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DIRECTIONAL TRANSITION MATRIX

HIROSHI KOKUBU
Department of Mathematics, Kyoto University
Kyoto 606-8502, Japan
E-mail: kokubu@kusm.kyoto-u.ac.jp

KONSTANTIN MISCHAIKOW
School of Mathematics, Georgia Institute of Technology
Atlanta, GA 30332, U.S.A.
E-mail: mischaik@math.gatech.edu

HIROE OKA
Department of Applied Mathematics and Informatics
Faculty of Science and Technology, Ryukoku University
Seta Otsu 520-2194, Japan
E-mail: oka@rins.ryukoku.ac.jp

Abstract. We present a generalization of topological transition matrices introduced in [6].

1. Introduction. This paper deals with connecting orbit problems for flows. Namely, given a flow on $\mathbb{R}^n$ having invariant sets called a repellor R and an attractor A, we are interested in the existence of an orbit whose α-limit set is contained in R and whose ω-limit set is contained in A. Such an orbit, if it exists, is called a *connecting orbit* from R to A. The Conley index theory [1, 2] provides us with a topological method for the connecting orbit problem. For this theory, one is assumed to have an isolated invariant set which contains the repeller and an attractor. The isolated invariant set S has an attractor-repeller decomposition if the attractor A and the repeller R are isolated invariant subsets in S and moreover it satisfies the following property: if there is an $x \in S \setminus (R \cup A)$, the orbit of x must be a connecting orbit from R to A, namely it holds that

$$S = R \cup A \cup C(R, A),$$

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where $C(R,A)$ is the union of all (possible) connecting orbits from R to A. Notice that the unions in the above equation are all disjoint. The simplest way of using Conley indices for detecting connecting orbits is in the following observation: suppose there is no connecting orbit from R to A, then S is the disjoint union of the attractor and the repeller, and hence, the corresponding Conley indices must be the direct sum:

$$CH^*(S) \cong CH^*(A) \oplus CH^*(R).$$

Therefore, if one can show that $CH^*(S)$ is not the direct sum $CH^*(A) \oplus CH^*(R)$, it implies that there must exist a connecting orbit from the repeller to the attractor.

This idea has been generalized to the theory of connection matrices [3]. If one has a Morse decomposition [1, 2] of an isolated invariant set S:

$$\mathcal{M} = \{M(p) \mid p \in \mathcal{P}\},$$

then the connection matrix Δ is given by the matrix representation of a degree one lower triangular homomorphism from the cohomology group $\bigoplus_{p\in\mathcal{P}} CH^*(M(p))$ to itself, such that its square is identically zero, and that, if it has a non-zero (p,q)-entry, there exists a sequence of connecting orbits from the Morse component $M(p)$ to $M(q)$. Note that since Conley indices remain the same under sufficiently small perturbation of the flow, connecting orbits detected by the above methods persist under perturbation. On the other hand, the connection matrix is not in general unique. See [9] for more detail.

Using ideas of Conley, Reineck [10] for the first time applied the idea of connection matrix to a broader class of connecting orbit problems, namely those of detecting connecting orbits which are not persistent under perturbation. The basic idea is to put a one-parameter family of flows into a slow-fast system by introducing an artificial slow drift in the parameter space. More precisely, he considered equation of the following form:

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= \varepsilon y(1-y). \end{aligned} \tag{1.1}$$

When $\varepsilon = 0$, this equation reduces to a one-parameter family of vector fields, whereas for $\varepsilon > 0$, the parameter y evolves slowly from $y = 0$ at $t = -\infty$ to $y = 1$ at $t = +\infty$. Suppose the parametrized system at $\varepsilon = 0$ has an isolated invariant set S_y for each $y \in [0,1]$ which continues over the parameter interval $[0,1]$, together with its Morse decomposition

$$\mathcal{M}_y = \{M_y(p) \mid p \in \mathcal{P}\}.$$

One can then consider the entire system with small $\varepsilon > 0$ and show that the connection matrix for (1.1) has the following decomposition:

$$\Delta = \begin{pmatrix} \Delta_0 & 0 \\ T & \Delta_1 \end{pmatrix},$$

where Δ_j is the connection matrix for the Morse decompositions at $y = j = 0, 1$, respectively. If one has a nonzero (p,q)-entry of the submatrix T above, then the system with $\varepsilon > 0$ possesses a connecting orbit from $M_0(p)$ to $M_1(q)$ for any $\varepsilon > 0$, since it is a part of the connection matrix Δ. Moreover, one can show from this that the connecting orbit even converges to a connected invariant subset in the parametrized system with $\varepsilon = 0$ in the Hausdorff metric as $\varepsilon \to 0$. This connected invariant subset contains $M_0(p)$ and $M_1(q)$, and hence one concludes that the parametrized system at $\varepsilon = 0$ has an increasing sequence of parameter values $\{y_i\}_{i=0}^{k+1}$ with $y_0 = 0, y_{k+1} = 1$, and connecting orbits from

$M_{y_i}(p_i)$ to $M_{y_i}(p_{i+1})$ for $i = 1, \ldots, k$ with $p_1 = p,\ p_{k+1} = q$, thereby showing the existence of connecting orbits which are not in general persistent under perturbation. This submatrix T is called a (singular) *transition matrix.*

One disadvantage of this formulation of the transition matrix is that it depends (at least formally) on the form of the slow parameter drift, although in general the slow parameter drift should be irrelevant to the form of connection matrices as well as the existence of connecting orbits which can be detected by these methods. In order to remove the artificial dependence on the slow parameter drift, McCord and Mischaikow [6] introduced the notion of topological transition matrix. The topological transition matrix can be defined only from the parametrized system at $\varepsilon = 0$, and detects the change of the topological nature of connecting orbits among Morse sets when the parameter varies from $y = 0$ to $y = 1$. More precisely, the topological transition matrix is defined as follows:

Since each Morse set $M_y(p)$ continues over $[0, 1]$, there are continuation isomorphisms

$$F^*_{1,0}(p) : CH^*(M_1(p)) \to CH^*(M_0(p)).$$

Similarly, since S_y continues over $[0, 1]$ there is an isomorphism

$$F^*_{1,0}(S) : CH^*(S_1) \to CH^*(S_0).$$

If $S_y = \bigcup_{p\in\mathcal{P}} M_y(p)$, i.e. the set of connecting orbits is empty at y, then there exists an index isomorphism

$$\Phi^*_y : CH^*(S_y) \to \bigoplus_{p\in\mathcal{P}} CH^*(M_y(p)). \tag{1.2}$$

Suppose there are no connections at either $y = 0$ or $y = 1$, then we can construct the following diagram

$$\begin{array}{ccc}
\bigoplus\limits_{p\in\mathcal{P}} CH^*(M_1(p)) & \xrightarrow{\bigoplus\limits_{p\in\mathcal{P}} F^*_{1,0}(p)} & \bigoplus\limits_{p\in\mathcal{P}} CH^*(M_0(p)) \\
\Phi^*_1 \uparrow & & \Phi^*_0 \uparrow \\
CH^*(S_1) & \xrightarrow{F^*_{1,0}(S)} & CH^*(S_0)
\end{array}$$

Even though every map is an isomorphism this diagram is not, in general, commutative. Furthermore, it is the failure of commutativity that gives information concerning connecting orbits. The *topological transition matrix*

$$T^{1,0} : \bigoplus_{p\in\mathcal{P}} CH^*(M_1(p)) \to \bigoplus_{p\in\mathcal{P}} CH^*(M_0(p))$$

is defined by

$$T^{1,0} = \Phi^*_0 \circ F^*_{1,0}(S) \circ (\Phi^*_1)^{-1}.$$

Note that the diagram

$$\begin{array}{ccc}
\bigoplus\limits_{p\in\mathcal{P}} CH^*(M_1(p)) & \xrightarrow{T^{1,0}} & \bigoplus\limits_{p\in\mathcal{P}} CH^*(M_0(p)) \\
\Phi^*_1 \uparrow & & \Phi^*_0 \uparrow \\
CH^*(S_1) & \xrightarrow{F^*_{1,0}(S)} & CH^*(S_0)
\end{array} \tag{1.3}$$

commutes by definition. The topological transition matrix is lower triangular and shares the same property as the singular transition matrix, namely its off diagonal nonzero entry implies the existence of connecting orbits between appropriate Morse sets for various $y \in (0,1)$. See [6] for more details.

Furthermore McCord and Mischaikow [7] showed the equivalence of the singular and topological transition matrices. It implies that the change of the connecting orbit structure from $y = 1$ to $y = 0$ in the system (1.1) with $\varepsilon < 0$ is given by the inverse of the singular transition matrix for $\varepsilon > 0$. This is proven by going through the topological transition matrix for which the inverse operation is well-defined and makes a good sense, whereas it cannot be directly applied to singular transition matrices since the isolation of the system is completely lost at $\varepsilon = 0$. This fact was used to show the existence of infinitely many connecting orbits of a slow-fast system. See [5].

In all these cases, the transition matrices provide information about how connecting orbit structure changes as the parameter y moves in one direction, say from $y = 0$ to $y = 1$. In this paper we want to extend the applicability of the idea of transition matrices to even broader class of problems. We consider the slow-fast systems of the form

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= \varepsilon g(x,y), \end{aligned} \tag{1.4}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$. Notice that this form of the slow-fast system is more general than (1.1) in that the equation for the slow variable y depends also on the fast variable x and hence, for $\varepsilon > 0$, different Morse components may have different directions of slow drift. We assume that when $\varepsilon = 0$ the parametrized system has an isolated invariant set S_y for each y that continues over the interval $[0,1]$ in the y-space, together with the Morse decomposition

$$\mathcal{M}_y = \{M_y(p) | p \in \mathcal{P}\}$$

parametrized by $y \in \mathbb{R}$. We assume that $g(M_y(p), y) \neq 0$ for any $y \in (0,1)$ and any $p \in \mathcal{P}$, but do not assume that the slow dynamics introduced when $\varepsilon > 0$ goes in the same direction for the Morse components, and we define the notion of box as follows.

Definition 1.1. A set $\mathcal{B}$ is a *box* if:

(1) There exists an isolating neighborhood $\mathcal{B} \subset \mathbb{R}^n \times [0,1]$ for the parameterized flow $\psi^{\mathcal{B}}$ defined by

$$\begin{aligned} \psi^{\mathcal{B}} : \mathbb{R} \times \mathbb{R}^n \times [0,1] &\to \mathbb{R}^n \times [0,1], \\ (t,x,y) &\mapsto (\psi_y(t,x), y), \end{aligned}$$

where ψ_y is the flow of $\dot{x} = f(x,y)$ with fixed y.

(2) Let $S(\mathcal{B}) := \mathrm{Inv}(\mathcal{B}, \psi^{\mathcal{B}})$. There exists a Morse decomposition

$$\mathcal{M}(S(\mathcal{B})) := \{M(p,\mathcal{B}) \mid p = 1, \ldots, P_{\mathcal{B}}\},$$

with the usual ordering on the integers as the admissible ordering. Let $\mathcal{B}_y = \mathcal{B} \cap (\mathbb{R}^n \times \{y\})$, $S_y(\mathcal{B}) := \mathrm{Inv}(\mathcal{B}_y, \psi_y)$ and let $\{M_y(p,\mathcal{B}) \mid p = 1, \ldots, P_{\mathcal{B}}\}$ be the corresponding Morse decomposition of $S_y(\mathcal{B})$. Then

$$S_0(\mathcal{B}) := \bigcup_{p=1}^{P_{\mathcal{B}}} M_0(p,\mathcal{B}) \quad \text{and} \quad S_1(\mathcal{B}) := \bigcup_{p=1}^{P_{\mathcal{B}}} M_1(p,\mathcal{B}).$$

(3) There are isolating neighborhoods $V(p,\mathcal{B})$ for $M(p,\mathcal{B})$ such that

$$V(p,\mathcal{B}) \subset \mathcal{B} \quad \text{and} \quad V(p,\mathcal{B}) \cap V(q,\mathcal{B}) = \emptyset$$

for $p \neq q$ with $p,q = 1,\ldots,P_{\mathcal{B}}$, and for every $y \in [0,1]$

$$V_y(p,\mathcal{B}) \subset \mathrm{Int}(\mathcal{B}_y).$$

Furthermore, there are $\delta(p,\mathcal{B}) \in \{\pm 1\}$, $p = 1,\ldots,P_{\mathcal{B}}$, such that

$$\delta(p,\mathcal{B}) g(x,y) > 0 \qquad \text{for all } (x,y) \in V(p,\mathcal{B})$$

From the last property, one can decompose the finite index set of the Morse decomposition as

$$\mathcal{P} = \mathcal{P}_+ \cup \mathcal{P}_-$$

where

$$\mathcal{P}_\pm = \{p \in \mathcal{P} \mid \pm\delta(p) > 0\},$$

and accordingly, one can define $M_{in}(p,\mathcal{B})$ and $M_{out}(p,\mathcal{B})$ as follows:

$$M_{in}(p,\mathcal{B}) = \begin{cases} M_0(p,\mathcal{B}) & \text{if } p \in \mathcal{P}_+, \\ M_1(p,\mathcal{B}) & \text{if } p \in \mathcal{P}_-; \end{cases} \tag{1.5}$$

$$M_{out}(p,\mathcal{B}) = \begin{cases} M_1(p,\mathcal{B}) & \text{if } p \in \mathcal{P}_+, \\ M_0(p,\mathcal{B}) & \text{if } p \in \mathcal{P}_-. \end{cases} \tag{1.6}$$

Notice that there are no connecting orbits among the Morse sets at $y = 0$ and at $y = 1$, and by the construction, the sets $S_0(\mathcal{B})$ and $S_1(\mathcal{B})$ are related by continuation. A box with bi-directional slow dynamics can naturally occur in various problems, for instance, in the FitzHugh-Nagumo equation. See [4] for more explanation. For this situation, either singular or topological transition matrix is not useful since they are both essentially uni-directional. In order to illustrate the difficulty, let us consider a variant of the connecting orbit problem studied in [6]. See Figure 1. Consider a one-parameter family of planar vector fields with three Morse sets indexed by 1,2,3 with the admissible ordering $3 > 2 > 1$, which continue over the parameter interval $[0,1]$. In transition from $y = 0$ to $y = 1$, there is a chance of connections among these Morse components which may be detected by the topological transition matrix. However, if the slow dynamics on each of the Morse sets is as indicated in Figure 1, none of the known transition matrices can provide us with information about orbits connecting, say $M_0(3)$ and $M_1(1)$, since the middle Morse component goes in the opposite direction.

The goal of this paper is to set up a version of transition matrix called *directional transition matrix* given as an isomorphism

$$D : \bigoplus_{p\in\mathcal{P}} CH^*(M_{out}(p)) \to \bigoplus_{p\in\mathcal{P}} CH^*(M_{in}(p)),$$

whose non-zero (p,q)-entry implies the existence of connecting orbits from $M_{in}(p)$ to $M_{out}(q)$. In the next section, we give a definition of the directional transition matrix and show that it has the desired property. The essential part of the proof relies on our recent results [4]. In Section 3, we illustrate how the directional transition matrix can be applied to the situation as in Figure 1 and be used to detect connecting orbits.

2. Directional transition matrix. Recall that the topological transition matrix

$$T : \bigoplus_{p\in\mathcal{P}} CH^*(M_1(p)) \to \bigoplus_{p\in\mathcal{P}} CH^*(M_0(p))$$

is lower triangular. Namely, if its (p,q)-entry $T(p,q)$ is nonzero, then $p > q$ with respect to an admissible ordering of the index set $\mathcal{P}$ for the Morse decomposition of a box in the slow-fast system (1.4). This means that there exist a finite increasing sequence $\{y_i\}_{i=1}^k$ in $[0,1]$ and a sequence $\{p_i\}_{i=1}^k$ in $\mathcal{P}$ satisfying

$$p = p_1 > p_2 > \ldots > p_k = q$$

such that the corresponding parametrized flow at $y = y_i$ has a connecting orbit from $M_{y_i}(p_i)$ to $M_{y_i}(p_{i+1})$.

We shall show an analogous but more general statement for the existence of connecting orbits from a nonzero entry of the directional transition matrix. Let us begin by giving a precise definition of the directional transition matrix.

Lemma 2.1. *Let V, V' and W, W' are mutually isomorphic finitely generated free Abelian groups, and let*

$$A : V \oplus W \to V' \oplus W'$$

be an isomorphism. Suppose A is lower triangular with the following block decomposition:

$$A = \begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$$

where $X : V \to V'$ and $Z : W \to W'$ are isomorphisms, then the following maps are all lower triangular isomorphisms:

$$\begin{aligned} A_1 &= \begin{pmatrix} X & 0 \\ -Z^{-1}Y & Z^{-1} \end{pmatrix} &&: \quad V \oplus W' \to V' \oplus W, \\ A_2 &= \begin{pmatrix} X^{-1} & 0 \\ YX^{-1} & Z \end{pmatrix} &&: \quad V' \oplus W \to V \oplus W', \\ A_3 &= \begin{pmatrix} X^{-1} & 0 \\ -Z^{-1}YX^{-1} & Z^{-1} \end{pmatrix} &&: \quad V' \oplus W' \to V \oplus W. \end{aligned}$$

The proof is straightforward. Since the topological transition matrix is lower triangular with respect to an admissible ordering, one can repeatedly apply the above lemma and obtain an isomorphism

$$D : \bigoplus_{p\in\mathcal{P}} CH^*(M_{out}(p)) \to \bigoplus_{p\in\mathcal{P}} CH^*(M_{in}(p)).$$

The matrix representation of this isomorphism, called the *directional transition matrix*, has the following property.

Theorem 2.2. *Let D be the directional transition matrix for a box in the slow-fast system (1.4). If its (p,q)-entry $D(p,q)$ is nonzero, then there exist a finite sequence $\{y_i\}_{i=1}^{k+1}$ in $[0,1]$ and a sequence $\{p_i\}_{i=1}^k$ in $\mathcal{P}$ satisfying*

$$\delta(p_{i+1})(y_{i+1} - y_i) > 0 \textit{ for all } i = 1, \ldots, k-1$$

and

$$p = p_1 > p_2 > \ldots > p_k > p_{k+1} = q$$

such that the corresponding parametrized flow at $y = y_i$ has a connecting orbit from $M_{y_i}(p_i)$ to $M_{y_i}(p_{i+1})$.

This theorem is proven by applying the TBC collection theorem in [4, Theorem 1.10]. Let us first recall some definitions from [4].

DEFINITION 2.3. $\mathcal{T}$ is a *tube* if:

(1) There exists an interval $[a, b]$ such that $\mathcal{T} \subset \mathbb{R}^n \times [a, b]$ and $\mathcal{T}$ is an isolating neighborhood for

$$\begin{array}{rcl} \psi^{\mathcal{T}} : \mathbb{R} \times \mathbb{R}^n \times [a,b] & \to & \mathbb{R}^n \times [a,b], \\ (t,x,y) & \mapsto & (\psi_y(t,x), y). \end{array}$$

(2) There exists $\delta(\mathcal{T}) \in \{\pm 1\}$ such that for all $(x, y) \in \mathcal{T}$ we have

$$\delta(\mathcal{T}) g(x, y) > 0.$$

DEFINITION 2.4. A set $\mathcal{C}(R)$ ($\mathcal{C}(A)$) is a *repelling (attracting) cap* if:

(1) There exists an interval $[e, f]$ such that $\mathcal{C} \subset \mathbb{R}^n \times [e, f]$ and $\mathcal{C}$ is an isolating neighborhood for

$$\begin{array}{rcl} \psi^{\mathcal{C}} : \mathbb{R} \times \mathbb{R}^n \times [e,f] & \to & \mathbb{R}^n \times [e,f] \\ (t,x,y) & \mapsto & (\psi_y(t,x), y) \end{array}$$

(2)

$$\begin{array}{rcl} x \in \mathcal{C}_e(R) & \Rightarrow & g(x,e) < 0 \\ x \in \mathcal{C}_f(R) & \Rightarrow & g(x,f) > 0 \\ x \in \mathcal{C}_e(A) & \Rightarrow & g(x,e) > 0 \\ x \in \mathcal{C}_f(A) & \Rightarrow & g(x,f) < 0, \end{array}$$

where $\mathcal{C}_y(R) := \mathcal{C}(R) \cap \{y\}$ and $\mathcal{C}_y(A) := \mathcal{C}(A) \cap \{y\}$.

The following definition is a special case of the TBC collections defined in [4]. In this paper we only need the case where the number of boxes is one (and hence the number of tubes is two), hence the definition is adapted to such cases for simplicity.

DEFINITION 2.5. A *tubes, box and caps collection* (TBC collection) is a collection of tubes $\mathcal{T}(1), \mathcal{T}(2)$, a box $\mathcal{B} \subset \mathbb{R}^n \times [0, 1]$, and caps $\mathcal{C}(R)$ and $\mathcal{C}(A)$ such that:

(1) (a) $\mathcal{T}(1) \cap (\mathbb{R}^n \times [0, 1]) \subset V(1, \mathcal{B})$ and $\mathcal{T}(1) \cap \mathcal{B})$ isolates $M(1)$.
  (b) $\mathcal{T}(2) \cap (\mathbb{R}^n \times [0, 1]) \subset V(P_{\mathcal{B}}, \mathcal{B})$ and $\mathcal{T}(2) \cap \mathcal{B}$ isolates $M(P_{\mathcal{B}})$.

(2) either

$$\delta(\mathcal{T}(2)) > 0 \text{ and } \delta(P_{\mathcal{B}}, \mathcal{B}) > 0 \text{ in which case } b_2 = 1$$

or

$$\delta(\mathcal{T}(2)) < 0 \text{ and } \delta(P_{\mathcal{B}}, \mathcal{B}) < 0 \text{ in which case } a_2 = 0$$

where a, b are as in Definition 2.3.

(3) either
$$\delta(\mathcal{T}(1)) > 0 \text{ and } \delta(1, \mathcal{B}) > 0 \text{ in which case } a_1 = 0$$
or
$$\delta(\mathcal{T}(1)) < 0 \text{ and } \delta(1, \mathcal{B}) < 0 \text{ in which case } b_1 = 1$$
where a, b are as in Definition 2.3.

(4) $\mathcal{C}(R) \cap \mathcal{T}(2) \neq \emptyset$ and $\mathcal{C}(A) \cap \mathcal{T}(1) \neq \emptyset$. Furthermore,
$$\begin{aligned} \mathcal{C}(R) \cap \mathcal{T}(2) \cap (\mathbb{R}^n \times \{y\}) \neq \emptyset \quad &\Rightarrow \quad \mathcal{C}_y(R) = \mathcal{T}_y(2), \\ \mathcal{C}(A) \cap \mathcal{T}(1) \cap (\mathbb{R}^n \times \{y\}) \neq \emptyset \quad &\Rightarrow \quad \mathcal{C}_y(A) = \mathcal{T}_y(1). \end{aligned}$$

Given a TBC collection, let
$$D(P_{\mathcal{B}}, 1) : CH^*(M_{out}(1, \mathcal{B})) \to CH^*(M_{in}(P_{\mathcal{B}}, \mathcal{B})) \tag{2.1}$$
denote the $(P_{\mathcal{B}}, 1)$-entry of the directional transition matrix of the box $\mathcal{B}$. The following theorem is the special case of Theorem 1.10 in [4]. See [4] for its complete statement as well as the proof.

Theorem 2.6. *Let $\mathcal{T}(1), \mathcal{T}(2)$, $\mathcal{C}(R), \mathcal{C}(A)$, and $\mathcal{B}$ be a TBC collection for the slow-fast system (1.4). Let*
$$\mathcal{N} := \mathcal{B} \cup \mathcal{T}(1) \cup \mathcal{T}(2) \cup \mathcal{C}(R) \cup \mathcal{C}(A).$$
Then, for $\varepsilon > 0$ sufficiently small,

(1) $\mathcal{N}$ is an isolating neighborhood for the flow φ^ε generated by (1.4);

(2) $(\mathrm{Inv}(\mathcal{C}(R), \varphi^\varepsilon), \mathrm{Inv}(\mathcal{C}(A), \varphi^\varepsilon))$ is an attractor-repeller pair for $\mathrm{Inv}(\mathcal{N}, \varphi^\varepsilon)$;

(3) If $D(P_{\mathcal{B}}, 1) \neq 0$, then
$$CH^*(\mathrm{Inv}(\mathcal{N}, \varphi^\varepsilon)) \not\cong CH^*(\mathrm{Inv}(\mathcal{C}(A), \varphi^\varepsilon)) \oplus CH^*(\mathrm{Inv}(\mathcal{C}(R), \varphi^\varepsilon)).$$
Therefore, for all sufficiently small $\varepsilon > 0$ there is a connecting orbit from $\mathrm{Inv}(\mathcal{C}(R), \varphi^\varepsilon)$ to $\mathrm{Inv}(\mathcal{C}(A), \varphi^\varepsilon)$ in $\mathcal{N}$ under the flow φ^ε.

In order to prove Theorem 2.2, one can apply the above theorem as follows: Suppose the directional transition matrix D associated to a box $\mathcal{B}$ has a nonzero (p, q)-entry. Then one can modify the slow-fast system outside the box in such a way that one can attach tubes as well as attracting and repelling caps to Morse components $M_{in}(p)$ and $M_{out}(q)$. Clearly the smallest interval in $\mathcal{P}$ that contains p, q gives rise to a subbox $\mathcal{B}'$ in $\mathcal{B}$, and together with the attached tubes and caps, they form a TBC collection. Therefore, from Theorem 2.6, one obtains a connecting orbit from $M_{in}(p)$ to $M_{out}(q)$ for any $\varepsilon > 0$. One can then apply the same reasoning as in [10] to show that the connecting orbit converges to a connected invariant set of the parametrized flow in the Hausdorff metric as ε tends to 0. This connecting invariant set consists of connecting orbits between some Morse sets $M_{y_i}(p_i)$ to $M_{y_i}(p_{i+1})$ at $y = y_i$ as well as intervals between y_i and y_{i+1} in the slow manifold corresponding to the p_i-th Morse component. Clearly the sequences y_i and p_i must satisfy the relation as in the assertion of Theorem 2.2, and hence the proof is completed.

3. Example. We shall illustrate how the directional transition matrix is computed and used in an example. Consider a slow-fast system on $\mathbb{R}^2 \times \mathbb{R}$ with a box as in Figure 1. If one attatches a repelling cap with a tube connecting the $\{y = 0\}$-side of the top Morse component $M_0(3)$ and an attracting cap with a tube connecting the $\{y = 1\}$-side of the bottom Morse component $M_1(1)$, then these caps, tubes and the box will form a TBC collection. Thus the problem is to find a connecting orbit from the repelling cap to the attracting cap. According to the slow directions along the Morse components in the box, if a connecting orbit exists, it should follow the top Morse component from $y = 0$ to some $y > 0$, then jumps down to the middle Morse component and follow it in the backward direction until it jumps further down to the bottom one and leaves the box through $y = 1$. In order to find such a connecting orbit, we must compute the directional transition matrix and the corresponding (3,1)-entry.

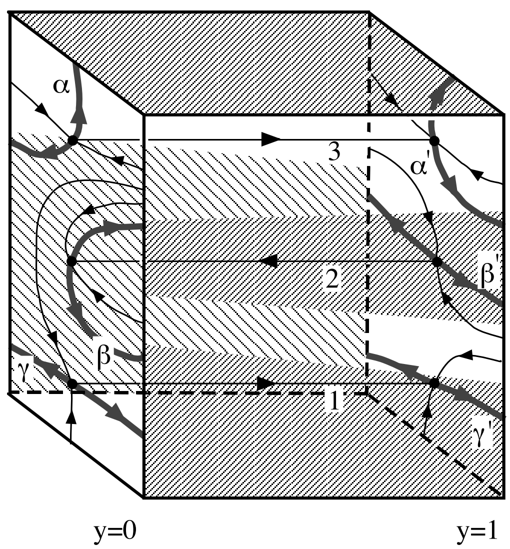

Figure 1: A box with three Morse components. Three horizontal lines represent the slow manifolds corresponding to the three Morse components $M(p)$ for $p = 1, 2, 3$. Slow dynamics in the slow manifolds are indicated by the arrows. At both sides of the box given by $y = 0, 1$, the fast dynamics change as indicated, where bold curves represent the unstable sets of the corresponding Morse components. These unstable sets labeled α, β, γ at $y = 0$ and α', β', γ' at $y = 1$ generate the cohomology Conley indices. Shaded regions in the side boundary of the box exhibit the (immediate) exit sets.

Here the unstable sets of each of the Morse components give the sets of basis for the corresponding Conley indices. To be more specific, let $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}$ be the elements in the $\mathbb{Z}_2$-coefficient cohomology Conley index $CH^*(S_0)$ at $y = 0$ corresponding to the unstable sets α, β, γ of the Morse component $M_0(p)$ for $p = 1, 2, 3$, respectively. Then they form a basis of $CH^*(S_0) \cong \bigoplus_{p=1,2,3} CH^*(M_0(p))$. Similarly, the elements $\widehat{\alpha}', \widehat{\beta}', \widehat{\gamma}'$ in the cohomology Conley index $CH^*(S_1) \cong \bigoplus_{p=1,2,3} CH^*(M_1(p))$ at $y = 1$ corresponding to the unstable sets α', β', γ' of the Morse component $M_1(p)$ for $p = 1, 2, 3$ form a basis of it. Figure 1 shows that these basis elements are related as

$$\begin{aligned} \widehat{\alpha}' &= \widehat{\alpha}, \\ \widehat{\beta}' &= \widehat{\alpha} + \widehat{\beta}, \\ \widehat{\gamma}' &= \widehat{\beta} + \widehat{\gamma}. \end{aligned} \tag{3.1}$$

Indeed, this follows from the duality between the homology and cohomology groups and a similar relation among homology classes generated by these unstable sets as follows:

$$\begin{aligned} \alpha' &= \alpha + \beta + \gamma, \\ \beta' &= \beta + \gamma, \\ \gamma' &= \gamma, \end{aligned}$$

which can be easily seen from Figure 1, where the same notation is used to indicate an unstable set and its homology class. From (3.1), one can compute the cohomology transition matrix

$$T^{1,0} : \bigoplus_{p=1,2,3} CH^*(M_1(p)) \to \bigoplus_{p=1,2,3} CH^*(M_0(p))$$

defined in §1 and obtains

$$T^{1,0} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Given the direction of the slow dynamics as in Figure 1, we can now compute the corresponding directional transition matrix. In this case, the directional transition matrix D is given as a homomorphism

$$\begin{aligned} D : \bigoplus_{p=1,2,3} CH^*(M_{out}(p)) &= CH^*(M_1(3)) \oplus CH^*(M_0(2)) \oplus CH^*(M_1(1)) \\ \to \bigoplus_{p=1,2,3} CH^*(M_{in}(p)) &= CH^*(M_0(3)) \oplus CH^*(M_1(2)) \oplus CH^*(M_0(1)). \end{aligned}$$

We first decompose the index set as

$$\begin{aligned} T^{1,0} &: CH^*(M_1(3)) \oplus [CH^*(M_1(2)) \oplus CH^*(M_1(1))] \\ &\to CH^*(M_0(3)) \oplus [CH^*(M_0(2)) \oplus CH^*(M_0(1))] \end{aligned}$$

in order to apply Lemma 2.1, and obtain

$$\begin{aligned} A_1 &: CH^*(M_1(3)) \oplus [CH^*(M_0(2)) \oplus CH^*(M_0(1))] \\ &\to CH^*(M_0(3)) \oplus [CH^*(M_1(2)) \oplus CH^*(M_1(1))], \end{aligned}$$

which is, as a matrix, given by

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

We then change the decomposition as

$$\begin{aligned} A_1 &: [CH^*(M_1(3)) \oplus CH^*(M_0(2))] \oplus CH^*(M_0(1)) \\ &\to [CH^*(M_0(3)) \oplus CH^*(M_1(2))] \oplus CH^*(M_1(1)), \end{aligned}$$

and apply Lemma 2.1 again. The resulting matrix

$$D : [CH^*(M_1(3)) \oplus CH^*(M_0(2))] \oplus CH^*(M_1(1))$$
$$\to [CH^*(M_0(3)) \oplus CH^*(M_1(2))] \oplus CH^*(M_0(1))$$

is the same as the above A_1, which gives the desired directional transition matrix in this case. Therefore

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

In particular, the (3,1)-entry of D is nonzero, and hence it follows from Theorem 2.2 that there exists a connecting orbit from $M_0(3)$ to $M_1(1)$. Similarly, there also exist connecting orbits from $M_0(3)$ to $M_0(2)$ and from $M_1(2)$ to $M_1(1)$, respectively, since the (3,2) and (2,1)-entries of D are nonzero.

One can view the above computation of the directional transition matrix as follows. Observe that $\widehat{\alpha}, \widehat{\beta}', \widehat{\gamma}$ form a basis of $\bigoplus_{p=1,2,3} CH^*(M_{in}(p))$, whereas $\widehat{\alpha}', \widehat{\beta}, \widehat{\gamma}'$ form a basis of $\bigoplus_{p=1,2,3} CH^*(M_{out}(p))$. From (3.1), we have a similar relation between these sets of basis, which are expressed as

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \widehat{\alpha}' \\ \widehat{\beta} \\ \widehat{\gamma}' \end{pmatrix} = \begin{pmatrix} \widehat{\alpha} \\ \widehat{\beta}' \\ \widehat{\gamma} \end{pmatrix}.$$

The matrix corresponding to this change of basis is in fact the directional transition matrix D.

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