zI - f_*|_{B_1})$$

factors as

$$\varphi_{3k}(z) \;=\; E_{3k}(z)U_{3k}(z),$$

where $E_{2k}(z)$ is a product of cyclotomic polynomials and $U_{2k}(z)$ is a complex Salem polynomial.

REMARK

Numerically this holds for $k \geq 3$.

Proof

The proof is similar to that for $\psi_{2k}(z)$.

$$\varphi_{3k}(z) = \frac{z^2+z+1}{z^3+1}(z^{3k+1}(z^3+z+1)-(-1)^k(z^3+z^2+1)).$$

Complex Pisot polynomial $P(z) = z^3+z+1$ is same as before.
The cubic roots of -1 are not roots of $\varphi_{3k}(z)$.

The polynomial $S_{3k+1,P,(-1)^{k+1}}(z)$ has at least $3k$ roots on $|z|=1$. For sufficiently large k, the number of unitary roots of $\varphi_{3k}(z)$ is exactly $3k-1$. (Numerically, this holds for $k \geq 3$.)
As $S'_{3k+1,P,(-1)^{k+1}}(\pm 1) \neq 0$, $z = \pm 1$ is at most simple root.

Next, we show that $\varphi_{3k}(z)$ has no real roots with $|z| \neq 1$.

If t is a root of $S_{3k+1,P,(-1)^{k+1}}(z)$, then t^{-1} is a root, too.

If $t > 1$, then

$$t^{3k+1}(t^3 + t + 1) > t^3 + t^2 + 1 > 0.$$

which implies $S_{3k+1,P,(-1)^{k+1}}(t) > 0$.

In the case of $t < -1$, we have two cases.

If k is even, recall that in this case,

$$\varphi_{3k}(z) \;=\; (z^2 + z + 1)(z^{3k+1} + z^{3k-1} - z^{3k-4} + \cdots - z^2 - 1).$$

As all terms in the second factor are negative, we see that $S_{3k+1,P,(-1)^{k+1}}(t) < 0$. Hence $\varphi_{3k}(t) < 0$.

If k is odd, recall that in this case,

$$\varphi_{3k}(z) \;=\; (z^2 + z + 1)(z^{3k+1} + z^{3k-1} - z^{3k-4} + \cdots + z^2 + 1).$$

As all terms in the second factor are positive, we see that $S_{3k+1,P,(-1)^{k+1}}(t) > 0$. Hence $\varphi_{3k}(t) > 0$.

These complete the proof.

Summary of characteristic polynomials

$\mathcal{V}_n$, $F : \mathcal{S} \to \mathcal{S}$. (Salem)

$$\chi_n(z) = z^{n+1}(z^3 - z - 1) + z^3 + z^2 - 1.$$

$\mathcal{V}_n\Gamma_1$, $f : \mathcal{R} \to \mathcal{R}$. (negative Salem)

$$\phi_n(z) = \frac{1}{z+1}\{z^{n+1}(z^3 - z + 1) - (-1)^n(z^3 - z^2 + 1)\}.$$

$\mathcal{V}_{2k}\Gamma_2$, $f : \mathcal{R} \to \mathcal{R}$. (complex Salem)

$$\psi_{2k}(z) = \frac{z+1}{z^2+1}\{z^{2k+1}(z^3 + z + 1) + (-1)^k(z^3 + z^2 + 1)\}.$$

$\mathcal{V}_{3k}\Gamma_3$, $f : \mathcal{R} \to \mathcal{R}$. (complex Salem)

$$\varphi_{3k}(z) = \frac{z^2+z+1}{z^3+1}\{z^{3k+1}(z^3 + z + 1) - (-1)^k(z^3 + z^2 + 1)\}.$$

Remark

If all the roots of the characteristic polynomial are on the unit circle, then all the eigenvalues are roots of unity.

In this case, the operation on the cohomology is periodic.

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[DK] Jeffrey Diller and Kyounghee Kim, Entropy of real rational surface automorphisms, arXiv:1710.07665v1,2017.