# Pointwise convergence to initial data

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# Summary

- Part 1: Set-up and introduction to the PDEs.
- Part 2: Convergence for the heat equation.
- Part 3: Convergence for the Schrödinger equation.
- Part 4: Counterexample for the Schrödinger equation.
- Part 5: Decay of the Fourier transform of fractal measures.
- Part 6: Convergence for the wave equation.

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# Part 1:
# Set-up and introduction to the PDEs

# The heat equation

$$\begin{cases} \partial_t u &= \Delta u \quad \text{in} \quad \mathbb{R}^n \times [0,\infty) \\ u &= u_0 \quad \text{in} \quad \mathbb{R}^n \times \{0\}. \end{cases}$$

Taking the Fourier transform of the equation we obtain

$$\begin{cases} \partial_t \widehat{u}(\xi) &= -|\xi|^2 \widehat{u}(\xi) \\ \widehat{u}(\xi) &= \widehat{u_0}(\xi). \end{cases}$$

Solving the ODE this yields

$$\widehat{u}(\xi) = e^{-t|\xi|^2} \widehat{u_0}(\xi)\,.$$

Inverting the Fourier transform, we write

$$u(x,t) = e^{t\Delta} u_0(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t|\xi|^2} \widehat{u_0}(\xi)\, d\xi\,.$$

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$$u(\cdot, t) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1\,.$$

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# The initial data

We take the initial data $u_0$ in the Bessel potential space

$$\begin{aligned} H^s(\mathbb{R}^n) &:= (1-\Delta)^{-s/2}L^2(\mathbb{R}^n) \\ &:= \{\, f \,:\, \widehat{f} = (1+|\cdot|^2)^{-s/2}\widehat{g}, \quad \widehat{g} \in L^2(\mathbb{R}^n)\,\} \end{aligned}$$

with norm

$$\|f\|_{H^s} = \left( \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 \, d\xi \right)^{1/2} = \|g\|_{L^2(\mathbb{R}^n)} \,.$$

or in the Riesz potential space

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## Lemma (Pointwise convergence for smooth data)

*Let $u$ solve the heat or Schrödinger equation with $u_0 \in \dot{H}^s(\mathbb{R}^n)$ with $n/2 < s < n/2 + 2$. Then*

$$\lim_{t \to 0} u(x,t) = u_0(x) \quad \text{for all } x \in \mathbb{R}^n.$$

**Proof:** $\widehat{u}_0 = |\cdot|^{-s}\widehat{g}$ with $g \in L^2$

$$\begin{aligned}(2\pi)^{n/2}|e^{t\Delta}u_0(x) - u_0(x)| &= \left|\int \frac{\widehat{g}(\xi)\, e^{ix\cdot\xi}(e^{-t|\xi|^2} - 1)}{|\xi|^s} d\xi\right| \\ &\le \|\widehat{g}\,\|_2 \Big(\int \frac{|e^{-t|\xi|^2} - 1|^2}{|\xi|^{2s}} d\xi\Big)^{1/2} \\ &= t^{s/2-n/4}\|g\|_2 \Big(\int \frac{|e^{-|y|^2} - 1|^2}{|y|^{2s}} dy\Big)^{1/2} \\ &= t^{s/2-n/4}\|f\|_{\dot{H}^s} \Big(\int \frac{\min\{|y|^2, 1\}^2}{|y|^{2s}} dy\Big)^{1/2} \\ &\le C_s t^{s/2-n/4}\|f\|_{\dot{H}^s}\,.\end{aligned}$$

The same calculation works for the Schrödinger equation.

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&= t^{s/2-n/4}\|f\|_{\dot{H}^s} \Big( \int \frac{\min\{|y|^2, 1\}^2}{|y|^{2s}} dy \Big)^{1/2} \\
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## Lemma (Pointwise convergence for smooth data)

*Let $u$ solve the heat or Schrödinger equation with $u_0 \in \dot{H}^s(\mathbb{R}^n)$ with $n/2 < s < n/2 + 2$. Then*

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Recall that the Hardy–Littlewood maximal operator $M$ is defined by

$$Mf = \sup_{r>0} \tfrac{1}{|B(0,r)|} \mathbf{1}_{B(0,r)} * |f|,$$

and that it is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

This allows one to conclude that

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So the $L^2$-bound for $M$ gives an $L^2$ maximal estimate for the heat equation which allows us to conclude that
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Let $A \subseteq \mathbb{R}^n$ be a borel set, $0 < \alpha < n$ and

$$\mathcal{H}^\alpha_\delta(A) := \inf\left\{ \sum_i \delta_i^\alpha \ : \ A \subset \bigcup_i B(x_i, \delta_i), \quad \delta_i < \delta \right\}.$$

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## Remark

*There exists a unique* $\alpha_0$ *such that*

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*Let $A \subset \mathbb{R}^n$ be a Borel set. The following are equivalent:*

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# Control of singularities

**Lemma**

*Let $0 < s < n/2$ and $\alpha > n - 2s$. Then, for all $\alpha$-dimensional $\mu$,*

$$\|f\|_{L^1(d\mu)} \leq C_\mu \|f\|_{\dot{H}^s}.$$

**Proof:** $f = I_s * g$ with $g \in L^2$ and $\widehat{I_s} = |\cdot|^{-s}$. Suffices to prove

$$\|I_s * g\|_{L^1(d\mu)} \lesssim \sqrt{E_{n-2s}(\mu)}\, \|g\|_{L^2(\mathbb{R}^n)}.$$

By Fubini's theorem and the Cauchy–Schwarz inequality,

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$$\begin{aligned}\|I_s * g\|_{L^1(d\mu)} &\le \int\int I_s(x-y)\, d\mu(x)\, |g(y)|\, dy \\ &\le \|I_s * \mu\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

# Control of singularities

**Lemma**

*Let $0 < s < n/2$ and $\alpha > n - 2s$. Then, for all $\alpha$-dimensional $\mu$,*

$$\|f\|_{L^1(d\mu)} \leq C_\mu \|f\|_{\dot{H}^s}.$$

**Proof:** $f = I_s * g$ with $g \in L^2$ and $\widehat{I_s} = |\cdot|^{-s}$. Suffices to prove

$$\|I_s * g\|_{L^1(d\mu)} \lesssim \sqrt{E_{n-2s}(\mu)}\, \|g\|_{L^2(\mathbb{R}^n)}.$$

By Fubini's theorem and the Cauchy–Schwarz inequality,

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Thus it remains to prove that

$$\|I_s * \mu\|^2_{L^2(\mathbb{R}^n)} \lesssim E_{n-2s}(\mu).$$

By Plancherel's theorem,

$$\|I_s * \mu\|^2_{L^2(\mathbb{R}^n)} = \|\widehat{I_s}\widehat{\mu}\|^2_{L^2(\mathbb{R}^n)} = \int \widehat{\mu}(\xi)\,\overline{\widehat{\mu}(\xi)}\,\widehat{I}_{2s}(\xi)\,d\xi.$$

Recalling that $I_{2s}(x) = C_{n,s}|x|^{-(n-2s)}$,

$$\begin{aligned}\|I_s * \mu\|^2_{L^2(\mathbb{R}^n)} &= \int \mu * I_{2s}(y)\,d\mu(y)\\ &= C_{n,s}\iint \frac{d\mu(x)d\mu(y)}{|x-y|^{n-2s}} = C_{n,s}E_{n-2s}(\mu),\end{aligned}$$

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# Optimality of the control of singularities lemma

If $\dim(A) = \alpha$ with $\alpha < n - 2s$, then we can take a $\gamma$ such that

$$\alpha < \gamma < n - 2s.$$

Then

$$\mathbf{1}_{B(0,1)} \mathrm{dist}(\cdot, A)^{-\gamma} \in L^2(\mathbb{R}^n)$$

but on the other hand

$$u_0 := I_s * \left[ \mathbf{1}_{B(0,1)} \mathrm{dist}(\cdot, A)^{-\gamma} \right] = \infty \quad \text{on } A.$$

So there is initial data $u_0 \in \dot{H}^s(\mathbb{R}^n)$ which is singular on a set of dimension $\alpha < n - 2s$.

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## Proposition (Maximal estimates imply convergence)

*Let $\alpha > \alpha_0 \geq n - 2s$. Suppose that, for all $\alpha$-dimensional $\mu$,*

$$\Big\| \sup_{0<t<1} |u(\cdot,t)| \Big\|_{L^1(d\mu)} \leq C_\mu \|u_0\|_{\dot{H}^s}.$$

*Then, for all $u_0 \in \dot{H}^s$,*

$$dim \Big\{ x \in \mathbb{R}^n \quad \lim_{t\to 0} u(t,x) \neq u_0(x) \Big\} \leq \alpha_0.$$

**Proof:** We are required to prove that for all $\alpha > \alpha_0$,

$$\mathcal{H}^\alpha \Big\{ x \in \mathbb{R}^n \quad \lim_{t\to 0} u(t,x) \neq u_0(x) \Big\} = 0$$

whenever $u_0 \in \dot{H}^s$. By Frostman's lemma, this follows by showing

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Take $h \in \dot{H}^{n/2+1}$ such that $\|u_0 - h\|_{\dot{H}^s} < \varepsilon$, and note that

$$|u(\cdot, t) - u_0| \leq |u(\cdot, t) - u_h(\cdot, t)| + |u_h(\cdot, t) - h| + |h - u_0|,$$

where $u_h$ denotes the solution with initial data $h$.

Take $h \in \dot{H}^{n/2+1}$ such that $\|u_0 - h\|_{\dot{H}^s} < \varepsilon$, and note that

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We use the maximal estimate for the first term, the second term is zero by the smooth data lemma, and the third term can be bounded by the control of singularities lemma so that

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# Part 2:

# Convergence for the heat equation

**Theorem (Maximal estimate for the heat equation)**

*Let $0 < s < n/2$ and $\alpha > n - 2s$. Then, for all $\alpha$-dimensional $\mu$,*

$$\Big\| \sup_{0<t<1} |e^{t\Delta} f| \Big\|_{L^1(d\mu)} \le C_\mu \|f\|_{\dot{H}^s}.$$

**Proof:** By linearising the operator, it will suffice to prove

$$\left| \int\!\!\int e^{ix\cdot\xi} e^{-t(x)|\xi|^2}\, \widehat{f}(\xi)\, d\xi\, w(x)\, d\mu(x) \right|^2 \lesssim E_{n-2s}(\mu)\, \|f\|_{\dot{H}^s}^2,$$

whenever $t : \mathbb{R}^n \to (0,\infty)$ and $w : \mathbb{R}^n \to \mathbb{S}^1$ are measurable. Now, by Fubini and Cauchy–Schwarz, the LHS is bounded by

$$\int |\widehat{f}(\xi)|^2 |\xi|^{2s} d\xi \int \left| \int e^{ix\cdot\xi} e^{-t(x)|\xi|^2} w(x)\, d\mu(x) \right|^2 \frac{d\xi}{|\xi|^{2s}}.$$

Squaring out the integral, it will suffice to show that

$$\int\!\!\int\!\!\int e^{i(x-y)\cdot\xi} e^{-(t(x)+t(y))|\xi|^2} \frac{d\xi}{|\xi|^{2s}} w(x) w(y)\, d\mu(x) d\mu(y) \lesssim E_{n-2s}(\mu).$$

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Squaring out the integral, it will suffice to show that

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*Let* $0 < s < n/2$ *and* $\alpha > n - 2s$. *Then, for all* $\alpha$*-dimensional* $\mu$*,*

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uniformly for all choices of $t(x), t(y) > 0$. Recalling that $\widehat{|\cdot|^{-2s}} = C_{n,s}|\cdot|^{2s-n}$, this would follow from

$$\frac{1}{\lambda^{n/2}} e^{-|\cdot|^2/\lambda} * \frac{1}{|\cdot|^{n-2s}} \lesssim \frac{1}{|\cdot|^{n-2s}}.$$

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*Let* $0 < s < n/2$ *and let* $u$ *be a solution to the heat equation with initial data* $u_0 \in \dot{H}^s$*. Then*

$$dim\left\{ x \in \mathbb{R}^n \quad \lim_{t\to 0} u(t,x) \neq u_0(x) \right\} \leq n - 2s.$$

As we saw before, $u_0 \in \dot{H}^s$ can be singular on a set of dimension less than $n - 2s$ and so this is optimal.

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# Part 3:

# Convergence for the Schrödinger equation

# Lebesgue a.e. convergence for Schrödinger

Studied by many authors:

Carleson (1979), Dahlberg–Kenig (1982), Cowling (1983), Carbery (1985), Sjölin (1987), Vega (1988), Bourgain (1992/95), Moyua–Vargas–Vega (1996/99), Tao–Vargas (2000), Tao (2003), Lee (2006), Bourgain (2013).

Best known sufficient condition for Lebesgue a.e. convergence:

- $s \geq 1/4$ in dimension $n = 1$ (Carleson);
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**Theorem (Barceló–Bennett–Carbery–R.)**

*Let $n/4 \leq s < n/2$ and $\alpha > n - 2s$. Then, for all $\alpha$-dimensional $\mu$,*

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Again this is sharp in the range $s \geq n/4$.

We cannot go below this regularity in one dimension due to the necessary condition of Dahlberg–Kenig.

In the next section we will see that we cannot go below this regularity in higher dimensions either via a fractal version of the Lucà–R.-necessary condition.

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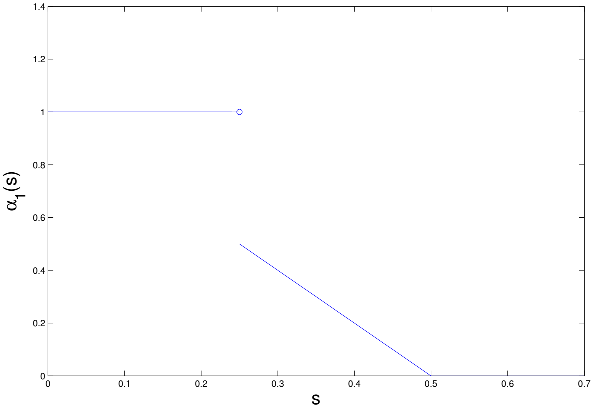
$$\alpha_n(s) := \sup_{u_0 \in H^s(\mathbb{R}^n)} \dim \left\{ \ x \in \mathbb{R}^n \ : \ \lim_{t \to 0} u(x,t) \neq u_0(x) \ \right\}$$

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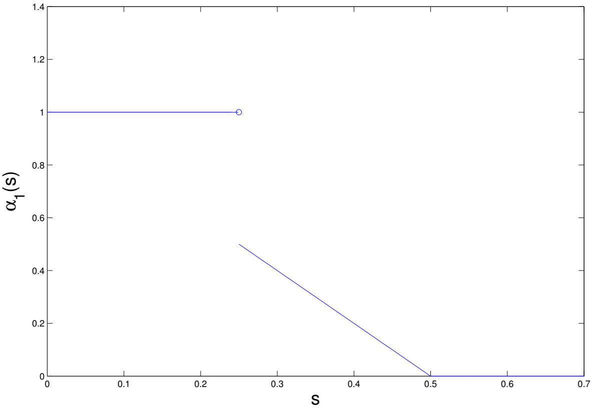


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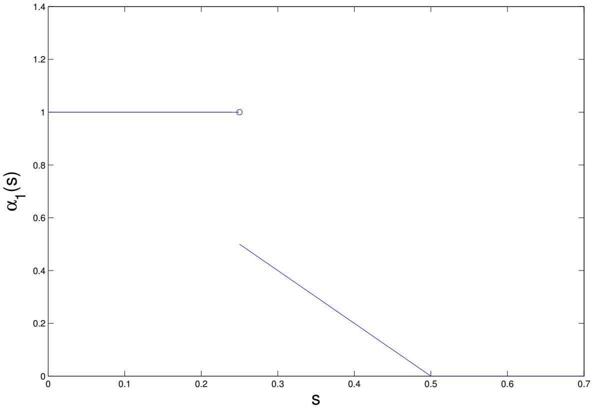


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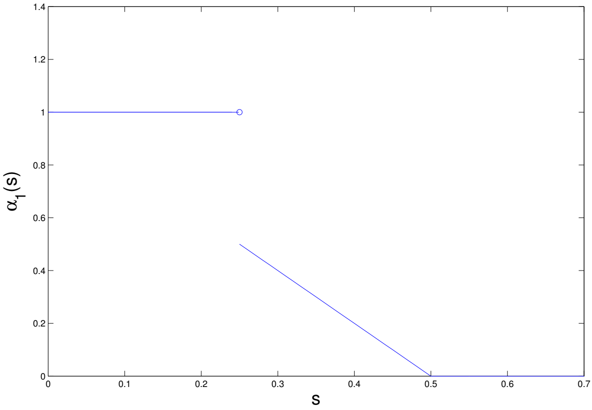


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# Part 4:

# Counterexample for the Schrödinger equation:

# lower bounds for $\alpha_n$

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**Theorem (Lucà–R.)**

$$\alpha_n(s) \geq \begin{cases} n & , \quad \textit{when} \qquad\qquad s \leq \frac{n}{2(n+2)} \\ n+1-\frac{2(n+2)s}{n}, & \quad \textit{when} \quad \frac{n}{2(n+2)} \leq s \leq \frac{n}{4} \\ n-2s & , \quad \textit{when} \quad \frac{n}{4} \leq s \leq \frac{n}{2} \\ 0 & , \quad \textit{when} \quad \frac{n}{2} \leq s \end{cases}$$

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# $\alpha_n(s) \geq n$ when $s < \frac{n}{2(n+2)}$

This bound follows from:

**Theorem (Lucà–R.)**

*Suppose that*

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**Lemma (Nikišin–Stein maximal principle)**

*Modulo endpoints:*

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*for all $u_0 \in H^s(\mathbb{R}^n)$ if and only if there is a constant $C$ such that*

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So it suffices to show that $s \geq \frac{n}{2(n+2)}$ is necessary for

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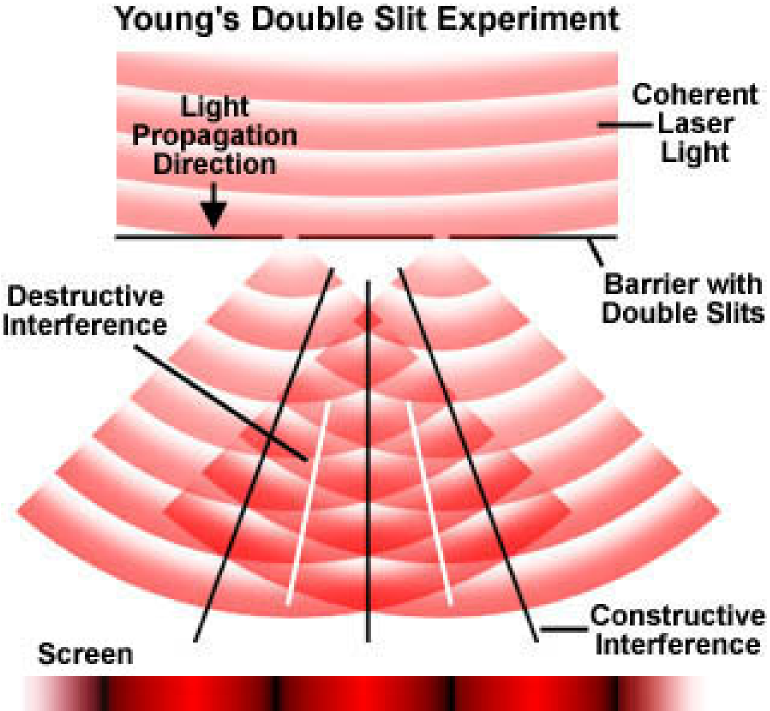
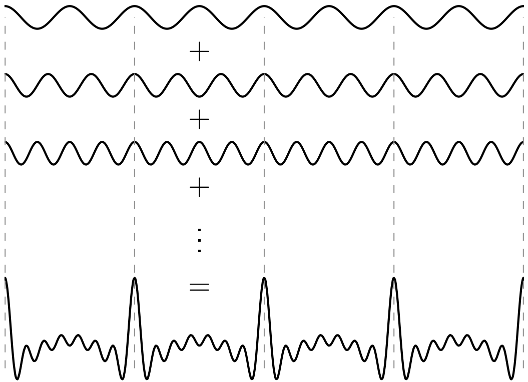

Young's Double Slit Experiment
Light Propagation Direction
Coherent Laser Light
Barrier with Double Slits
Destructive Interference
Constructive Interference
Screen

# Constructive interference



# The initial data

We consider the frequencies

$$\Omega := \left\{\xi \in 2\pi R^{1-\kappa}\mathbb{Z}^n \,:\, |\xi| \le R\right\} + B(0, \tfrac{1}{10}),$$

for $0 < \kappa < \frac{1}{n+2}$,

and initial data defined by

$$\widehat{f} = \frac{1}{\sqrt{|\Omega|}}\chi_\Omega, \quad \text{so that} \quad \|f\|_2 = 1.$$

This data was introduced in the context of Mattila's question by Barceló–Bennett–Carbery–Ruiz–Vilela (2007).

Note that

$$|\Omega| \simeq \text{number of frequencies} \simeq R^{n\kappa}.$$

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This data was introduced in the context of Mattila's question by Barceló–Bennett–Carbery–Ruiz–Vilela (2007).

Note that

$$|\Omega| \simeq \text{number of frequencies} \simeq R^{n\kappa}.$$

# The initial data

We consider the frequencies

$$\Omega := \left\{\xi \in 2\pi R^{1-\kappa}\mathbb{Z}^n \,:\, |\xi| \le R\right\} + B(0, \tfrac{1}{10}),$$

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The constructive interference reappears periodically in time:

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So that there is no cancellation in the integral:

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# Travelling periodically coherent solutions

Instead we take

$$f_\theta(x) = e^{i\frac{1}{2}\theta\cdot x} f(x), \quad \text{where} \quad \theta \in \mathbb{S}^{n-1}$$

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*Let $0 < \kappa < \frac{1}{n+2}$. Then there exists $\theta \in \mathbb{S}^{n-1}$ such that*

$$B(0, 1/2) \subset \bigcup_{t \in T} X + t\theta.$$

This is optimal in the sense that it is not true for $\kappa \geq \frac{1}{n+2}$.

After scaling and quotienting out $\mathbb{Z}^n$, this follows from quantitive ergodic theory on the torus $\mathbb{T}^n$.

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Plugging into the maximal estimate,

$$\left\| \sup_{0<t<1} |e^{it\Delta} f_\theta| \right\|_{L^2(B(0,1))} \lesssim R^s \|f_\theta\|_2,$$

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## $\alpha_n(s) \geq n + 1 - \frac{2(n+2)s}{n}$ when $\frac{n}{2(n+2)} \leq s \leq \frac{n}{4}$

This follows from:

**Theorem (Lucà–Rogers)**

*Let $n/2 \leq \alpha \leq n$ and suppose that, for all $u_0 \in H^s(\mathbb{R}^n)$,*

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The Nikišin–Stein maximal principle does not hold in this context, and so we first give a direct proof of the Lebesgue measure result.

We consider a sum of the previous initial data

$$f := \sum_{j>1} f_{\theta_j}, \qquad \theta_j \in \mathbb{S}^{d-1},$$

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Then if $s < \frac{n\kappa}{2} - \varepsilon$ we can sum so that $f \in H^s$.

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If $\kappa < \frac{1}{n+2}$, then $B(0, 1/2) \subset \bigcap_{j>1} E_j$, and we are done.

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$$\limsup_{j \to \infty} E_j := \bigcap_{j>1} \bigcup_{k>j} E_k$$

and prove that this is $\alpha$-dimensional.

For this we use that the limit is '$\alpha$–Hausdorff dense'.

# Falconer's density theorem

Consider the Hausdorff content $\mathcal{H}^\alpha_\infty$ defined by

$$\mathcal{H}^\alpha_\infty(E) := \inf\left\{ \sum_i \delta_i^\alpha \;:\; E \subset \bigcup_i B(x_i, \delta_i)\right\}.$$

**Theorem (Falconer (1985))**

*Suppose that, for all balls $B_r \subset B(0,1)$ of radius $r$,*

$$\liminf_{j\to\infty} \mathcal{H}^\alpha_\infty(E_j \cap B(x,r)) \geq cr^\alpha. \qquad (\dagger)$$

*Then* $\dim\left(\limsup_{j\to\infty} E_j\right) \geq \alpha$.

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# Part 5:
# Decay for the Fourier transform of fractal measures

$\widehat{\delta_{x_n=0}}\big(R(\overline{\xi},\xi_n)\big) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^{n-1}} e^{iR\overline{x}\cdot\overline{\xi}}\,d\overline{x}$ is independent of $\xi_n$.

Thus, the Fourier transform of certain $(n-1)$-dimensional measures do not decay in every direction.

But perhaps they decay on average......

Let $\beta_n(\alpha)$ denote the supremum of the numbers $\beta$ for which

$$\|\widehat{\mu}(R\,\cdot)\|^2_{L^2(\mathbb{S}^{n-1})} \lesssim c_\alpha(\mu)\|\mu\|R^{-\beta}$$

whenever $R > 1$ and $\mu$ is $\alpha$-dimensional and supported in $B(0,1)$.

**Question (Mattila (1987))**

*Who is $\beta_n(\alpha)$ ?*

Equivalently $\beta_n(\alpha)$ is the supremum of the numbers $\beta$ for which

$$\big\|(gd\sigma)^{\vee}(R\,\cdot)\big\|_{L^1(d\mu)} \lesssim \sqrt{c_\alpha(\mu)\|\mu\|}\,R^{-\beta/2}\|g\|_{L^2(\mathbb{S}^{n-1})}.$$

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# Previous results

$$\beta_2(\alpha) = \begin{cases} \alpha, & \alpha \in (0, 1/2], \\ 1/2, & \alpha \in [1/2, 1], \\ \alpha/2, & \alpha \in [1, 2], \end{cases}$$

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$$\beta_n(\alpha) \geq \begin{cases} \alpha, & \alpha \in (0, \frac{n-1}{2}], \\ \frac{n-1}{2}, & \alpha \in [\frac{n-1}{2}, \frac{n}{2}], \\ \alpha - 1 + \frac{n+2-2\alpha}{4}, & \alpha \in [\frac{n}{2}, \frac{n+2}{2}], \\ \alpha - 1, & \alpha \in [\frac{n+2}{2}, n], \end{cases}$$

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## Lemma (Bridging lemma)

*Let $u$ be a solution to $\partial_t u = i(-\Delta)^{m/2} u$ with initial data $u \in \dot{H}^s(\mathbb{R}^n)$ with $0 < s < n/2$. Then if $\beta_n(\alpha) > n - 2s$, then*

$$\dim \left\{ \, x \in \mathbb{R}^n \; : \; \lim_{t \to 0} u(x,t) \neq u_0(x) \, \right\} \leq \alpha.$$

**Proof:** It will suffice to prove, for all $\alpha$-dimensional $\mu$,

$$\Big\| \sup_{0<t<1} |e^{it(-\Delta)^{m/2}} f| \Big\|_{L^1(d\mu)} \lesssim C_\mu \|f\|_{\dot{H}^s}.$$

Writing $\widehat{f} = |\cdot|^{-s} \widehat{g}$ and using polar coordinates,

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$$\Big\|\sup_{t\in\mathbb{R}}|e^{it(-\Delta)^{m/2}}f|\Big\|_{L^1(d\mu)} \leq C_\mu\int_0^\infty \frac{R^{n-1-s}}{(1+R)^{\beta/2}}\|\widehat{g}(R\,\cdot\,)\|_{L^2(\mathbb{S}^{n-1})}dR.$$

Finally, by Cauchy–Schwarz,

$$\leq C_\mu\left(\int_0^\infty \frac{R^{n-1-2s}}{(1+R)^\beta}dR\right)^{1/2}\left(\int_0^\infty \|\widehat{g}(R\,\cdot\,)\|^2_{L^2(\mathbb{S}^{n-1})}R^{n-1}dR\right)^{1/2}$$

$$\leq C_\mu\,\|g\|_{L^2(\mathbb{R}^n)},$$

where for the final inequality we must take $\beta > n-2s$. $\square$

# Part 6:

# Convergence for the wave equation

Recall that, with initial data $u(\cdot,0)=u_0$ and $\partial_t u(\cdot,0)=u_1$, the solution to the wave equation satisfies

$$
\begin{aligned}
\widehat{u}(\xi) &= \cos(t|\xi|)\widehat{u_0}(\xi) && + \frac{\sin(t|\xi|)}{|\xi|}\widehat{u_1}(\xi) \\
&= \frac{1}{2}(e^{it|\xi|}+e^{-it|\xi|})\widehat{u_0}(\xi) && + \frac{1}{2}\frac{(e^{it|\xi|}-e^{-it|\xi|})}{i|\xi|}\widehat{u_1}(\xi) \\
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With this notation, we can write

$$u(\cdot,t) = e^{it(-\Delta)^{1/2}}f_+ + e^{-it(-\Delta)^{1/2}}f_-.$$

If the initial data is in $\dot{H}^s \times \dot{H}^{s-1}$, both $f_+$ and $f_-$ belong to $\dot{H}^s$.

Thus convergence of $e^{it(-\Delta)^{1/2}}f$ to $f$ for all $f\in\dot{H}^s$ implies convergence of $u(\cdot,t)$ to $u_0$ for all $(u_0,u_1)\in\dot{H}^s\times\dot{H}^{s-1}$

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## Corollary (of bridging lemma and Sjölin's estimate)

*Let $u$ be a solution to the Schrödinger equation with initial data in $\dot{H}^s$ or to the wave equation with initial data in $\dot{H}^s \times \dot{H}^{s-1}$. Then*

$$\dim \left\{ x \in \mathbb{R}^n \, : \, \lim_{t \to 0} u(x,t) \neq u_0(x) \right\} \leq n - 2s + 1.$$

**Proof:** By the result of Sjölin, $\beta(\alpha) \geq \alpha - 1$ so that $\beta(\alpha) > n - 2s$ as long as $\alpha > n - 2s + 1$. Thus, by the bridging lemma,

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**Theorem (Lucà–R.)**

*Let $n \geq 3$. Then*

$$\beta_n(\alpha) \geq \alpha - 1 + \frac{(n-\alpha)^2}{(n-1)(2n-\alpha-1)}.$$

This is an improvement in the range $n/2 + 1 \leq \alpha < n$.

The proof takes advantage of:

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Arigatou gozaimasu!