Existence of infinitely many homoclinic doubling bifurcations from some codimension three homoclinic orbits

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Abstract

An inclination-flip homoclinic orbit of weak type on $\mathbb{R}^3$ is a homoclinic orbit given as intersection of a special one-dimensional C^2-weak stable manifold and the one-dimensional unstable manifold of a hyperbolic singularity with three real eigenvalues. In this paper, we show that in a generic unfolding of such a homoclinic orbit, there appear curves in the parameter space that correspond to ordinary inclination-flip homoclinic orbit of order N for any integer N. As a consequence, there exist infinitely many homoclinic doubling bifurcation curves emanating from the codimension three degenerate point corresponding to the inclination-flip homoclinic orbit of weak type.

Key words

homoclinic orbit, homoclinic doubling bifurcation, inclination-flip of weak type, codimension

1 Introduction

It has been known that, under some conditions, an orbit homoclinic to a hyperbolic singularity can undergo a homoclinic doubling bifurcation like the well-known period doubling bifurcation for periodic orbits. Namely, it is a change

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of a homoclinic orbit into twice rounding one in a neighborhood of the original homoclinic orbit. The first such results have been obtained by [5] for the case of complex principal eigenvalues, and by [18] for the case of real principal eigenvalues.

In fact the homoclinic and period doubling bifurcations have some similarity for three-dimensional vector fields, because both of them are for a loop orbit which becomes doubled after the bifurcations. It has been discovered by Feigenbaum [6] and Coullet-Tresser [3] that the period doubling bifurcations can occur for a 2^n-periodic orbit giving rise to a 2^{n+1}-periodic orbit for any n successively, and the cascade of the period doubling bifurcations can accumulate with a geometric convergence rate at an onset of complicated dynamics, which is now known as the period-doubling route to chaos. The main discovery of Feigenbaum and Coullet-Tresser is that the way of accumulation is universal in the sense that the convergence rate does not depend on the choice of families of dynamical systems in which the cascade of period doubling bifurcations occur but just depends on the local geometry at the bifurcation point.

A natural question would then be asking if similar doubling bifurcations can occur infinitely many times for homoclinic orbits as well. Namely it is the bifurcation where a 2^n-homoclinic orbit turns to a 2^{n+1}-homoclinic orbit for any n successively, and the cascade of the homoclinic doubling bifurcations accumulate to an onset of complicated dynamics, analogously to the Feigenbaum-Coullet-Tresser phenomenon for the period doubling bifurcations. In fact, according to numerical experiments for a family of piecewise-linear vector fields, cascade of homoclinic doubling bifurcations can occur and does accumulate with a geometric convergence rate. See [9] for the details of the numerical experiments. This paper is a first attempt toward answering this question theoretically. Since the complex eigenvalue case seems to be rather well-understood and is not likely to have such accumulation of successive homoclinic doubling bifurcations in a way similar to the Feigenbaum-Coullet-Tresser case, here we focus on real eigenvalue case.

In the real eigenvalue case, several mechanisms for homoclinic doubling bifurcations have been known by [18], [2], [9], and [16]. All these homoclinic doubling bifurcations occur in a codimension 2 manner: we need two parameters to have such bifurcations. Furthermore, if we want to find successive homoclinic doubling bifurcations, we must trace the bifurcation branches globally in the 2-parameter space, which is in general a very difficult task. In order to get around this difficulty, we consider a similar problem in a local situation. Namely, we put the problem into a more degenerate situation that involves three parameters, and try to find homoclinic doubling bifurcations locally in a neighborhood of the degenerate point of codimension 3. To be more precise, we consider what is called the inclination-flip homoclinic orbit, a codimension 2 homoclinic orbit through which one type of homoclinic doubling bifurcation may undergo. We take, as a codimension 3 point, a degenerate version of inclination-flip homoclinic orbit, called that of weak type, and study bifurcations occurring in a neighborhood of this degenerate homoclinic orbit. In this paper, we shall show the existence of inclination-flip homoclinic orbits of order N for every integer N, namely, an inclination-flip homoclinic orbit rounding N times in a neighborhood of the original homoclinic orbit, which bifurcates from the inclination-flip homoclinic orbit of weak type.

We shall recall some definitions in order to state our main result. Let X be a smooth vector field on $\mathbb{R}^3$, O being a singularity of saddle type such that dX_O, the linearized vector field at the singularity, satisfies $dX_O = \lambda_u x \frac{\partial}{\partial x} - \lambda_{ss} y \frac{\partial}{\partial y} - \lambda_s z \frac{\partial}{\partial z}$, where $-\lambda_{ss} < -\lambda_s < 0 < \lambda_u$. Put $\alpha = \lambda_{ss}/\lambda_u$ and $\beta = \lambda_s/\lambda_u$. The stable (resp. unstable) manifold of the saddle is of dimension 2 (resp. 1), which we denote by $W^{s,ss}$ (resp. W^u). Also, there exists the one-dimensional strong stable manifold W^{ss}, which is a part of $W^{s,ss}$, whose tangent space at O is spanned by the eigenvector associated with $-\lambda_{ss}$.

There also exists an *extended unstable manifold* $W^{u,s}$: it is an invariant manifold whose tangent space at O is spanned by the eigenspaces associated with $-\lambda_s$ and λ_u. In general, this manifold is not unique and is only of C^1, but its tangent space along the unstable manifold do not depend on the choice of $W^{u,s}$. See [7] for the proof of existence and uniqueness of these manifolds.

Throughout we shall assume that X is of at least C^3 *uniformly linearizable*. More precisely, for any unfolding X_γ of X, $\gamma \in \mathbb{R}^d$, there exists a neighborhood $\mathcal{W}_O$ of the singularity and a C^3 diffeomorphism g_γ depending on the parameter in C^3 manner, such that for all γ near 0, $g_\gamma^*(X_\gamma) = d(X_\gamma)_O$ holds on $\mathcal{W}_O$. This property can be obtained if we consider some non-resonance conditions for the eigenvalues at the singularity. See [1], [15] for more details.

Under this condition, and under the eigenvalue condition $1/2 < \beta < \alpha < 1$ which we assume throughout the paper, there exists a unique C^2 *weak stable manifold*, which is an invariant one dimensional manifold tangent at O to the eigenspace associated with $-\lambda_s$. This manifold is given as follows: Using the local linearizing coordinates, any one dimensional invariant manifold in the local stable manifold, except the strong stable manifold, has the form: $z = cy^{\frac{\alpha}{\beta}}$ for some constant c. Since $\alpha/\beta < 2$ from the assumption, such a manifold can not be C^2 except in the case where $c = 0$, which is precisely the desired weak stable manifold, and hence the uniqueness of such a C^2 invariant manifold is guaranteed. See [13] for more details. We denote this manifold by W^s.

Definition 1.1. We say that Γ is an *inclination-flip homoclinic orbit* if it satisfies the following properties:

1. Γ is a homoclinic orbit;
2. $W^{ss} \cap \Gamma = \emptyset$;
3. $W^{s,ss}$ is tangent to $W^{u,s}$ along Γ.

Under the eigenvalue condition $1/2 < \beta < \alpha < 1$, we also say that Γ is an *inclination-flip homoclinic orbit of weak type* if moreover $\Gamma \cap W^s \neq \emptyset$.

Definition 1.2. Let X_γ, $\gamma \in \mathbb{R}^d$, be a smooth family of vector fields. We assume that, at $\gamma = 0$, the vector field X_0 has a homoclinic orbit Γ. We say that Γ_N is a *homoclinic orbit of order N bifurcating from* Γ if for a sufficiently small disc Σ through which Γ passes transversely, there exists a neighborhood $V_\Sigma \subset \mathbb{R}^d$ of 0 such that the vector field X_γ for some γ in V_Σ possesses a homoclinic orbit Γ_N with the property that $\Gamma_N \cap \Sigma$ consists of N points.

Note that this definition does not depend on the choice of Σ. Now, we state the main result of this paper.

Theorem 1.3. *Let X be a smooth vector field on $\mathbb{R}^3$, O being a singularity of saddle type such that the eigenvalues $-\lambda_{ss}$, $-\lambda_s$, λ_u at the singularity satisfy $1/2 < \beta = \lambda_s/\lambda_u < \alpha = \lambda_{ss}/\lambda_u < 1$. Moreover, we assume that X possesses an inclination-flip homoclinic orbit of weak type. Let X_γ, $\gamma \in \mathbb{R}^d$ $(d \geq 3)$, be a smooth generic unfolding of X. Then, for any integer N, there exists an inclination-flip homoclinic orbit Γ_N of order N bifurcating from Γ.*

Let us now recall some results concerning bifurcations from inclination-flip homoclinic orbits. Kisaka, Kokubu and Oka [9] studied unfoldings of an inclination-flip homoclinic orbit in the case where $1/2 < \beta < 1 < \alpha$, proving the existence of the homoclinic doubling bifurcation. Deng [4] observed that an inclination-flip homoclinic orbit can give rise to chaotic dynamics under some condition of eigenvalues. Homburg, Kokubu and Krupa [8] then proved the existence of suspension of the Smale's horseshoe in unfoldings of an inclination-flip homoclinic orbit with $\beta < 1/2$, $2\beta < \alpha$ and with quadratic tangency of $W^{s,ss}$ and W^u. They described completely how homoclinic orbits of order N are created and destroyed in the unfolding. Sandstede [17] showed the existence of a shift dynamics in the unfolding of an inclination-flip homoclinic orbit with $\alpha < 1$, $\alpha < 2\beta$ using Lin's methods [11].

Recently, the existence of Hénon-like strange attractors was proved in [14], using a result of Mora and Viana [12], in the case where $1 < \alpha + \beta,\ \beta < 1/2,\ K\beta < \alpha$ with some large enough K (in fact $K = 3$ is enough, see [13]). Moreover, Hénon-like attractors also appear in the unfolding of an inclination-flip homoclinic orbit of weak type when the eigenvalues at the singularity satisfy $1 < \alpha + \beta,\ \alpha < 2\beta < 1$.

It was proved in [13] that an inclination-flip homoclinic orbit of weak type with $\alpha < 2\beta < 1$ can be generically unfolded in a codimension 3 manner, and that suspension of the Smale's horseshoe appears in the unfolding of such a homoclinic orbit. Note that the mechanism of creating the suspended horseshoe is different from that in [17].

Sandstede also obtained various results concerning so-called the *orbit-flip homoclinic orbit*, another codimension 2 homoclinic orbit undergoing the homoclinic doubling bifurcation which is given as a connection between the strong stable manifold W^{ss} and the unstable manifold W^u. See [16] as well as [10].

This paper is organized as follows. In Section 2, we prepare some definition and notation for later use, and in Section 3, we prove the main theorem. Finally, we make some remarks concerning the successive homoclinic doubling bifurcations and accumulation of infinitely many homoclinic doublings in Section 4.

2 Preliminaries

Let X be a smooth vector field that possesses an inclination-flip homoclinic orbit Γ of weak type, let X_γ, $\gamma \in \mathbb{R}^d$ ($d \geq 3$), be a smooth generic unfolding of X, $X_0 = X$, and let $\mathcal{W}_O$ be a neighborhood of O where X_γ is C^3-linearizable. Up to some smooth change of coordinates, we may suppose that for all γ:

$$\begin{aligned} W^{s,ss}(\gamma) \cap \mathcal{W}_O &\subset \{(x,y,z) \in \mathbb{R}^3 \mid x = 0\}, \\ W^{s}(\gamma) \cap \mathcal{W}_O &\subset \{(x,y,z) \in \mathbb{R}^3 \mid y = 0,\ x = 0\}, \\ W^{u}(\gamma) \cap \mathcal{W}_O &\subset \{(x,y,z) \in \mathbb{R}^3 \mid y = 0,\ z = 0\}, \\ W^{s,u}(\gamma) \cap \mathcal{W}_O &\subset \{(x,y,z) \in \mathbb{R}^3 \mid y = 0\}. \end{aligned}$$

We define the following two cross sections, which are transverse to Γ,

$$\begin{aligned} \Sigma &= \{(x,y,z) \in \mathcal{W}_O \mid x = 1\}, \\ S^+ &= \{(x,y,z) \in \mathcal{W}_O \mid x \geq 0,\ z = 1\}. \end{aligned}$$

Let $f_\gamma : S^+ \to \Sigma;\ (x,y,1) \mapsto (1, f_Y, f_Z)$ and $h_\gamma : \Sigma \to S^+;\ (1,Y,Z) \mapsto (h_x, h_y, 1)$ be the local and global maps whose composition defines the Poincaré map along Γ associated to X_γ. As X_γ is locally linear in $\mathcal{W}_O$, we have $f_Y = yx^\alpha$ and $f_Z = x^\beta$. Moreover, we have the following expressions for h_γ:

$$h_x = \epsilon(\gamma) + a(\gamma)Y + \mu(\gamma)Z + Q_x(Y,Z), \tag{2.1}$$

$$h_y = \omega(\gamma) + c(\gamma)Y + d(\gamma)Z + Q_y(Y,Z), \tag{2.2}$$

where $\epsilon(0) = \mu(0) = \omega(0) = 0$, and $Q_x,\ Q_y$ are higher order terms. The family X_γ unfolds an inclination-flip homoclinic orbit of weak type if the map $\gamma \in \mathbb{R}^d \mapsto (\epsilon, \mu, \omega) \in \mathbb{R}^3$ is a submersion at 0. The parameter ϵ breaks the saddle connection, and μ, the tangency between $W^{s,u}$ and $W^{s,ss}$. Both parameters hence unfold the inclination-flip homoclinic orbit, and that of the "weak" type is obtained when $(\epsilon, \mu, \omega) = 0$. From now on, we may consider without loss of generality $\gamma = (\epsilon, \mu, \omega)$. Moreover, as the Poincaré return map needs to preserve the orientation, we have $a(0) \cdot d(0) < 0$. Note also that both Q_x and Q_y consist of higher order terms, hence at least quadratic.

Definition 2.1. For any integer $n \geq 1$, we define $h_{n,\gamma} : \Sigma \to S^+,\ h_{n,\gamma}(Y,Z) = (h_{n,x}(Y,Z), h_{n,y}(Y,Z))$, such that:

$$h_{1,\gamma} = h_\gamma, \quad h_{n+1,\gamma} = h_\gamma \circ f \circ h_{n,\gamma}.$$

Define also:

$$\epsilon_n(\gamma) = h_{n,x}(0,0), \quad \mu_n(\gamma) = \frac{\partial h_{n,x}}{\partial Z}(0,0), \quad \omega_n(\gamma) = h_{n,y}(0,0).$$

We will use the following notation for the matrix H_n associated to $Dh_n(0,0)$:

$$H_n = \begin{pmatrix} a_n & \mu_n \\ c_n & d_n \end{pmatrix}. \tag{2.3}$$

All the entries depend on the parameter and may not be bounded as one of the parameters tends to 0. With these definitions, we can say that X_γ has an inclination-flip homoclinic orbit of order n if and only if

$$\epsilon_n(\gamma) = 0 \text{ and } \mu_n(\gamma) = 0.$$

3 Proof of Main Theorem

For any integer n, we shall find a curve of inclination-flip homoclinic orbit of order n (abbrev. $\mathbf{IF}_n$) in the (ϵ, μ, ω)-space with the form:

$$\mu = M_n(\epsilon) \cdot \epsilon^{1-\beta}, \quad \omega = \Omega_n(\epsilon) \cdot \epsilon^{1-\alpha}, \tag{3.1}$$

where both $M_n(\epsilon)$ and $\Omega_n(\epsilon)$ are continuous functions and:

$$\lim_{\epsilon \to 0} M_n(\epsilon) < 0, \quad \lim_{\epsilon \to 0} \Omega_n(\epsilon) \neq 0. \tag{3.2}$$

We will assume that there exist functions $M_n(\epsilon)$ and $\Omega_n(\epsilon)$ satisfying (3.2) such that if both μ and ω satisfy (3.1), we have $\epsilon_n = \mu_n = 0$ and $\epsilon_i \neq 0,\ \mu_i \neq 0$ for all $0 < i < n$. More precisely if we define A_n and B_n as smooth functions on M, Ω and ϵ by

$$\epsilon_n = A_n(M, \Omega, \epsilon) \cdot \epsilon, \quad \mu_n = B_n(M, \Omega, \epsilon) \cdot \mu, \tag{3.3}$$

by introducing new auxiliary parameters M and Ω as

$$\mu = M \cdot \epsilon^{1-\beta}, \quad \omega = \Omega \cdot \epsilon^{1-\alpha}. \tag{3.4}$$

We intend to find a parametric expression $(M_n(\epsilon), \Omega_n(\epsilon))$ satisfying

$$A_n(M_n(\epsilon), \Omega_n(\epsilon), \epsilon) = B_n(M_n(\epsilon), \Omega_n(\epsilon), \epsilon) = 0 \text{ for } \forall \epsilon > 0,$$

and

$$\lim_{\epsilon \to 0} A_i(M_n(\epsilon), \Omega_n(\epsilon), \epsilon) > 0, \quad \lim_{\epsilon \to 0} B_i(M_n(\epsilon), \Omega_n(\epsilon), \epsilon) > 0, \tag{3.5}$$

for all $0 < i < n$.

Lemma 3.1. *If the auxiliary parameter Ω lies in a region given by $|\Omega| = |\omega \epsilon^{\alpha-1}| > K > 0$ for some constant K, then $\omega_n = \omega\{1 + O(\epsilon^{2\beta-1})\}$.*

Proof. By definition, we know that $\omega_n = h_{n,y}(0,0)$. Assume this holds up to an integer k, and show it for $k+1$. We have $\omega_{k+1} = h_y \circ f \circ h_k(0,0)$ and $h_k(0,0) = (\epsilon_k, \omega_k)$. Using (2.2), it follows that

$$\omega_{k+1} = \omega + c \cdot \epsilon_k^{\alpha} \omega_k + d \cdot \epsilon_k^{\beta} + Q_y(\epsilon_k^{\alpha} \omega_k, \epsilon_k^{\beta}).$$

As Q_x and Q_y are at least quadratic, it follows that for all (Y, Z) near $(0,0)$, there exists a constant $C > 0$ such that

$$\|Q_x(Y,Z), Q_y(Y,Z)\| < C\|Y, Z\|^2,$$

where $\|Y, Z\| = \sup\{|Y|, |Z|\}$. So, as $\alpha > \beta$, we have:

$$|Q_y(\epsilon_k^\alpha \omega_k, \epsilon_k^\beta)| < C\epsilon_k^{2\beta}. \tag{3.6}$$

(This inequality holds also for Q_x.) Using (3.3) and (3.4), we obtain

$$\left|\frac{\omega_{k+1} - \omega}{\omega}\right| < \left|\frac{c}{\Omega}\right| |A_k|^\alpha \epsilon^{2\alpha-1}|\omega_k| + \left|\frac{d}{\Omega}\right| |A_k|^\beta \epsilon^{\alpha+\beta-1} + \left|\frac{C}{\Omega}\right| |A_k|^{2\beta}\epsilon^{\alpha+2\beta}.$$

As $\omega_k = \omega\{1 + O(\epsilon^{2\beta-1})\}$ and as $1/2 < \beta < \alpha < 1$, it follows that

$$\left|\frac{\omega_{k+1} - \omega}{\omega}\right| < O(\epsilon^{2\beta-1}).$$

Here we have used that $|\Omega|$ is bounded away from 0. This ends the proof of the lemma.

Proposition 3.2. *Let M and Ω be the auxiliary parameters given by (3.4) with $|\Omega| > K$ for some $K > 0$, and define A_n and B_n by (3.3), then there exists a continuous function $f_n(M, \Omega, \epsilon)$ such that*

$$A_{n+1} = 1 + a\Omega A_n^\alpha + MA_n^\beta + f_n(M, \Omega, \epsilon)\epsilon^{2\beta-1}.$$

This equality will be used to obtain the first condition for $\mathbf{IF}_{n+1}$, namely $A_{n+1} = 0$.

Proof. We know that $h_{n+1,x}(0,0) = h_x \circ f \circ h_{n,\gamma}(0,0)$. Using (2.1), it follows that

$$\epsilon_{n+1} = h_{n+1,x}(0,0) = \epsilon + a\epsilon_n^\alpha\omega_n + \mu\epsilon_n^\beta + Q_x(\epsilon_n^\alpha\omega_n, \epsilon_n^\beta).$$

From (3.4) and Lemma 3.1, we have

$$\begin{aligned} \epsilon_{n+1} &= \epsilon + aA_n^\alpha\Omega\epsilon(1 + O(\epsilon^{2\beta-1})) + MA_n^\beta\epsilon + Q_x(\epsilon_n^\alpha\omega_n, \epsilon_n^\beta) \\ &= \epsilon\left\{1 + aA_n^\alpha\Omega + MA_n^\beta + \frac{1}{\epsilon}Q_x(\epsilon_n^\alpha\omega_n, \epsilon_n^\beta)\right\}. \end{aligned}$$

Using an estimate similar to (3.6), we finally obtain:

$$\epsilon_{n+1} = \epsilon\{1 + aA_n^\alpha\Omega + MA_n^\beta + O(\epsilon^{2\beta-1})\},$$

which is the desired expression.

We consider the situation at $\epsilon = 0$ and introduce the functions

$$\varphi_n : \mathbb{R}^+ \to \mathbb{R}, \quad \varphi_n(x) = A_n(-x, \frac{k_0}{a} x^{\frac{\alpha}{\beta}}, 0),$$

$$\psi_n : \mathbb{R}^+ \to \mathbb{R}^+, \quad \psi_n(x) = x\varphi_n(x)^\beta,$$

where $k_0 = \frac{\beta}{\alpha}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\beta}-1}$. Note that from Proposition 3.2, the function $\varphi_n(x)$ satisfies the functional equation as follows:

$$\varphi_{n+1}(x) = 1 + k_0\psi_n(x)^{\frac{\alpha}{\beta}} - \psi_n(x) = \varphi_2(\psi_n(x)). \tag{3.7}$$

Lemma 3.3. *There exists an increasing sequence $\{x_n\}_{n\geq 2}$ satisfying*

$$\varphi_n(x_n) = 0,\ \varphi_n'(x_n) = 0$$

$$\psi_n(x_n) = 0,\ \psi_n'(x_n) = 0$$

$$\varphi_n(x) > 0,\ \varphi_n'(x) > 0 \quad \text{for} \quad x > x_n$$

$$\psi_n(x) > 0,\ \psi_n'(x) > 0 \quad \text{for} \quad x > x_n.$$

Proof. We prove it by induction on n. It is easy to see that $x_2 = \frac{\alpha}{\alpha-\beta}$ satisfies $\varphi_2(x_2) = \varphi_2'(x_2) = 0$ and $\varphi_2'(x) > 0$ for $x > x_2$. From the definition of $\psi_2(x)$, we also have $\psi_2(x_2) = \psi_2'(x_2) = 0$ and $\psi_2(x) > 0$ for $x > x_2$. Recalling that $\varphi_{n+1}(x) = \varphi_2(\psi_n(x))$ from (3.7), the desired solution x_n ($n \geq 3$) is given as a solution to $\psi_{n-1}(x_n) = x_2 = \frac{\alpha}{\alpha-\beta}$. This gives an increasing sequence, since $\psi_{n-1}(x_{n-1}) = 0$ and $\psi_{n-1}(x)$ is a strictly increasing function for $x > x_{n-1}$ from the induction hypothesis. This sequence satisfies the conclusion, because

$$\varphi_n(x_n) = \varphi_2(\psi_{n-1}(x_n)) = \varphi_2(x_2) = 0,$$

$$\varphi_n'(x_n) = \varphi_2'(x_2)\psi_{n-1}'(x_n) = 0,$$

$$\varphi_n'(x) = \varphi_2'(\psi_{n-1}(x))\psi_{n-1}'(x) > 0 \quad \text{for} \quad x > x_n.$$

Here the last inequality follows since $\psi_{n-1}(x) > \psi_{n-1}(x_n) = x_2$ and hence $\varphi_2'(\psi_{n-1}(x)) > 0$. Similar inequalities hold for $\psi_n(x)$ as well. This completes the proof.

Remark 3.4. We can even show that the sequence $\{x_n\}$ converges. See §4.2.

Now, we shall derive a recursive formula for B_{n+1}. First, we need to estimate a part of the matrix H_n given by (2.3) for all n. Recall that we want to find

$\mathbf{IF}_{n+1}$, knowing the existence of $\mathbf{IF}_n$ for some values M_n and Ω_n. Let y_n denote $\frac{k_0}{a} x_n^{\frac{\alpha}{\beta}}$ from here on.

Lemma 3.5. *For M near $-x_n$ and Ω near y_n, d_n is bounded away from 0.*

Proof. We prove the lemma by induction on n, and so we suppose that there exists $n > 0$ such that d_n is bounded away from 0 as ϵ tends to 0 and show that d_{n+1} remains bounded as well, assuming (3.3)-(3.4) and (3.5) with all $0 < i < m+1$. We denote by $E(\epsilon)$ the matrix associated to $Dh_x(f \circ h_n(0,0))$. As $h_n(0,0) = (\epsilon_n, \omega_n)$, we have $f \circ h_n(0,0) = (\epsilon_n^\alpha \omega_n, \epsilon_n^\beta)$, and then, we get

$$E(M, \Omega, \epsilon) = \begin{pmatrix} a + \theta_1 & \mu + \theta_2 \\ c + \theta_3 & d + \theta_4 \end{pmatrix},$$

where each $\theta_j = \theta_j(M, \Omega, \epsilon)$ consists of higher order terms in E coming from Q_x and Q_y. Using (3.4), we have

$$|\theta_j| < O(\epsilon^\beta)$$

for any j. Now, we compute

$$H_{n+1} = E \cdot \begin{pmatrix} \alpha \epsilon_n^{\alpha-1} \omega_n & \epsilon_n^\alpha \\ \beta \epsilon_n^{\beta-1} & 0 \end{pmatrix} \cdot H_n.$$

A straightforward computation shows that

$$\mu_{n+1} = (a + \theta_1)(\alpha \epsilon_n^{\alpha-1} \omega_n \mu_n + d_n \epsilon_n^\alpha) + \beta(\mu + \theta_2) \mu_n \epsilon_n^{\beta-1}, \tag{3.8}$$

$$d_{n+1} = (c + \theta_3)(\alpha \epsilon_n^{\alpha-1} \omega_n \mu_n + d_n \epsilon_n^\alpha) + \beta(d + \theta_4) \mu_n \epsilon_n^{\beta-1}. \tag{3.9}$$

It follows from this together with (3.3)-(3.4) and Lemma 3.1 that

$$\begin{aligned} d_{n+1} &= (c + \theta_3)\{\alpha A_n^{\alpha-1} \epsilon^{\alpha-1} \omega (1 + O(\epsilon^{2\beta-1})) B_n \mu + d_n A_n^\alpha \epsilon^\alpha\} \\ &\quad + \beta(d + \theta_4) A_n^{\beta-1} B_n \mu \epsilon^{\beta-1}. \end{aligned}$$

Using (3.4), we finally have

$$\begin{aligned} d_{n+1} &= (c + \theta_3)\{\alpha A_n^{\alpha-1} B_n \Omega (1 + O(\epsilon^{2\beta-1})) \mu + d_n A_n^\alpha \epsilon^\alpha\} \\ &\quad + \beta(d + \theta_4) A_n^{\beta-1} B_n M. \end{aligned}$$

Therefore, since M near $-x_{n+1}$ and Ω near y_{n+1}, in which case $A_n(M, \Omega, \epsilon) > 0$ for $x_{n+1} > x_n$, we conclude that d_{n+1} is bounded. This ends the proof of the lemma.

Next we have a recursive formula for B_n as in the following proposition.

Proposition 3.6. *Let M be near $-x_n$ and Ω be near y_n, and define A_n and B_n by (3.3) for each n, then there exists a continuous function $g_n(M, \Omega, \epsilon)$ such that:*

$$B_{n+1} = \frac{B_n}{A_n}\left\{\alpha a \Omega A_n^\alpha + \beta M A_n^\beta + g_n(M, \Omega, \epsilon)\epsilon^{2\beta-1}\right\}.$$

This will give us another condition for $\mathbf{IF}_{n+1}$.

Proof. Using (3.8) we know that

$$\mu_{n+1} = (a+\theta_1)(\alpha\epsilon_n^{\alpha-1}\omega_n\mu_n + d_n\epsilon_n^\alpha) + \beta(\mu+\theta_2)\mu_n\epsilon_n^{\beta-1}.$$

From (3.3), (3.4), and Lemma 3.1, it follows that:

$$\mu_{n+1} = (a+\theta_1)\{\alpha A_n^{\alpha-1}\Omega(1+O(\epsilon^{2\beta-1}))\mu_n + d_n A_n^\alpha \epsilon^\alpha\} + \beta(\mu+\theta_2)\mu_n A_n^{\beta-1}\epsilon^{\beta-1}.$$

Moreover, as $|\theta_2| < O(\epsilon^{2\beta})$, it follows that there exists a continuous function $\tilde{\theta} = \tilde{\theta}(M, \Omega, \epsilon)$, such that $\mu + \theta_2 = \mu(1+\tilde{\theta})$, where

$$\left|\tilde{\theta}\right| < \left|\frac{1}{M}\right| O(\epsilon^{2\beta-1}).$$

Also, we have

$$d_n A_n^\alpha \epsilon^\alpha (a+\theta_1) = \frac{\mu_n d_n A_n^\alpha \epsilon^{\alpha+\beta-1}(a+\theta_1)}{B_n M}.$$

Using Lemma 3.5, we finally obtain

$$\mu_{n+1} = \frac{B_n}{A_n}\mu\{\alpha a A_n^\alpha \Omega + \beta M A_n^\beta + h_n(M, \Omega, \epsilon)\} \tag{3.10}$$

where

$$h_n(M, \Omega, \epsilon) = \tilde{\theta} + \frac{1}{B_n M} d_n A_n^\alpha \epsilon^{\alpha+\beta-1}(a+\theta_1).$$

We claim that $h_n(M, \Omega, \epsilon) = \epsilon^{2\beta-1} g_n(M, \Omega, \epsilon)$ for some continuous function $g_n(M, \Omega, \epsilon)$. First we consider the part $\alpha a A_n(M, \Omega, \epsilon)^\alpha \Omega + \beta M A_n(M, \Omega, \epsilon)^\beta$.

Let $M = -x,\ \Omega = \frac{k_0}{a} x^{\frac{\alpha}{\beta}}$ and $\epsilon = 0$, then

$$\begin{aligned} \alpha a A_n^\alpha \Omega + \beta M A_n^\beta &= \beta\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\beta}-1} x^{\frac{\alpha}{\beta}}\varphi_n(x)^\alpha - \beta x \varphi_n(x)^\beta \\ &= \beta\left\{\left(\frac{\alpha-\beta}{\alpha}\psi_n(x)\right)^{\frac{\alpha}{\beta}} - \left(\frac{\alpha-\beta}{\alpha}\psi_n(x)\right)\right\}. \end{aligned}$$

We can thus conclude that

$$\alpha a A_n^\alpha \Omega + \beta M A_n^\beta \Big|_{M=-x, \Omega=\frac{k_0}{a}x^{\frac{\alpha}{\beta}}, \epsilon=0} \geq 0 \tag{3.11}$$

if and only if

$$\psi_n(x) \geq x_2 = \frac{\alpha}{\alpha - \beta} \text{ or equivalently } x \geq x_{n+1}. \tag{3.12}$$

Next we shall prove, by induction, the following two assertions, both for M near $-x_n$, Ω near $y_n = \frac{k_0}{a} x_n^{\frac{\alpha}{\beta}}$, and ϵ small enough:

(i)$_n$ B_n is bounded away from 0;

(ii)$_n$ $h_n(M, \Omega, \epsilon) = \epsilon^{2\beta-1} g_n(M, \Omega, \epsilon)$ for some $g_n(M, \Omega, \epsilon)$.

Since $B_1 = 1$, the assertion (i)$_1$ follows trivially. Then the assertion (ii)$_1$ is also true, since

$$h_2(M, \Omega, \epsilon) = O(\epsilon^{2\beta-1}) + \frac{1}{M} d_1 \epsilon^{\alpha+\beta-1}(a + \theta_1).$$

As induction hypotheses, we assume (i)$_n$ and (ii)$_n$, and consider the case for $n+1$. From (3.10) and (ii)$_n$, we have

$$B_{n+1} = \frac{B_n}{A_n} \left\{ \alpha a \Omega A_n^\alpha + \beta M A_n^\beta + g_n(M, \Omega, \epsilon) \epsilon^{2\beta-1} \right\}.$$

Noting that now M is near $-x_{n+2}$ and Ω is near y_{n+2}, since we are considering the case for $n+1$. It then follows from the equivalence of (3.11) and (3.12) that B_{n+1} is bounded away from 0, and hence the assertion (i)$_{n+1}$ is proven. The assertion (ii)$_{n+1}$ now easily follows from (i)$_{n+1}$, since

$$h_{n+2}(M, \Omega, \epsilon) = O(\epsilon^{2\beta-1}) + \frac{1}{B_{n+1} M} d_{n+1} A_{n+1}^\alpha \epsilon^{\alpha+\beta-1}(a + O(\epsilon^\beta)).$$

We have completed the induction, and therefore the proof of Proposition 3.6.

Completion of the proof of Theorem 1.3: We have obtained the increasing sequence $\{x_n\}$ given by $\psi_{n-1}(x_n) = x_2$. Let $y_n = \frac{k_0}{\alpha} x_n^{\frac{\alpha}{\beta}}$, then from Lemma 3.3, they satisfy

$$A_n(-x_n, y_n, 0) = \varphi_n(x_n) = 0$$

and

$$A_i(-x_n, y_n, 0) = \varphi_i(x_n) > 0 \text{ for } 0 < i < n.$$

Furthermore, from the proof of Proposition 3.6, they also satisfy

$$B_n(-x_n, y_n, 0) = 0$$

and

$$B_i(-x_n, y_n, 0) > 0 \text{ for } 0 < i < n.$$

Therefore $(M, \Omega) = (-x_n, y_n)$ satisfies the condition of $\mathbf{IF}_n$ when $\epsilon = 0$. Now we use the implicit function theorem to

$$A_n(M, \Omega, \epsilon) = 0 \text{ and } B_n(M, \Omega, \epsilon) = 0$$

around $(M, \Omega, \epsilon) = (-x_n, y_n, 0)$ as follows: A straightforward computation gives

$$\begin{aligned}
&\det \left(\frac{\partial(A_{n+1}, B_{n+1})}{\partial(M, \Omega)} \right) \Bigg|_{(M,\Omega,\epsilon)=(-x_{n+1}, y_{n+1}, 0)} \\
&\quad = (\alpha - \beta) a B_n A_n^{\alpha+\beta-2} \left(A_n + \beta M \frac{\partial A_n}{\partial M} + \alpha \Omega \frac{\partial A_n}{\partial \Omega} \right) \Bigg|_{(M,\Omega,\epsilon)=(-x_{n+1}, y_{n+1}, 0)} \\
&\quad = \{\varphi_n(x_{n+1}) + \beta x_{n+1} \varphi_n'(x_{n+1})\} (\alpha - \beta) a \; B_n A_n^{\alpha+\beta-2} \Big|_{(M,\Omega,\epsilon)=(-x_{n+1}, y_{n+1}, 0)}.
\end{aligned}$$

From Lemma 3.3, we know that $\varphi_n(x_{n+1})$ and $\varphi_n'(x_{n+1})$ are positive, and hence that $\det \left(\frac{\partial(A_{n+1}, B_{n+1})}{\partial(M,\Omega)} \right) \Big|_{(M,\Omega,\epsilon)=(-x_{n+1}, y_{n+1}, 0)} > 0$. We thus obtain a solution $(M_n(\epsilon), \Omega_n(\epsilon), \epsilon)$ with

$$(M_n(0), \Omega_n(0), 0) = (-x_n, y_n, 0).$$

This gives a desired parametric expression (3.1) for $\mathbf{IF}_n$. The proof of Theorem 1.3 is thus completed.

4 Remarks

4.1 First Two Successive Homoclinic Doublings

In this subsection, we shall compute the first two *successive* homoclinic doubling bifurcations emanating from an inclination-flip homoclinic orbit of weak type. To be more precise, in the three dimensional parameter space of (ϵ, μ, ω) which generically unfolds an inclination-flip homoclinic orbit of weak type, we locate two curves of homoclinic doubling bifurcations corresponding to $\mathbf{IF}_1$ and $\mathbf{IF}_2$ that connect the bifurcation surfaces for 1-homoclinic, 2-homoclinic and

4-homoclinic orbits. This result together with the main theorem of this paper suggests the existence of cascade of infinitely many homoclinic doubling bifurcations from inclination-flip homoclinic orbits of weak type, which will be studied in more detail in a forthcoming paper. See also [10]. Here we only show simple calculations for the first two homoclinic doublings.

We consider a family of vector fields which satisfies all the properties given in Section 2. In particular, when $(\epsilon, \mu, \omega) = (0,0,0)$, it has an inclination-flip homoclinic orbit of weak type associated with the saddle singularity O. Moreover, when $\epsilon = 0$, the vector field possesses a homoclinic orbit and it is of non-twisted (resp. twisted) if and only if $\mu > 0$ (resp. $\mu < 0$). We start from this bifurcation surface $\mathbf{Hom}_1 = \{(\epsilon, \mu, \omega) \mid \epsilon = 0\}$ of homoclinic orbit, which we regard as 1-homoclinic orbits, and we shall trace successive homoclinic doubling bifurcations. The inclination-flip homoclinic orbits in $\mathbf{Hom}_1$ appear when $\epsilon = \mu = 0$, hence giving $\mathbf{IF}_1 = \{(\epsilon, \mu, \omega) \mid \epsilon = \mu = 0\}$, from which 2-homoclinic orbits bifurcate. Let us first compute the bifurcation surface $\mathbf{Hom}_2$ for the 2-homoclinic orbits.

In order to compute $\mathbf{Hom}_2$, we use notations introduced in Section 3, namely,

$$h_2 = h \circ f \circ h = (h_{2,x}, h_{2,y})$$

for the local map $f(x,y) = (x^\alpha y, x^\beta)$ and the global map $h(Y,Z)$ along the 1-homoclinic orbit. Define

$$\epsilon_2 = h_{2,x}(0,0), \quad \mu_2 = \frac{\partial h_{2,x}}{\partial Z}(0,0), \quad \omega_2 = h_{2,y}(0,0),$$

then $\mathbf{Hom}_2$ and $\mathbf{IF}_2$ are given by

$$\mathbf{Hom}_2 = \{(\epsilon, \mu, \omega) \mid \epsilon_2 = 0\}$$

and

$$\mathbf{IF}_2 = \{(\epsilon, \mu, \omega) \mid \epsilon_2 = \mu_2 = 0\}.$$

From the results in Section 3, we have

$$\epsilon_2 = A_2(M, \Omega, \epsilon)\epsilon = (1 + a\Omega + M + O(\epsilon^{2\beta-1}))\epsilon$$

and

$$\mu_2 = B_2(M, \Omega, \epsilon)\mu = (\alpha a \Omega + \beta M + O(\epsilon^{2\beta-1}))M \cdot \epsilon^{1-\beta},$$

where $\mu = M\epsilon^{1-\beta}$ and $\omega = \Omega\epsilon^{1-\alpha}$. Here we assume $a > 0$ (inward twisted case), the other case being treated similarly. Then the surface $\mathbf{Hom}_2$ given by the equation

$$\mu = -a\omega\epsilon^{\alpha-\beta} - \epsilon^{1-\beta} + h.o.t.$$

becomes as in Figure 4.1. In particular, 2-homoclinic orbits bifurcating from $\mathbf{IF}_1$ are twisted because $\mu_2 < 0$ if they are near the bifurcation. The curve $\mathbf{IF}_2$ is given by solving

$$1 + a\Omega + M + O(\epsilon^{2\beta-1}) = 0, \quad M(\alpha a\Omega + \beta M) + O(\epsilon^{2\beta-1}) = 0,$$

and hence

$$(a\Omega, M) \approx \left(\frac{\alpha}{\alpha-\beta}, -\frac{\beta}{\alpha-\beta}\right) \text{ or } (-1, 0).$$

These two solutions correspond to semi-curves $\mathbf{IF}_2^\pm$ in $\mathbf{Hom}_2$ emanating from the origin as indicated in Figure 4.1, and the 2-homoclinic orbits in the region in $\mathbf{Hom}_2$ bounded by $\mathbf{IF}_2^\pm$ are non-twisted, since $\mu_2 > 0$. From these two curves $\mathbf{IF}_2^\pm$ there must bifurcate 4-homoclinic orbits, and the bifurcation surface $\mathbf{Hom}_4$ can be computed in a similar way. We, however, stop the computation here, since it is more tedious to go further. We need a more systematic way of computing these curves if we want to show the successive homoclinic doubling bifurcations $\mathbf{IF}_{2^n}$, but this will be a subject of our future work.

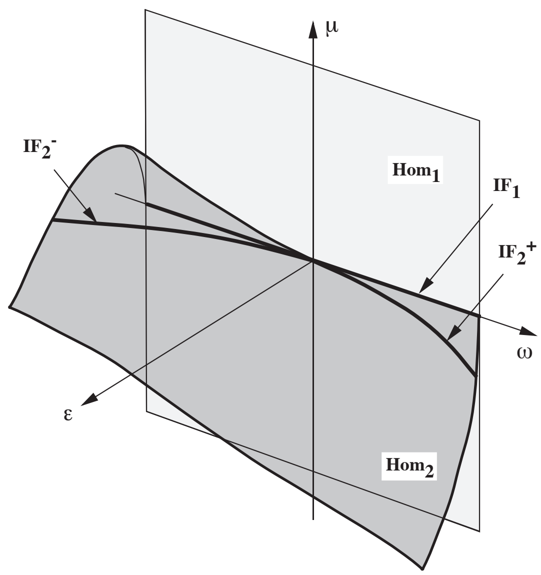

Figure 4.1: Bifurcation sets for $\mathbf{Hom}_i$ and $\mathbf{IF}_i$ $(1 = 1, 2)$.

4.2 Convergence of Curves of $\mathbf{IF}_n$

Now we are concerned with the convergence of the leading term of exponential expressions of curves $M_n(\epsilon)$ and $\Omega_n(\epsilon)$. In order to be complete, we really need to show that, for a given ϵ_0 close enough to 0, the sequences $M_n(\epsilon)$ and $\Omega_n(\epsilon)$ converge for all $0 < \epsilon < \epsilon_0$. Our goal here is to show that the sequence x_n converges as n tends to ∞, which then implies that the sequence y_n also does. This result gives us a hope that the curves of $\mathbf{IF}_n$ converge to some universal curve $\mathbf{IF}_\infty$ in the parameter space that corresponds to accumulation of inclination-flip homoclinic doubling bifurcations, where we may expect to have a sort of universality as that in the accumulation of period doubling bifurcations.

Proposition 4.1. *The sequence $\{x_n\}$ converges.*

Recall that the sequence $\{x_n\}$ is given by

$$\psi_{n-1}(x_n) = x_2 = \frac{\alpha}{\alpha - \beta}.$$

Since the sequence $\{x_n\}$ is increasing, it suffices to show that it is bounded. It is easy to see that ψ_2 has a fixed point p given by $\varphi_2(p) = 1$. This is also a fixed point of ψ_n for all n, since inductively we have

$$\varphi_n(p) = \varphi_2(\psi_{n-1}(p)) = \varphi_2(p) = 1.$$

From Lemma 3.3, we know that ψ_n is an increasing function on (x_n, ∞), and $\psi_n(x_n) = 0$. Proposition 4.1 follows from the next lemma.

Lemma 4.2.

$$p > x_n \quad \textit{for all} \quad n \geq 2.$$

Proof. First we show $p > x_2$. Indeed, since

$$p^{\frac{\alpha}{\beta}-1} = \frac{1}{k_0} \text{ where } k_0 = \frac{\beta}{\alpha}\left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\beta}-1},$$

it suffices to show

$$p^{\frac{\alpha}{\beta}-1} = \frac{1}{k_0} > x_2^{\frac{\alpha}{\beta}-1} \text{ or equivalently } k_0 < \left(\frac{\alpha-\beta}{\alpha}\right)^{\frac{\alpha}{\beta}-1},$$

which is trivially true as $\alpha > \beta$. Suppose there exists an integer n with $x_n \geq p$, then, since ψ_{n-1} is increasing,

$$\psi_{n-1}(x_n) = x_2 \geq \psi_{n-1}(p) = p,$$

which is absurd. Therefore we have $p > x_n$ for all $n \geq 2$, and this proves the convergence of the sequence $\{x_n\}$.

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Figure caption

Figure 4.1: Bifurcation sets for $\mathbf{Hom}_i$ and $\mathbf{IF}_i$ $(1 = 1, 2)$