SEMIPOSITIVITY THEOREMS FOR MODULI PROBLEMS

OSAMU FUJINO

Abstract. We prove some semipositivity theorems for singular varieties coming from graded polarizable admissible variations of mixed Hodge structure. As an application, we obtain that the moduli functor of stable varieties is semipositive in the sense of Kollár. This completes Kollár's projectivity criterion for the moduli spaces of higher-dimensional stable varieties.

Contents

1. Introduction

One of the main purposes of this paper is to give a proof of the following "folklore statement" (see, for example, [A2], [Kr], [Kv], [AH], [K5], [F3]) based on [FF] and [F4]. In general, the (quasi-) projectivity of some moduli space is a subtle problem and is harder than it looks (see, for example, [ST], [K2], [V3], and so on).

Theorem 1.1 (Projectivity of moduli spaces of stable varieties)**.** *Every complete subspace of a coarse moduli space of stable varieties is* projective*.*

To the best knowledge of the author, Theorem 1.1 is new for stable n-folds with $n \geq 3$ (see the comments in 1.9 below). Note that a

Date: 2012/10/29, version 1.61.
2010 *Mathematics Subject Classification.* Primary 14J10; Secondary 14E30.
Key words and phrases. semipositivity theorems, mixed Hodge structures, semi log canonical pairs, moduli spaces.

stable n-fold is an n-dimensional projective semi log canonical variety with ample canonical divisor and is called an *n-dimensional semi log canonical model* in [K5]. For the details of semi log canonical varieties, see, for example, [F4] and [K7]. We also note that the coarse moduli space of stable varieties is first constructed in the category of algebraic spaces (cf. [KM]). As a corollary of Theorem 1.1, by using the boundedness result obtained through works of Tsuji, Hacon–McKernan and Takayama (cf. [HM], [T]), we have:

Corollary 1.2 (cf. [K1, 5.6. Corollary], [Kr, Corollary 1.2], and [AH, Proposition 6.1.2]). *The moduli functor $\mathcal{M}^{sm}_H$ of smoothable stable varieties with Hilbert function H is coarsely represented by a* projective *algebraic scheme.*

For the precise statement of the boundedness, see, for example, [HK, Theorem 13.6]. See also the definition of *smoothable* stable varieties in Definition 5.1 below, which is more general than [Kr, Definition 2.8]. Therefore, we obtain:

Corollary 1.3 (cf. [V2, Theorem 1.11]). *The moduli functor $\mathcal{M}_H$ of canonically polarized smooth projective varieties with Hilbert function H is coarsely represented by a* quasi-projective *algebraic scheme.*

More generally, we have:

Corollary 1.4. *The moduli functor $\mathcal{M}^{can}_H$ of canonically polarized normal projective varieties having only canonical singularities with Hilbert function H is coarsely represented by a* quasi-projective *algebraic scheme.*

Theorem 1.1 is a direct consequence of Theorem 1.5 below by Kollár's projectivity criterion (cf. [K1, Sections 2 and 3]).

Theorem 1.5 (Semipositivity of $\mathcal{M}^{stable}$). *Let $\mathcal{M}^{stable}$ be the moduli functor of stable varieties. Then $\mathcal{M}^{stable}$ is semipositive in the sense of Kollár.*

For the reader's convenience, let us recall the definition of the semipositivity of $\mathcal{M}^{stable}$, which is a special case of [K1, 2.4. Definition].

Definition 1.6 (cf. [K1, 2.4. Definition]). The moduli functor $\mathcal{M}^{stable}$ of stable varieties is said to be *semipositive* (*in the sense of Kollár*) if the following condition holds:

There is a fixed m_0 such that if C is a smooth projective curve and $(f : X \to C) \in \mathcal{M}^{stable}(C)$, then $f_*\omega^{[m]}_{X/C}$ is a semipositive locally free sheaf on C for every $m \geq m_0$.

As the culmination of the works of several authors, we have:

Corollary 1.7. *If the moduli functor $\mathcal{M}_H^{stable}$ of stable varieties with Hilbert function H is bounded, then $\mathcal{M}_H^{stable}$ is coarsely represented by a* projective *algebraic scheme.*

We think that the boundedness of the moduli functor $\mathcal{M}_H^{stable}$ will be established in [HMX]. Note that the boundedness of the moduli functor $\mathcal{M}_H^{stable}$ for stable surfaces holds true (see [A1] and [AM]).

Theorem 1.5 follows almost directly from the definition of the semipositivity of $\mathcal{M}^{stable}$ in Definition 1.6 (cf. [K1, 2.4. Definition]) and the main semipositivity theorem of this paper:

Theorem 1.8 (Main Theorem)**.** *Let X be an equi-dimensional variety which satisfies Serre's S_2 condition and is Gorenstein in codimension one. Let $f : X \to C$ be a projective surjective morphism onto a smooth projective curve C such that every irreducible component of X is dominant onto C. Assume that there exists a non-empty Zariski open set U of C such that $f^{-1}(U)$ has only semi log canonical singularities. Then $f_*\omega_{X/C}$ is semipositive.*

Assume further that $\omega_{X/C}^{[k]}$ is locally free and f-generated for some positive integer k. Then $f_*\omega_{X/C}^{[m]}$ is semipositive for every $m \geq 1$.

1.9 (Comments)**.** Theorem 1.8 is a reformulation of [K1, 4.12. Theorem]. Kollár has pointed out that the assumption that the fibers are surfaces was inadvertently omitted from its statement. He is really claiming [K1, 4.12. Theorem] for $f : Z \to C$ with $\dim Z = 3$ (see [K1, 1. Introduction]). Therefore, Theorem 1.8 is new when $\dim X \geq 4$. Likewise, Theorem 1.5 and Theorem 1.1 are new when the dimension of the stable varieties are greater than or equal to three.

We feel that the arguments in [K1, 4.14] only work when the fibers are surfaces. In other words, we needed some new ideas and techniques to prove Theorem 1.8. Our arguments heavily depend on the recent advances on the semipositivity theorems of Hodge bundles ([FF]) and the construction of quasi-log resolutions for quasi-projective semi log canonical pairs (cf. [F4]).

Remark 1.10. In general, we have to prove Theorem 1.8 for non-normal (reducible) varieties X in order to see that some moduli functor is semipositive in the sense of Kollár even if we are only interested in the moduli spaces of *smoothable* stable varieties. Roughly speaking, if the curve C is contained in $M_H^{sm} \setminus M_H^{can}$, where M_H^{sm} (resp. M_H^{can}) is the coarse moduli space of $\mathcal{M}_H^{sm}$ (resp. $\mathcal{M}_H^{can}$), then a general fiber of $f : X \to C$ may be non-normal and reducible.

For the general theory of Kollár's projectivity criterion, see [K1, Sections 2 and 3] and [V2, Theorem 4.34]. We do not discuss the technical details of the construction of moduli spaces of stable varieties in this paper. We mainly treat various semipositivity theorems. Note that the projectivity criterion discussed here is independent of the existence problem of moduli spaces. We recommend the reader to see [K1, Section 2] and [K5, Section 5] for Kollár's program for constructing moduli spaces of stable varieties. Our paper is related to the topic in [K5, **41** (Projectivity)].

In this paper, we prove Theorem 1.8 in the framework of [F4] and [FF]. Note that [F4] and [FF] heavily depend on the theory of *mixed* Hodge structures on cohomology groups with compact support. A key ingredient of this paper is the following semipositivity theorem, which is essentially contained in [FF]. It is a generalization of Fujita's semipositivity theorem (see [Ft, (0.6) Main Theorem]). We note that a Hodge theoretic approach to the original Fujita semipositivity theorem was introduced by Zucker (see [Z]).

Theorem 1.11 (cf. [FF, Section 5])**.** *Let (X, D) be a simple normal crossing pair such that D is reduced. Let $f : X \to C$ be a projective surjective morphism onto a smooth projective curve C. Assume that every stratum of X is dominant onto C. Then $f_*\omega_{X/C}(D)$ is semipositive.*

Although we do not know what is the best formulation of the semipositivity theorem for moduli problems, we think Theorem 1.11 will be one of the most fundamental results for application to Kollár's projectivity criterion for moduli spaces. We prove Theorem 1.8 by using Theorem 1.11. By the same proof as that of Theorem 1.8, we obtain a generalization of Theorem 1.8, which contains both Theorem 1.8 and Theorem 1.11.

Theorem 1.12. *Let X be an equi-dimensional variety which satisfies Serre's S_2 condition and is Gorenstein in codimension one. Let $f : X \to C$ be a projective surjective morphism onto a smooth projective curve C such that every irreducible component of X is dominant onto C. Let D be a reduced Weil divisor on X such that no irreducible component of D is contained in the singular locus of X. Assume that there exists a non-empty Zariski open set U of C such that $(f^{-1}(U), D|_{f^{-1}(U)})$ is a semi log canonical pair. Then $f_*\omega_{X/C}(D)$ is semipositive.*

We further assume that $\mathcal{O}_X(k(K_X+D))$ is locally free and f-generated for some positive integer k. Then $f_*\mathcal{O}_X(m(K_{X/C}+D))$ is semipositive for every $m \geq 1$.

By combining Theorem 1.12 with Viehweg's covering trick, we obtain the following theorem: Theorem 1.13, which is an answer to the question in [A2, 5.6]. Although we do not discuss the moduli spaces of *stable pairs* here, Theorem 1.12 and Theorem 1.13 play important roles in the proof of the projectivity of the moduli spaces of stable pairs (see [A2], [FP], and 4.4 below).

Theorem 1.13. *Let X be an equi-dimensional variety which satisfies Serre's S_2 condition and is Gorenstein in codimension one. Let $f : X \to C$ be a projective surjective morphism onto a smooth projective curve C with connected general fibers such that every irreducible component of X is dominant onto C. Let Δ be an effective $\mathbb{Q}$-Weil divisor on X such that no irreducible component of the support of Δ is contained in the singular locus of X. Assume that there exists a non-empty Zariski open set U of C such that $(f^{-1}(U), \Delta|_{f^{-1}(U)})$ is a semi log canonical pair. We further assume that $\mathcal{O}_X(k(K_X + \Delta))$ is locally free and f-generated for some positive integer k. Then $f_*\mathcal{O}_X(k(K_{X/C} + \Delta))$ is semipositive. Therefore, $f_*\mathcal{O}_X(kl(K_{X/C} + \Delta))$ is semipositive for every $l \geq 1$.*

Remark 1.14. In this paper, we do not use algebraic spaces for the proof of the semipositivity theorems. We only treat *projective* varieties. Note that Theorem 1.11 follows from the theory of variations of mixed Hodge structure. The variations of mixed Hodge structure discussed in [FF] are *graded polarizable* and *admissible*. Therefore, we can not directly apply the results in [FF] to the variations of (mixed) Hodge structure arising from families of algebraic spaces. We need some polarization to obtain various semipositivity theorems in our framework. We also note that the admissibility assures us the existence of (canonical) extensions of Hodge bundles, which does not always hold for abstract graded polarizable variations of mixed Hodge structure (see [FF, Example 1.6]).

We do not use the Fujita–Kawamata semipositivity theorem coming from the theory of polarized variations of Hodge structure. We think we need some semipositivity theorems obtained by the theory of graded polarizable admissible variations of mixed Hodge structure for Kollár's projectivity criterion of moduli spaces of stable varieties.

Remark 1.15. (1) As explained in [K1] and [K3], it is difficult to directly check the *quasi-projectivity* of non-complete singular spaces. This is because there is no good ampleness criterion for non-complete spaces. In this paper, we adopt Kollár's framework in [K1, Sections 2 and 3], where we use the Nakai–Moishezon criterion to check the projectivity of complete algebraic spaces. Note that Viehweg discusses the *quasi-projectivity* of *non-complete* moduli spaces (cf. [V2], [V3]). On the other hand, Kollár and we prove the *projectivity* of *complete* moduli spaces (cf. [K1]).

(2) Although we repeatedly use Viehweg's covering arguments, we do not use the notion of *weak positivity*, which was introduced by Viehweg and plays crucial roles in his works (cf. [V1], [V2], [V3]). We just treat the *semipositivity* on smooth projective curves (cf. [K1]).

(3) From the Hodge theoretic viewpoint, our approach is based on the theory of *mixed* Hodge structures (cf. [FF]). The arguments in [V2], [K1], and [V3] use only *pure* Hodge structures. It is one of the main differences between our approach and the others.

We summarize the contents of this paper. In Section 2, we collect some basic definitions. In Section 3, we quickly review the moduli functor $\mathcal{M}^{stable}$ of stable varieties and its coarse moduli space. Section 4 is the main part of this paper, where we prove the theorems in Section 1. Our proofs depend on [FF], [F4], and Viehweg's covering arguments. In Section 5, we prove the corollaries in Section 1.

Acknowledgments. The author was partially supported by the Grant-in-Aid for Young Scientists (A) ♯24684002 from JSPS. He thanks Professor János Kollár for answering his questions and giving him many useful comments and Professor Steven Zucker for giving him useful comments and advice. He also thanks Professors Varely Alexeev, Taro Fujisawa and Sándor Kovács for comments. Finally, he thanks Professor Shigefumi Mori for warm encouragement.

We will work over $\mathbb{C}$, the complex number field, throughout this paper. Note that, by the Lefschetz principle, all the results in this paper hold over any algebraically closed field k of characteristic zero. We will freely use the notation and terminology in [FF] and [F4]. For the standard notations and conventions of the log minimal model program, see [F2].

2. Preliminaries

Let us recall the definition of *semismooth* varieties.

Definition 2.1 ([K1, 4.1. Definition]). An algebraic variety X is called *semismooth* if all of its closed points are analytically (or formally) isomorphic to one of the following:

- a smooth point;
- a double normal crossing point: $x_1x_2 = 0 \in \mathbb{C}^{n+1}$; or

- a pinch point: $x_1^2 - x_2^2 x_3 = 0 \in \mathbb{C}^{n+1}$.

Let us recall the definition of *semipositive locally free sheaves*. For the details, see, for example, [V2, Section 2].

Definition 2.2 (Semipositive locally free sheaves)**.** A locally free sheaf of finite rank $\mathcal{E}$ on a complete variety X is *semipositive* (or *nef*) if the following equivalent conditions are satisfied:

(i) $\mathcal{O}_{\mathbb{P}_X(\mathcal{E})}(1)$ is nef on $\mathbb{P}_X(\mathcal{E})$.
(ii) For every map from a smooth projective curve $f : C \to X$, every quotient line bundle of $f^*\mathcal{E}$ has non-negative degree.

In this paper, we only discuss various semipositivity theorems for locally free sheaves on a smooth projective curve.

The following well-known lemma is very useful. We omit the proof of Lemma 2.3 because it is an easy exercise.

Lemma 2.3. *Let C be a smooth projective curve and let $\mathcal{E}_i$ be a locally free sheaf on C for $i = 1, 2$. Assume that $\mathcal{E}_1 \subset \mathcal{E}_2$, $\mathcal{E}_1$ is semipositive, and that $\mathcal{E}_1$ coincides with $\mathcal{E}_2$ over some non-empty Zariski open set of C. Then $\mathcal{E}_2$ is semipositive.*

We need the notion of *simple normal crossing pairs* for Theorem 1.11. Note that a simple normal crossing pair is sometimes called a *semi-snc* pair in the literature (cf. [BP, Definition 1.1]).

Definition 2.4 (Simple normal crossing pairs)**.** We say that the pair (X, D) is *simple normal crossing* at a point $a \in X$ if X has a Zariski open neighborhood U of a that can be embedded in a smooth variety Y, where Y has regular local coordinates $(x_1, \cdots, x_p, y_1, \cdots, y_r)$ at $a = 0$ in which U is defined by a monomial equation

$$x_1 \cdots x_p = 0$$

and

$$D = \sum_{i=1}^{r} \alpha_i (y_i = 0)|_U, \quad \alpha_i \in \mathbb{R}.$$

We say that (X, D) is a *simple normal crossing pair* if it is simple normal crossing at every point of X.

For the reader's convenience, we recall the notion of *semi log canonical pairs*.

Definition 2.5 (Semi log canonical pairs)**.** Let X be an equi-dimensional algebraic variety which satisfies Serre's S_2 condition and is normal crossing in codimension one. Let Δ be an effective $\mathbb{R}$-divisor on X

such that no irreducible component of $\operatorname{Supp}\Delta$ is contained in the singular locus of X. The pair (X, Δ) is called a *semi log canonical pair* (an *slc pair*, for short) if

(1) $K_X + \Delta$ is $\mathbb{R}$-Cartier, and
(2) (X^ν, Θ) is log canonical, where $\nu : X^\nu \to X$ is the normalization and $K_{X^\nu} + \Theta = \nu^*(K_X + \Delta)$.

If $(X, 0)$ is a semi log canonical pair, then we simply say that X is a *semi log canonical variety* or X has only *semi log canonical singularities.*

For the details of semi log canonical pairs and the basic notations, see [F4] and [K7].

2.6 ($\mathbb{Q}$-divisors)**.** Let D be a $\mathbb{Q}$-divisor on an equi-dimensional variety X, that is, D is a finite formal $\mathbb{Q}$-linear combination

$$D = \sum_i d_i D_i$$

of irreducible reduced subschemes D_i of codimension one. We define the *round-up* $\lceil D \rceil = \sum_i \lceil d_i \rceil D_i$ (resp. *round-down* $\lfloor D \rfloor = \sum_i \lfloor d_i \rfloor D_i$), where every real number x, $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the integer defined by $x \leq \lceil x \rceil < x + 1$ (resp. $x - 1 < \lfloor x \rfloor \leq x$). We set

$$D^{<0} = \sum_{d_i < 0} d_i D_i, \quad D^{>0} = \sum_{d_i > 0} d_i D_i, \quad \text{and} \quad D^{=1} = \sum_{d_i = 1} D_i.$$

We close this section with the definition of $\omega^{[m]}_{X/C}$.

Definition 2.7. In Theorem 1.8, $\omega^{[m]}_{X/C}$ is the *m-th reflexive power* of $\omega_{X/C}$. It is the double dual of the m-th tensor power of $\omega_{X/C}$:

$$\omega^{[m]}_{X/C} := (\omega^{\otimes m}_{X/C})^{**}.$$

3. A quick review of $\mathcal{M}^{stable}$

In this section, we quickly review the moduli space of stable varieties (see [K5]). First, let us recall the definition of *stable varieties.*

Definition 3.1 (Stable varieties)**.** Let X be a connected projective semi log canonical variety with ample canonical divisor. Then X is called a *stable variety* or a *semi log canonical model.*

In order to obtain the boundedness of the moduli functor of stable varieties, we have to fix some numerical invariants. So we introduce the notion of the *Hilbert function* for stable varieties.

Definition 3.2 (Hilbert function of stable varieties)**.** Let X be a stable variety. The *Hilbert function* of X is

$$H_X(m) := \chi(X, \omega_X^{[m]})$$

where $\omega_X^{[m]} := (\omega_X^{\otimes m})^{**} \simeq \mathcal{O}_X(mK_X)$. By [F4, Corollary 1.9], we see that

$$H_X(m) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(mK_X)) \geq 0$$

for every $m \geq 2$.

The following definition of the moduli functor of stable varieties is mainly due to Kollár. Note that a stable variety X is not necessarily Cohen–Macaulay when $\dim X \geq 3$. We think that it is one of the main difficulties when we treat families of stable varieties.

Definition 3.3 (Moduli functor of stable varieties)**.** Let $H(m)$ be a $\mathbb{Z}$-valued function. The moduli functor of stable varieties with Hilbert function H is

$$\mathcal{M}_H^{stable}(S) := \left\{ \begin{array}{c} \text{Flat, proper families } X \to S \text{, fibers are stable} \\ \text{varieties with ample canonical divisor} \\ \text{and Hilbert function } H(m)\text{, } \omega_{X/S}^{[m]} \text{ is flat over } S \\ \text{and commutes with base change for every } m\text{,} \\ \text{modulo isomorphisms over } S. \end{array} \right\}.$$

Remark 3.4. We consider $(f : X \to S) \in \mathcal{M}_H^{stable}(S)$. By the base change theorem and [F4, Corollary 1.9], we obtain that $f_*\omega_{X/S}^{[m]}$ is a locally free sheaf on S with $\mathrm{rank} f_*\omega_{X/S}^{[m]} = H(m)$ for every $m \geq 2$.

Let us quickly review the construction of the coarse moduli space of stable varieties following [K5].

3.5 (Coarse moduli space of $\mathcal{M}^{stable}$)**.** Let us consider the moduli functor

$$\mathcal{M}^{stable}(S) := \left\{ \begin{array}{c} \text{Flat, proper families } X \to S \text{, fibers are stable} \\ \text{varieties with ample canonical divisor,} \\ \omega_{X/S}^{[m]} \text{ is flat over } S \\ \text{and commutes with base change for every } m\text{,} \\ \text{modulo isomorphisms over } S. \end{array} \right\}$$

of stable varieties. It is obvious that $\mathcal{M}_H^{stable}$ is an open and closed subfunctor of $\mathcal{M}^{stable}$. It is known that the moduli functor $\mathcal{M}^{stable}$ is well-behaved, that is, $\mathcal{M}^{stable}$ is locally closed. For the details, see [K4, Corollary 25]. We have already known that the moduli functor $\mathcal{M}^{stable}$

satisfies the valuative criterion of separatedness and the valuative criterion of properness by Kollár's gluing theory and the existence of log canonical closures (see [K6, Theorem 26] and [HX, Section 7]). Moreover, it is well known that the automorphism group $\mathrm{Aut}(X)$ of a stable variety X is a finite group (for a more general result, see [F4, Corollary 6.16]). Then, by using [KM, 1.2 Corollary], we obtain a coarse moduli space M^{stable} of $\mathcal{M}^{stable}$ in the category of algebraic spaces (see, for example, [K5, **39** (Existence of coarse moduli spaces)]). Note that M^{stable} is a separated algebraic space which is locally of finite type. Since $\mathcal{M}^{stable}$ satisfies the valuative criterion of properness, M^{stable}_H is proper if and only if it is of finite type.

4. Proof of theorems

Let us start the proof of Theorem 1.11. Theorem 1.11 is essentially contained in [FF, Section 5]. We need no extra assumptions on D and local monodromies since C is a curve.

Proof of Theorem 1.11. There is a closed subset Σ of C such that every stratum of (X, D) is smooth over $C_0 = C \setminus \Sigma$. Apply [BP, Theorem 1.2] to $(X, \mathrm{Supp}(D + f^*\Sigma))$. Then we obtain a birational morphism $g : X' \to X$ from a projective simple normal crossing variety X' such that g is an isomorphism outside $\mathrm{Supp}\, f^*\Sigma$ and that $g^{-1}_* D + g^* f^* \Sigma$ has a simple normal crossing support on X'. Let D' be the horizontal part of $g^{-1}_* D$. We may assume that D' is Cartier (cf. [BP, Section 8]). Then (X', D') is a simple normal crossing pair and we obtain an inclusion

$$f_* g_* \omega_{X'/C}(D') \to f_* \omega_{X/C}(D),$$

which is an isomorphism over C_0. Therefore, it is sufficient to prove that $f_* g_* \omega_{X'/C}(D')$ is semipositive by Lemma 2.3. By replacing (X, D) with (X', D'), we may assume that every stratum of (X, D) is dominant onto C. We note that every local monodromy on $R^d f_{0*} \iota_! \mathbb{Q}_{X_0 \setminus D_0}$ around Σ is quasi-unipotent, where $d = \dim X - 1$, $X_0 = f^{-1}(C_0)$, $f_0 = f|_{X_0}$, $D_0 = D|_{X_0}$, and $\iota : X_0 \setminus D_0 \hookrightarrow X_0$. We take a unipotent reduction $\pi : C' \to C$ of $R^d f_{0*} \iota_! \mathbb{Q}_{X_0 \setminus D_0}$. We may assume that π is a Kummer cover. By shrinking C_0, we may further assume that $\pi : C' \to C$ is étale over C_0. Let $\mathcal{G}$ be the canonical extension of

$$\mathrm{Gr}^0_F (\pi^*_0 R^d f_{0*} \iota_! \mathbb{Q}_{X_0 \setminus D_0} \otimes \mathcal{O}_{C'_0})$$

where $\pi_0 = \pi|_{C'_0} : C'_0 := \pi^{-1}(C_0) \to C_0$. Note that $\pi^*_0 R^d f_{0*} \iota_! \mathbb{Q}_{X_0 \setminus D_0}$ underlies an admissible graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure. Then $\mathcal{G}$ is locally free and $\mathcal{G}^*$ is a semipositive locally free sheaf on C' (cf. [FF, Theorem 5.1]). Since $R^d f_* \mathcal{O}_X(-D) \simeq (\pi_* \mathcal{G})^G$, where G is the Galois group of $\pi : C' \to C$, we obtain a nontrivial map $\pi^* R^d f_* \mathcal{O}_X(-D) \to \mathcal{G}$, which is an isomorphism on C'_0. Note that $R^d f_* \mathcal{O}_X(-D)$ is the lower canonical extension of

$$\mathrm{Gr}^0_F(R^d f_{0*} \iota_! \mathbb{Q}_{X_0 \setminus D_0} \otimes \mathcal{O}_{C_0}).$$

See Step 4 in the proof of Theorem 5.1 and Theorem 5.3 in [FF]. Therefore, by taking the dual, we obtain an inclusion $0 \to \mathcal{G}^* \to \pi^* f_* \omega_{X/C}(D)$, which is an isomorphism on C'_0. Thus, $\pi^* f_* \omega_{X/C}(D)$ is semipositive by Lemma 2.3. So we obtain that $f_* \omega_{X/C}(D)$ is semipositive because π is surjective. □

Remark 4.1. When X is smooth in Theorem 1.11, the semipositivity theorem obtained in [F1, Theorem 3.9] is sufficient for the proof of Theorem 1.11. Note that [F1, Theorem 3.9] also follows from the theory of graded polarizable admissible variations of mixed Hodge structure.

Before we prove Theorem 1.8, we treat the following easier semipositivity theorem: Theorem 4.2. It is essentially the same as [K1, 4.13. Lemma]. Note that if X is semismooth then X has only semi log canonical singularities. Therefore, Theorem 4.2 is a special case of Theorem 1.8.

Theorem 4.2 (Semipositivity for semismooth varieties)**.** *Let X be a semismooth variety and let $f : X \to C$ be a projective surjective morphism onto a smooth projective curve. Assume that every irreducible component of X is dominant onto C. Then $f_* \omega_{X/C}$ is semipositive.*

By using the arguments in [F4, Section 4], we reduce Theorem 4.2 to Theorem 1.11. Hence we recommend the reader to see [F4, Section 4] before reading the rest of this section. Although Theorem 4.2 is a special case of Theorem 1.8, we give a proof for the reader's convenience. It will help the reader to understand more sophisticated arguments in the proof of Theorem 1.8.

Proof of Theorem 4.2. By Step 2, Step 3, and Step 4 in the proof of [F4, Theorem 1.2], we can find a smooth projective variety M with $\dim M = \dim X + 1$, a semismooth variety X', which is a divisor on M, such that $\alpha : X' \to X$ is birational and $\mathrm{Sing}\, X'$ maps birationally onto $\mathrm{Sing}\, X$. Since $\alpha_* \omega_{X'} \simeq \omega_X$, we can replace X with X'. Therefore, we may assume that X is a divisor on a smooth projective variety M. Let X^{snc} denote the simple normal crossing locus of X. Take a blow-up along an irreducible component of $X \setminus X^{\mathrm{snc}}$ and replace X with its total transform with the reduced structure. After finitely many steps, we may further assume that X is a simple normal crossing divisor on a smooth projective variety (see Step 6 in the proof of [F4, Theorem

1.2]). If there is a stratum S of X which is not dominant onto C, then we take a blow-up along that stratum. We can replace X with its strict transform by Lemma 2.3. Note that if S is an irreducible component of X then we replace X with $\overline{X \setminus S}$ in the above process. After finitely many steps, we may assume that every stratum of X is dominant onto C. By Theorem 1.11, we obtain that $f_*\omega_{X/C}$ is semipositive. □

From now on, we prove Theorem 1.8. The proof of Theorem 1.8 is essentially the same as that of Theorem 4.2 when $m = 1$. The case $m \geq 2$ can be reduced to the case when $m = 1$ by using Viehweg's covering arguments.

Proof of Theorem 1.8. In Step 1, we prove the semipositivity of $f_*\omega_{X/C}$.

Step 1. We apply the proof of Theorem 1.2 in [F4, Section 4] to X. Before we apply Step 6 in [F4, Section 4], we add the following step.

> **Step 5.5.** If there is an irreducible component S of $\operatorname{Sing} X_3$ in $(f \circ f_1 \circ f_2 \circ f_3)^{-1}(\Sigma)$, where $\Sigma = C \setminus U$, then we take a blow-up of X_3 along S and replace X_3 with its strict transform. By repeating this process finitely many times, we may assume that $(f \circ f_1 \circ f_2 \circ f_3)^{-1}(\Sigma)$ contains no irreducible components of $\operatorname{Sing} X_3$.

Then we obtain a smooth projective variety $\overline{M}$ and a simple normal crossing divisor $\overline{Z}$ on $\overline{M}$, a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\overline{B}$ on $\overline{M}$, and a projective surjective morphism $\overline{h} : \overline{Z} \to X$ with the following properties.

(1) $\overline{B}$ is a subboundary $\mathbb{Q}$-divisor, that is, $\overline{B}^{\leq 1} = \overline{B}$.
(2) $\overline{B}$ and $\overline{Z}$ have no common irreducible components.
(3) $\operatorname{Supp}(\overline{Z} + \overline{B})$ is a simple normal crossing divisor on $\overline{M}$.

We set $Z := (\overline{h})^{-1} f^{-1}(U)$ and $h := \overline{h}|_Z : Z \to f^{-1}(U)$. Then $h : Z \to f^{-1}(U)$ is a quasi-log resolution as in [F4, Theorem 1.2]. More precisely, we have the following properties.

(4) $K_Z + \Delta_Z \sim_{\mathbb{Q}} h^* K_{f^{-1}(U)}$ such that $\Delta_Z = \overline{B}|_Z$.
(5) $h_*\mathcal{O}_Z(\lceil -\Delta_Z^{<1} \rceil) \simeq \mathcal{O}_{f^{-1}(U)}$.
(6) The set of slc strata of $f^{-1}(U)$ gives the set of qlc centers of $[f^{-1}(U), K_{f^{-1}(U)}]$.

By taking more blow-ups, if necessary, we may assume that there are no strata of $(\overline{Z}, \operatorname{Supp} \Delta_{\overline{Z}})$ in $\overline{Z} \setminus Z$, where $\Delta_{\overline{Z}} = \overline{B}|_{\overline{Z}}$. By the construction of (Z, Δ_Z), we can check that

$$h_*\omega_Z(\Delta_Z^{=1}) \simeq \omega_{f^{-1}(U)} \quad \text{and} \quad \overline{h}_*\omega_{\overline{Z}}(\Delta_{\overline{Z}}^{=1}) \subset \omega_X.$$

Therefore, it is sufficient to prove that $(f \circ \overline{h})_*\omega_{\overline{Z}}(\Delta_{\overline{Z}}^{=1})$ is semipositive by Lemma 2.3. It is nothing but Theorem 1.11.

Step 2 (cf. Proof of [V2, Corollary 2.45])**.** By Viehweg's covering trick, we can prove that $f_*\omega^{[m]}_{X/C}$ is semipositive for every $m \geq 2$ by using the case when $m=1$. It is essentially [V2, Corollary 2.45]. Here, we closely follow the proof of [V2, Corollary 2.45].

Let $\mathcal{H}$ be an ample line bundle on C. We set

$$r = \min\{\mu \in \mathbb{Z}_{>0} \,|\, (f_*\omega^{[k]}_{X/C}) \otimes \mathcal{H}^{\mu k-1} \text{ is semipositive}\}.$$

By the assumption, we have that the natural map $f^*f_*\omega^{[k]}_{X/C} \to \omega^{[k]}_{X/C}$ is surjective. Since $(f_*\omega^{[k]}_{X/C}) \otimes \mathcal{H}^{rk-1}$ is semipositive, $(f_*\omega^{[k]}_{X/C}) \otimes \mathcal{H}^{rk}$ is ample. Therefore, $S^N((f_*\omega^{[k]}_{X/C}) \otimes \mathcal{H}^{rk})$ is generated by its global sections for some positive integer N. Hence we see that $\omega_{X/C} \otimes f^*\mathcal{H}^r$ is semi-ample. More precisely, $\omega^{[k]}_{X/C} \otimes f^*\mathcal{H}^{rk}$ is locally free and semi-ample. By the usual covering argument (see Remark 4.3), $(f_*\omega^{[k]}_{X/C}) \otimes \mathcal{H}^{r(k-1)}$ is semipositive (cf. [V2, Proposition 2.43]). This is only possible if $(r-1)k-1 < r(k-1)$. It is equivalent to $r \leq k$. Therefore, $(f_*\omega^{[k]}_{X/C}) \otimes \mathcal{H}^{k^2-1}$ is semipositive. The same holds true if we take any base change by $\pi : C' \to C$ such that π is a finite morphism from a smooth projective curve and ramifies only over general points of C. Therefore, $f_*\omega^{[k]}_{X/C}$ is semipositive (cf. [V2, Lemma 2.15]). By the same argument as above, we see that $\omega_{X/C} \otimes f^*\mathcal{H}$ is semi-ample since $f_*\omega^{[k]}_{X/C}$ is semipositive. More precisely, $\omega^{[k]}_{X/C} \otimes f^*\mathcal{H}^k$ is locally free and semi-ample in the usual sense. By the covering argument (see Remark 4.3), $(f_*\omega^{[m]}_{X/C}) \otimes \mathcal{H}^{m-1}$ is semipositive for every $m > 0$ (cf. [V2, Proposition 2.43]). The same holds true if we take any base change by $\pi : C' \to C$ as in the above case. Therefore, $f_*\omega^{[m]}_{X/C}$ is semipositive for every $m > 0$ (cf. [V2, Lemma 2.15]). For more details, see [V2, Section 2].

We have finished the proof of Theorem 1.8. □

Remark 4.3 (cf. [K1, 4.15. Lemma and 4.16])**.** Let $\varphi : X' \to X$ be a cyclic cover associated to a general member $A \in |\omega^{[kl]}_{X/C} \otimes f^*\mathcal{H}^{rkl}|$ (resp. $|\omega^{[kl]}_{X/C} \otimes f^*\mathcal{H}^{kl}|$) for some positive integer l. Then $f' := f \circ \varphi : X' \to C$ satisfies all the assumptions for $f : X \to C$. Therefore, we have that $f'_*\omega_{X'/C}$ is semipositive. It is easy to see that $\omega^{[k]}_{X/C} \otimes f^*\mathcal{H}^{r(k-1)}$ (resp. $\omega^{[m]}_{X/C} \otimes f^*\mathcal{H}^{m-1}$) is a direct summand of $\varphi_*\omega_{X'/C}$.

Theorem 1.5 is almost obvious by Theorem 1.8.

Proof of Theorem 1.5. We consider $(f : X \to C) \in \mathcal{M}^{stable}(C)$ where C is a smooth projective curve. By Kawakita's inversion of adjunction (see [Kwk, Theorem]), we can easily check that X itself is a semi log canonical variety. By the definition of $\mathcal{M}^{stable}$, we can find a positive integer k such that $\omega^{[k]}_{X/C}$ is locally free and f-ample. Hence $f_*\omega^{[m]}_{X/C}$ is semipositive for every $m \geq 1$ by Theorem 1.8. It implies that $\mathcal{M}^{stable}$ is semipositive in the sense of Kollár. □

By Kollár's results (see [K5, Sections 2 and 3]), Theorem 1.1 follows from Theorem 1.5. For some technical details, see also [V2, Theorem 4.34 and Theorem 9.25].

Proof of Theorem 1.1. It is sufficient to prove this theorem for connected subspaces. Let Z be a connected complete subspace of M^{stable}. It is obvious that M^{stable} has an open subspace of finite type which contains Z. By replacing $\mathcal{M}^{stable}$ with the subfunctor given by this subspace, we get a new functor $\mathcal{N}$ which is bounded. By recalling the construction of the coarse moduli space, we know that there is a locally closed subscheme S of $\mathrm{Hilb}(\mathbb{P}^N)$ for some N such that Z is obtained as the geometric quotient $S/\operatorname{Aut}(\mathbb{P}^N)$. Let $f : X \to S$ be the universal family. By the proof of [K1, 2.6. Theorem], we see that $\det(f_*\omega^{[k]}_{X/S})^p$ descends to an ample line bundle on Z for a sufficiently large and divisible integer k and a sufficiently divisible positive integer p (see [V2, Lemma 9.26]). Note that [K1, 2.6. Theorem] needs the semipositivity of $\mathcal{M}^{stable}$. For the details, see [K1, Sections 2 and 3] and [V2]. □

The proof of Theorem 1.8 works for Theorem 1.12 with some minor modifications.

Proof of Theorem 1.12. Roughly speaking, all we have to do is to replace K_X (resp. ω_X) with $K_X + D$ (resp. $\omega_X(D)$) in the proof of Theorem 1.8. We leave the details as an exercise for the reader. □

Theorem 1.12 is useful for the projectivity of the moduli space of *stable maps* (cf. [FP]). For some related topics, see also [A2].

4.4 (Projectivity of the space of stable maps (cf. [FP]))**.** We freely use the notation in [FP]. Let $\mathcal{F} = (\pi, \mathcal{C} \to S, \{p_i\}, \mu)$ be a *stable family of maps* over S to $\mathbb{P}^r$. For the definition, see [FP, 1.1. Definitions]. We set

$$E_k(\pi) = \pi_*(\omega_\pi^k(\sum_{i=1}^n kp_i) \otimes \mu^*(\mathcal{O}(3k))).$$

In [FP, Lemma 3], it is proved that $E_k(\pi)$ is a semipositive locally free sheaf on S for $k \geq 2$ by using the results in [K1, Section 4]. This semipositivity is used for the projectivity of the moduli space of stable maps in [FP, 4.3. Projectivity]. The semipositivity of $E_k(\pi)$ can be checked as follows:

Since $k \geq 2$, by the base change theorem, we may assume that S is a smooth projective curve. We take a general member H of $|\mu^*\mathcal{O}(3)|$. Then $(\mathcal{C}, \sum_{i=1}^n p_i + H)$ is a semi log canonical surface and $K_{\mathcal{C}/S} + \sum_{i=1}^n p_i + H$ is π-ample. Therefore

$$\pi_*\mathcal{O}_{\mathcal{C}}(k(K_{\mathcal{C}/S} + \sum_{i=1}^n p_i + H)) \simeq E_k(\pi)$$

is semipositive for every $k \geq 2$ by Theorem 1.12.

Let us start the proof of Theorem 1.13.

Proof of Theorem 1.13. We use Viehweg's covering arguments (cf. [V1]) and Theorem 1.12.

Step 1. As in the proof of Theorem 1.8, we obtain a smooth projective variety $\overline{M}$ and a simple normal crossing divisor $\overline{Z}$ on $\overline{M}$, a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\overline{B}$ on $\overline{M}$, and a projective surjective morphism $\overline{h} : \overline{Z} \to X$ with the following properties.

(1) $\overline{B}$ is a subboundary $\mathbb{Q}$-divisor, that is, $\overline{B}^{\leq 1} = \overline{B}$.
(2) $\overline{B}$ and $\overline{Z}$ have no common irreducible components.
(3) $\operatorname{Supp}(\overline{Z} + \overline{B})$ is a simple normal crossing divisor on $\overline{M}$.

We set $Z := (\overline{h})^{-1}f^{-1}(U)$ and $h := \overline{h}|_Z : Z \to f^{-1}(U)$. Then $h : Z \to f^{-1}(U)$ is a quasi-log resolution as in [F4, Theorem 1.2]. More precisely, we have the following properties.

(4) $K_Z + \Delta_Z \sim_{\mathbb{Q}} h^*(K_{f^{-1}(U)} + \Delta_{f^{-1}(U)})$ such that $\Delta_Z = \overline{B}|_Z$ and $\Delta_{f^{-1}(U)} = \Delta|_{f^{-1}(U)}$.
(5) $h_*\mathcal{O}_Z(\lceil -\Delta_Z^{<1} \rceil) \simeq \mathcal{O}_{f^{-1}(U)}$.
(6) The set of slc strata of $(f^{-1}(U), \Delta_{f^{-1}(U)})$ gives the set of qlc centers of $[f^{-1}(U), K_{f^{-1}(U)} + \Delta_{f^{-1}(U)}]$.

By taking more blow-ups, if necessary, we may assume that $\overline{Z} \setminus Z$ is a simple normal crossing divisor on $\overline{Z}$, $(\overline{Z} \setminus Z) \cup \operatorname{Supp} \Delta_{\overline{Z}}$ is a simple normal crossing divisor on $\overline{Z}$, and therefore there are no strata of $(\overline{Z}, \operatorname{Supp} \Delta_{\overline{Z}})$ in $\overline{Z} \setminus Z$, where $\Delta_{\overline{Z}} = \overline{B}|_{\overline{Z}}$. In this case, by the construction of (Z, Δ_Z), we can check that

$$h_*\mathcal{O}_Z(k(K_Z + \Delta_Z^{\geq 0})) \simeq \mathcal{O}_{f^{-1}(U)}(k(K_X + \Delta))$$

and

$$\overline{h}_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}} + \Delta_{\overline{Z}}^{\geq 0})) \subset \mathcal{O}_X(k(K_X + \Delta)).$$

Hence it is sufficient to prove that $(f\circ \overline{h})_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C}+\Delta^{\geq 0}_{\overline{Z}}))$ is semipositive by Lemma 2.3. By shrinking U, we may further assume that every stratum of $(Z, \operatorname{Supp}\Delta_Z)$ is smooth over U.

Step 2. By the above construction, we have

$$K_Z+\Delta^{\geq 0}_Z = h^*(K_{f^{-1}(U)}+\Delta_{f^{-1}(U)})+(-\Delta^{\leq 0}_Z).$$

We apply the covering argument discussed in [C, Section 4.4], which is a modification of Viehweg's covering argument in [V1, Lemma 5.1 and Corollary 5.2]. We set $g=f\circ\overline{h}$. By taking more blow-ups over $\overline{Z}\setminus Z$ if necessary, we may assume that

$$\mathcal{F} := \operatorname{Image}(g^*g_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C}+\Delta^{\geq 0}_{\overline{Z}}))\to \mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C}+\Delta^{\geq 0}_{\overline{Z}})))$$

is a line bundle which is g-generated and that

$$\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C}+\Delta^{\geq 0}_{\overline{Z}}))\simeq \mathcal{F}\otimes \mathcal{O}_{\overline{Z}}(E)$$

such that E is an effective Cartier divisor on $\overline{Z}$ and that $\operatorname{Supp}E$ is a simple normal crossing divisor on $\overline{Z}$. We may further assume that $\operatorname{Supp}E\cup\operatorname{Supp}\Delta^{\geq 0}_{\overline{Z}}$ is a simple normal crossing divisor on $\overline{Z}$. By the construction, we see that $E=-k\Delta^{\leq 0}_Z$ over U. Let $\mathcal{H}$ be an ample line bundle on C. We set

$$r=\min\{\mu\in\mathbb{Z}_{>0}\,|\, g_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C}+\Delta^{\geq 0}_{\overline{Z}}))\otimes\mathcal{H}^{\mu k-1} \text{ is semipositive}\}.$$

The following lemma is essentially contained in [V1, Lemma 5.1 and Corollary 5.2].

Lemma 4.5 (see [C, Lemma 4.19])**.** *Let $g:\overline{Z}\to C$ be as above. Let $\mathcal{A}$ be an ample line bundle on C. Assume that*

$$S^N(g_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C}+\Delta^{\geq 0}_{\overline{Z}}))\otimes\mathcal{A}^k)$$

is generated by its global sections for some positive integer N. Then

$$g_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C}+\Delta^{\geq 0}_{\overline{Z}}))\otimes\mathcal{A}^{k-1}$$

is semipositive on C.

Proof of Lemma 4.5. Once we adopt Theorem 1.12, the arguments in the proof of [C, Lemma 4.9] works for our situation.

Since every stratum of $\overline{Z}$ is dominant onto C, $\overline{Z}$ is a simple normal crossing variety, and g has connected general fibers, we have $g_*\mathcal{O}_{\overline{Z}}\simeq\mathcal{O}_C$ (see, for example, [F4, Lemma 3.6]). By the definition of $\mathcal{F}$, we have

$$g_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C}+\Delta^{\geq 0}_{\overline{Z}}))\simeq g_*\mathcal{F}.$$

Therefore, $S^N(g_*\mathcal{F}\otimes\mathcal{A}^k)$ is generated by its global sections by the assumption. Hence $|(\mathcal{F}\otimes g^*\mathcal{A}^k)^N|$ is a free linear system on $\overline{Z}$. Note that $\mathcal{F}$ is g-generated. We set

$$\mathcal{L}=\mathcal{O}_{\overline{Z}}(K_{\overline{Z}/C}+k\Delta^{\geq 0}_{\overline{Z}})\otimes g^*\mathcal{A}.$$

Then we have

$$\mathcal{L}^k=\mathcal{O}_{\overline{Z}}(E+(k-1)(k\Delta^{\geq 0}_{\overline{Z}}))\otimes\mathcal{F}\otimes g^*\mathcal{A}^k.$$

Let H be a general member of the free linear system $|(\mathcal{F}\otimes g^*\mathcal{A}^k)^N|$. Then we obtain

$$\mathcal{L}^{kN}=\mathcal{O}_{\overline{Z}}(H+NE+N(k-1)(k\Delta^{\geq 0}_{\overline{Z}})).$$

We take a (kN)-fold cyclic cover $p:\widetilde{Z}\to\overline{Z}$ associated to

$$\mathcal{L}^{kN}=\mathcal{O}_{\overline{Z}}(H+NE+N(k-1)(k\Delta^{\geq 0}_{\overline{Z}})).$$

Note that $(\widetilde{Z},p^*\Delta^{=1}_{\overline{Z}})$ is a semi log canonical pair (cf. [K6, Theorem 26]). More explicitly, $\widetilde{Z}$ can be written as follows:

$$\widetilde{Z}=\mathrm{Spec}_{\overline{Z}}\bigoplus_{i=0}^{kN-1}(\mathcal{L}^{(i)})^{-1},$$

where

$$(\mathcal{L}^{(i)})^{-1}=\mathcal{L}^{-i}\otimes\mathcal{O}_{\overline{Z}}(\lfloor\frac{i}{k}(E+(k-1)(k\Delta^{\geq 0}_{\overline{Z}}))\rfloor).$$

We can easily see that $\omega_{\overline{Z}}\otimes\mathcal{L}^{(k-1)}$ is a direct summand of $p_*\omega_{\widetilde{Z}}$ by the construction, where

$$\mathcal{L}^{(k-1)}=\mathcal{L}^{k-1}\otimes\mathcal{O}_{\overline{Z}}(-\lfloor\frac{k-1}{k}(E+(k-1)(k\Delta^{\geq 0}_{\overline{Z}}))\rfloor).$$

By the calculation in the proof of [C, Lemma 4.9], we have a natural inclusion

$$g_*(\mathcal{O}_{\overline{Z}}(K_{\overline{Z}/C}+\Delta^{=1}_{\overline{Z}})\otimes\mathcal{L}^{(k-1)})\subset g_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C}+\Delta^{\geq 0}_{\overline{Z}}))\otimes\mathcal{A}^{k-1},$$

which is an isomorphism on U. We note that $\mathcal{O}_{\overline{Z}}(K_{\overline{Z}}+\Delta^{=1}_{\overline{Z}})\otimes\mathcal{L}^{(k-1)}$ is a direct summand of $p_*\mathcal{O}_{\widetilde{Z}}(K_{\widetilde{Z}}+p^*\Delta^{=1}_{\overline{Z}})$. By taking a suitable partial resolution of $(\widetilde{Z},p^*\Delta^{=1}_{\overline{Z}})$ (see [BP, Theorem 1.2]), we can construct a simple normal crossing pair (V,D) such that D is reduced and that $q_*\omega_V(D)\simeq\omega_{\widetilde{Z}}(p^*\Delta^{=1}_{\overline{Z}})$ where $q:V\to\widetilde{Z}$. Hence we have the following properties.

(1) $\pi=p\circ q:V\to\overline{Z}$ is a generically finite cover.
(2) there is a locally free sheaf $\mathcal{E}$ on C such that $\mathcal{E}$ is a direct summand of $(g\circ\pi)_*\omega_{V/C}(D)$.

(3) $\mathcal{E} \subset g_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C} + \Delta^{\geq 0}_{\overline{Z}})) \otimes \mathcal{A}^{k-1}$, which is an isomorphism over some non-empty Zariski open set of C.

By a special case of Theorem 1.12, we obtain that $(g \circ \pi)_*\omega_{V/C}(D)$ is semipositive. Therefore, the direct summand $\mathcal{E}$ is also a semipositive locally free sheaf on C. By Lemma 2.3, we obtain that

$$g_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C} + \Delta^{\geq 0}_{\overline{Z}})) \otimes \mathcal{A}^{k-1}$$

is semipositive. □

By the definition of r, $g_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C} + \Delta^{\geq 0}_{\overline{Z}})) \otimes \mathcal{H}^{rk-1}$ is semipositive. Therefore,

$$S^N(g_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C} + \Delta^{\geq 0}_{\overline{Z}})) \otimes \mathcal{H}^{rk})$$

is generated by its global sections for some positive integer N. Then, by Lemma 4.5, we obtain that

$$g_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C} + \Delta^{\geq 0}_{\overline{Z}})) \otimes \mathcal{H}^{r(k-1)}$$

is semipositive. This is only possible if $(r-1)k - 1 < r(k-1)$. It is equivalent to $r \leq k$. Therefore,

$$g_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C} + \Delta^{\geq 0}_{\overline{Z}})) \otimes \mathcal{H}^{k^2-1}$$

is semipositive. The same holds true if we take any base change by $\pi : C' \to C$ such that π is a finite morphism from a smooth projective curve and ramifies only over general points of C. Therefore,

$$g_*\mathcal{O}_{\overline{Z}}(k(K_{\overline{Z}/C} + \Delta^{\geq 0}_{\overline{Z}}))$$

is semipositive.

Hence we obtain that $f_*\mathcal{O}_X(k(K_{X/C} + \Delta))$ is semipositive on C. Since $\mathcal{O}_X(kl(K_X + \Delta))$ is f-generated, by replacing k with kl in the above arguments, we obtain that $f_*\mathcal{O}_X(kl(K_{X/C} + \Delta))$ is semipositive for every positive integer l. □

We close this section with comments on Kollár's arguments in [K1, Section 4] for the reader's convenience.

4.6 (Comments on Kollár's arguments in [K1, Section 4])**.** In [K1, Section 4], Kollár essentially claims Theorem 1.8 when $\dim X = 3$. However, it is not so obvious to follow his arguments. In the last part of [K1, 4.14], he says

> As in the proof of 4.13 the kernel of δ is a direct summand and thus semipositive.

In [K1, 4.14], E is not always smooth. Therefore, it is not clear what kind of variations of Hodge structure should be considered. The map

$$(f \circ g)_* \omega_{E/C} \xrightarrow{\delta} R^1(f \circ g)_* \omega_{X/C}$$

in [K1, 4.14] is different from the map

$$\delta' : (f \circ g)_* \omega_{D'/C} \to R^1(f \circ g)_* \omega_{Z'/C}$$

in the proof of [K1, 4.13. Lemma] from the Hodge theoretic viewpoint. Note that D' and Z' are smooth by the construction. In general, E and X are *singular* in [K1, 4.14]. Kollár informed the author that the semipositivity of $(f \circ g)_* \omega_{X/C}(E)$ can be checked with the aid of the classification of semi log canonical surface singularities. Note that his arguments only work for the case when the fibers are surfaces. Anyway, we do not pursue them here because the semipositivity of $(f \circ g)_* \omega_{X/C}(E)$ is a special case of Theorem 1.12 when $f \circ g$ is projective.

5. Proof of corollaries

In this section, we prove the corollaries in Section 1. Before we start the proof, let us recall the definition of *smoothable* stable varieties (cf. [K1, 5.4. Definition (ii)]). We use smoothable stable varieties in order to compactify some moduli spaces.

Definition 5.1 (Smoothable stable varieties)**.** A stable variety X_0 is *smoothable* if there is a flat projective morphism $\pi : X \to C$ to a smooth curve C such that $\omega_{X/C}^{[m]}$ is flat over C and commutes with base change for every $m \geq 1$, the special fiber of π is X_0, and the general fiber X_t is a canonically polarized normal projective variety with only canonical singularities.

We start the proof of the corollaries in Section 1.

Proof of Corollary 1.2. We check the boundedness of the moduli functor $\mathcal{M}_H^{sm}$. The arguments in [Kr] work for our situation with some suitable modifications. In our situation, we can not apply Matsusaka's big theorem because the general fiber X_t in Definition 5.1 is not assumed to be Gorenstein. Therefore we use the boundedness result obtained through works of Tsuji, Hacon–McKernan, and Takayama (see [HK, Theorem 13.6]). By the arguments in [Kr], we obtain the boundedness of $\mathcal{M}_H^{sm}$. By the existence of relative canonical models, which is established by Birkar–Cascini–Hacon–McKernan (see, for example, [HK, Part II]), we see that $\mathcal{M}_H^{sm}$ is a closed subfunctor of $\mathcal{M}_H^{stable}$. Therefore there is a coarse moduli space M_H^{sm} of $\mathcal{M}_H^{sm}$ which is a complete algebraic space. Since the moduli functor $\mathcal{M}_H^{sm}$ is semipositive in the sense of Kollár by Theorem 1.5 and Theorem 1.8, we see that M_H^{sm} is a projective algebraic scheme. □

Corollaries 1.3 and 1.4 are almost obvious by Corollary 1.2.

Proof of Corollary 1.3. The moduli functor $\mathcal{M}_H$ is an open subfunctor of $\mathcal{M}_H^{sm}$ because the smoothness is an open condition. Therefore Corollary 1.3 follows from Corollary 1.2. □

Proof of Corollary 1.4. Note that any small deformations of canonical singularities are canonical (see [Kwm, Main Theorem]). Therefore the moduli functor $\mathcal{M}_H^{can}$ is an open subfunctor of $\mathcal{M}_H^{sm}$. Hence, Corollary 1.4 follows from Corollary 1.2. □

We close this paper with the proof of Corollary 1.7.

Proof of Corollary 1.7. If the moduli functor $\mathcal{M}_H^{stable}$ is bounded, then we have a coarse moduli space M_H^{stable} of $\mathcal{M}_H^{stable}$ which is a complete algebraic space. Note that the moduli functor $\mathcal{M}_H^{stable}$ satisfies the valuative criterion of separatedness and the valuative criterion of properness. Since the moduli functor $\mathcal{M}_H^{stable}$ is semipositive in the sense of Kollár by Theorem 1.5 and Theorem 1.8, we see that M_H^{stable} is a projective algebraic scheme. □

The boundedness of the moduli functor $\mathcal{M}_H^{stable}$ is the last open problem for the construction of projective coarse moduli spaces of stable varieties, which will be fixed by [HMX].

References

[AH] D. Abramovich, B. Hassett, Stable varieties with a twist, *Classification of algebraic varieties*, 1–38, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011.

[A1] V. Alexeev, Boundedness and K^2 for log surfaces. Internat, J. Math. **5** (1994), no. 6, 779–810.

[A2] V. Alexeev, Moduli spaces $M_{g,n}(W)$ for surfaces, *Higher-dimensional complex varieties (Trento, 1994)*, 1–22, de Gruyter, Berlin, 1996.

[AM] V. Alexeev, S. Mori, Bounding singular surfaces of general type, *Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000)*, 143–174, Springer, Berlin, 2004.

[BP] E. Bierstone, F. Vera Pacheco, Resolution of singularities of pairs preserving semi-simple normal crossings, preprint (2011).

[C] F. Campana, Orbifolds, special varieties and classification theory, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 3, 499–630.

[F1] O. Fujino, Higher direct images of log canonical divisors, J. Differential Geom. **66** (2004), no. 3, 453–479.

[F2] O. Fujino, Fundamental theorems for the log minimal model program, Publ. Res. Inst. Math. Sci. **47** (2011), no. 3, 727–789.

[F3] O. Fujino, Recent developments in minimal model theory, Sugaku Expositions **24** (2011), no. 2, 205–237.

[F4] O. Fujino, Fundamental theorems for semi log canonical pairs, preprint (2012).

[FF] O. Fujino, T. Fujisawa, Variations of mixed Hodge structure and semipositivity theorems, preprint (2012).

[Ft] T. Fujita, On Kähler fiber spaces over curves, J. Math. Soc. Japan **30** (1978), no. 4, 779–794.

[FP] W. Fulton, R. Pandharipande, Notes on stable maps and quantum cohomology, *Algebraic geometry—Santa Cruz 1995*, 45–96, Proc. Sympos. Pure Math., **62**, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[HK] C. D. Hacon, S. J. Kovács, *Classification of higher dimensional algebraic varieties*, Oberwolfach Seminars, **41**. Birkhäuser Verlag, Basel, 2010.

[HM] C. D. Hacon, J. McKernan, Boundedness of pluricanonical maps of varieties of general type, Invent. Math. **166** (2006), no. 1, 1–25.

[HMX] C. D. Hacon, J. McKernan, C. Xu, Boundedness of moduli functors, in preparation.

[HX] C. D. Hacon, C. Xu, Existence of log canonical closures, to appear in Invent. Math.

[Kr] K. Karu, Minimal models and boundedness of stable varieties, J. Algebraic Geom. **9** (2000), no. 1, 93–109.

[Kwk] M. Kawakita, Inversion of adjunction on log canonicity, Invent. Math. **167** (2007), no. 1, 129–133.

[Kwm] Y. Kawamata, Deformations of canonical singularities, J. Amer. Math. Soc. **12** (1999), no. 1, 85–92.

[KM] S. Keel, S. Mori, Quotients by groupoids, Ann. of Math. (2) **145** (1997), no. 1, 193–213.

[K1] J. Kollár, Projectivity of complete moduli, J. Differential Geom. **32** (1990), no. 1, 235–268.

[K2] J. Kollár, Non-quasi-projective moduli spaces, Ann. of Math. (2) **164** (2006), no. 3, 1077–1096.

[K3] J. Kollár, Two examples of surfaces with normal crossing singularities, Sci. China Math. **54** (2011), no. 8, 1707–1712.

[K4] J. Kollár, Hulls and Husks, preprint (2008).

[K5] J. Kollár, Moduli of varieties of general type, preprint (2010).

[K6] J. Kollár, Sources of log canonical centers, preprint (2011).

[K7] J. Kollár, *Singularities of the Minimal Model Program*, preprint (2011).

[Kv] S. J. Kovács, Young person's guide to moduli of higher dimensional varieties, *Algebraic geometry—Seattle 2005*. Part 2, 711–743, Proc. Sympos. Pure Math., **80**, Part 2, Amer. Math. Soc., Providence, RI, 2009.

[ST] G. Schumacher, H. Tsuji, Quasi-projectivity of moduli spaces of polarized varieties, Ann. of Math. (2) **159** (2004), no. 2, 597–639.

[T] S. Takayama, Pluricanonical systems on algebraic varieties of general type, Invent. Math. **165** (2006), no. 3, 551–587.

[V1] E. Viehweg, Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces, *Algebraic varieties and analytic varieties (Tokyo, 1981)*, 329–353, Adv. Stud. Pure Math., **1**, North-Holland, Amsterdam, 1983.

[V2] E. Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], **30**, Springer-Verlag, Berlin, 1995.

[V3] E. Viehweg, Compactifications of smooth families and of moduli spaces of polarized manifolds, Ann. of Math. (2) **172** (2010), no. 2, 809–910.

[Z] S. Zucker, Remarks on a theorem of Fujita, J. Math. Soc. Japan **34** (1982), no. 1, 47–54.

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: `fujino@math.kyoto-u.ac.jp`