

Chaotic dynamics in  $\mathbb{Z}_2$ -equivariant unfoldings  
of codimension 3 singularities of vector fields  
in  $\mathbb{R}^3$

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## Abstract

We study the most generic nilpotent singularity of a vector field in  $\mathbb{R}^3$  which is equivariant under reflection with respect to a line, say the  $z$ -axis. We prove the existence of 8 equivalence classes for  $C^0$ -equivalence, all determined by the 2-jet. We also show that in certain cases, the  $\mathbb{Z}_2$ -equivariant unfoldings generically contain codimension one heteroclinic cycles which are comparable to the Shil'nikov-type homoclinic cycle in non-equivariant unfoldings. The heteroclinic cycles are accompanied by infinitely many horseshoes and have also quite reasonable possibility of generating suspensions of Hénon-like attractors, and even Lorenz-like attractors.

# 1 Introduction

In this paper we consider vector field singularities on  $\mathbb{R}^3$  which are equivariant under reflectional symmetry with respect to a line, say the  $z$ -axis, namely, which are equivariant under the linear map

$$\mathbb{R}^3 \rightarrow \mathbb{R}^3; (x, y, z) \mapsto (-x, -y, z).$$

We study the most generic nilpotent singularities with such  $\mathbb{Z}_2$ -symmetry. Those singularities have the 1-jet  $y \frac{\partial}{\partial x}$  and normal form calculation shows that the 2-jet takes the form

$$y \frac{\partial}{\partial x} + (axz + byz) \frac{\partial}{\partial y} + (dx^2 + ez^2) \frac{\partial}{\partial z}, \quad (1.1)$$

where  $a, b, d, e$  are constants. If  $ade \neq 0$ , one may normalize these coefficients as

$$a = -1, \quad d = +1, \quad \text{and } e = \pm 1$$

by rescaling the space and time variables. In particular, we need to reverse the time coordinate if  $ad > 0$ .

The purpose of this paper is to study the determinacy and unfoldings of these singularities within the  $\mathbb{Z}_2$ -equivariant vector fields. Recall that the  $k$ -jet of a vector field  $X$  is  $C^0$ -determining, if any vector field having the same  $k$ -jet is locally topologically equivalent to  $X$ .

**Theorem 1.1** (1) *The 2-jet (1.1) with  $a = -1, d = +1, e = \pm 1$  is  $C^0$ -determining if*

$$(b, e) \neq \pm\left(\frac{5}{2}, 1\right), \pm\left(\frac{1}{4}, 1\right).$$

*There are 4 topological equivalence classes of such singularities.*

(2) *Consider an unfolding of a singularity as above with  $e = 1$  of the form:*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \lambda x + \mu y - xz + (b + O(|\lambda, \mu, \nu|))yz + O(|x, y, z|^3), \\ \dot{z} &= \nu + x^2 + z^2 + O(|x, y, z|^3). \end{aligned}$$

*There exists a codimension 1 smooth hypersurface  $\mathcal{H}$  in the parameter space for  $(\lambda, \mu, \nu)$  such that the corresponding vector fields with parameters in  $\mathcal{H}$  possess a heteroclinic cycle connecting two singularities on the  $z$ -axis.*

The structure of the paper is as follows: In Section 2, we shall give a proof of the 2-jet determinacy of the singularity by using quasi-homogeneous blowing-up techniques. The existence of a heteroclinic cycle in the unfolding is shown in Section 3, and the dynamics around the heteroclinic cycle is discussed in Section 4. In Appendix, we give a proof of a variant of normal form theorem which we use in Section 3.

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## 2 Structure of the singularity and determinacy of 2-jet

In this section we show that the singularities having the following 2-jets are 2-determined for  $C^0$ -equivalence:

$$y \frac{\partial}{\partial x} + (-xz + byz) \frac{\partial}{\partial y} + (x^2 \pm z^2) \frac{\partial}{\partial z}, \quad (2.1)$$

if  $(b, e) \neq \pm(\frac{5}{2}, 1), \pm(\frac{1}{4}, 1)$ , and we give the classification of these singularities up to  $C^0$ -equivalence. This will be done by using the quasi-homogeneous blowing-ups [5, 6] and the results of [8].

We begin with the singularities

$$y \frac{\partial}{\partial x} + (-xz + byz + cx^3 + f(x, y, z)) \frac{\partial}{\partial y} + (x^2 + ez^2 + g(x, y, z)) \frac{\partial}{\partial z},$$

with  $e = \pm 1$  and where both  $f$  and  $g$  are  $O(|x, y, z|^3)$ , with  $f$  not containing an  $x^3$  term. From the form of the singularity, we shall perform the quasi-homogeneous blow-up using

$$x = r^3 \bar{x}, \quad y = r^5 \bar{y}, \quad z = r^4 \bar{z},$$

making the calculations in the following four different charts:

- (1) in  $\bar{z} = \pm 1$  charts;
- (2) in  $\bar{x} = 1$  chart;
- (3) in  $\bar{y} = 1$  chart.

Due to the symmetry, the results of blowing ups in the remaining charts are the same as one of the above. The resulted blown-up vector fields are given in the following lemmas:

**Lemma 2.1** *The blown-up vector field in  $\bar{z} = +1$  chart is*

$$\begin{aligned} \dot{r} &= \frac{1}{4}r(\bar{x}^2 + er^2 + O(r^3)), \\ \dot{\bar{x}} &= \bar{y} - \frac{3}{4}\bar{x}^3 - \frac{3e}{4}r^2\bar{x} + O(r^3), \\ \dot{\bar{y}} &= -\bar{x} - \frac{5}{4}\bar{x}^2\bar{y} + \left(b - \frac{5e}{4}\right)r^2\bar{y} + cr^2\bar{x}^3 + O(r^4). \end{aligned} \tag{2.2}$$

*The singularity on the blow-up locus  $\{r = 0\}$  is only at the origin  $\bar{x} = \bar{y} = 0$  and it is a weak stable focus with radial dynamics given by  $\dot{r} = \frac{e}{4}r^3$ .*

**Remark 2.2** In  $\bar{z} = +1$  chart, the function  $\bar{x}^2 + \bar{y}^2$  is a (weak) Lyapunov function in the blow-up locus, namely,

$$\frac{d}{dt}(\bar{x}^2 + \bar{y}^2) = -\frac{\bar{x}^2}{2}(3\bar{x}^2 + 5\bar{y}^2) \leq 0.$$

**Lemma 2.3** *The blown-up vector field in  $\bar{z} = -1$  chart is*

$$\begin{aligned} \dot{r} &= -\frac{1}{4}r(\bar{x}^2 + er^2 + O(r^3)), \\ \dot{\bar{x}} &= \bar{y} + \frac{3}{4}\bar{x}^3 + \frac{3e}{4}r^2\bar{x} + O(r^3), \\ \dot{\bar{y}} &= \bar{x} + \frac{5}{4}\bar{x}^2\bar{y} - \left(b - \frac{5e}{4}\right)r^2\bar{y} + cr^2\bar{x}^3 + O(r^4). \end{aligned} \tag{2.3}$$

*The singularities on the blow-up locus  $\{r = 0\}$  are either the origin  $(0, 0)$  or  $(\bar{x}_\pm, \bar{y}_\pm) = \pm \left(\sqrt[4]{\frac{16}{15}}, -\frac{4}{5}\sqrt[4]{\frac{15}{16}}\right)$ . The origin is a hyperbolic saddle with radial dynamics given by  $\dot{r} = -\frac{e}{4}r^3$ , whereas the other singularities  $(\bar{x}_\pm, \bar{y}_\pm)$  are hyperbolic unstable foci with hyperbolic attracting radial dynamics.*

It can be shown using the Bendixson's criterion that there is no limit cycle which is entirely contained in the  $\bar{z} = \pm 1$  charts, see [9], since the divergence of the vector fields on these charts have everywhere a constant sign. The results of blowing-ups in these charts depend on the constant  $e$ , and are summarized in Figure 1.

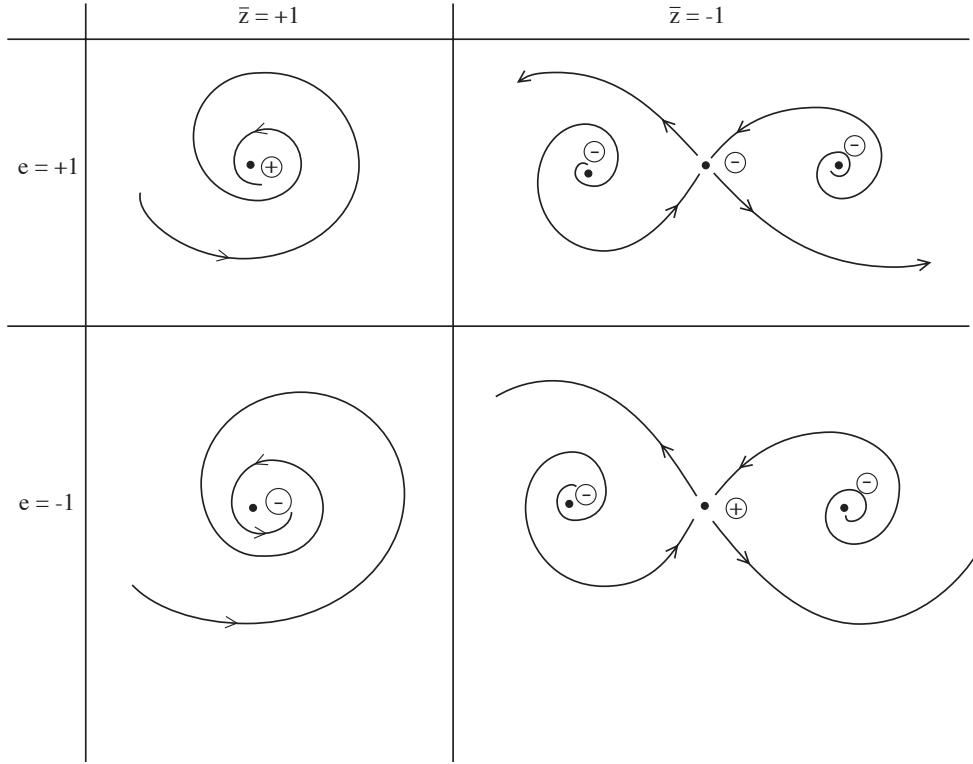


Figure 1: Blowing-ups in  $\bar{z} = \pm 1$  charts.

**Lemma 2.4** *The blown-up vector field in  $\bar{x} = 1$  chart is*

$$\begin{aligned}
 \dot{r} &= \frac{1}{3}r\bar{y}, \\
 \dot{\bar{y}} &= -\bar{z} - \frac{5}{3}\bar{y}^2 + br^2\bar{y}\bar{z} + cr^2 + O(r^4), \\
 \dot{\bar{z}} &= 1 - \frac{4}{3}\bar{y}\bar{z} + er^2\bar{z}^2 + O(r^4).
 \end{aligned} \tag{2.4}$$

*The vector field on the blow-up locus is transverse to the line  $\bar{z} = 0$ .*

**Lemma 2.5** *The blown-up vector field in  $\bar{y} = 1$  chart is*

$$\begin{aligned}
\dot{r} &= r \left( -\frac{1}{5}\bar{x}\bar{z} + \frac{b}{5}r^2\bar{z} + \frac{c}{5}r^2\bar{x}^3 + O(r^4) \right), \\
\dot{\bar{x}} &= 1 + \frac{3}{5}\bar{x}^2\bar{z} - \frac{b}{5}r^2\bar{x}\bar{z} - \frac{c}{5}r^2\bar{x}^4 + O(r^4), \\
\dot{\bar{z}} &= \bar{x}^2 + \frac{4}{5}\bar{x}\bar{z}^2 + \left( e - \frac{4b}{5} \right) r^2\bar{z}^2 - \frac{4c}{5}r^2\bar{x}^3\bar{z} + O(r^4).
\end{aligned} \tag{2.5}$$

*The vector field on the blow-up locus is transverse to the line  $\bar{z} = 0$ .*

In order to show the determinacy of the 2-jet, we only have to study the behavior near the singularity at  $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 1)$ , since

- $(\bar{x}_\pm, \bar{y}_\pm, -1)$  are hyperbolic singularities;
- $(0, 0, -1)$  has one-dimensional center manifold on which the dynamics is determined by  $-\frac{\epsilon}{4}r^3\frac{\partial}{\partial r}$ ;
- there is no limit cycle on the blow-up locus.

More detailed study of the singularity  $(0, 0, 1)$  requires normal form reduction and further blow-up.

In order to obtain the normal form of the singularity at  $(0, 0, 1)$ , we use the complex coordinates given by

$$\zeta = \bar{x} + i\bar{y}, \quad \bar{\zeta} = \bar{x} - i\bar{y}.$$

The result of the coordinate changes is

$$\begin{aligned}
\dot{\zeta} &= -i\zeta - \frac{1}{4}\zeta^3 - \frac{7}{16}\zeta^2\bar{\zeta} - \frac{1}{8}\zeta\bar{\zeta}^2 + \frac{1}{16}\bar{\zeta}^3 + \frac{b-2\epsilon}{2}r^2\zeta - \frac{\epsilon-2b}{4}r^2\bar{\zeta} \\
&\quad + \frac{c}{8}r^2(\zeta^3 + 3\zeta^2\bar{\zeta} + 3\zeta\bar{\zeta}^2 + \bar{\zeta}^3) + O(r^3), \\
\dot{r} &= r \left( \frac{1}{16}\zeta^2 + \frac{1}{8}|\zeta|^2 + \frac{1}{16}\bar{\zeta}^2 + \frac{\epsilon}{4}r^2 + O(r^3) \right).
\end{aligned}$$

Therefore the normal form is given by

$$\begin{aligned}
\dot{\zeta} &= \zeta \left\{ \left( -\frac{7}{16}|\zeta|^2 + \frac{b-2\epsilon}{2}r^2 \right) + \left( -1 + \frac{3c}{8}r^2|\zeta|^2 \right) i + \text{h.o.t.} \right\}, \\
\dot{r} &= r \left( \frac{1}{8}|\zeta|^2 + \frac{\epsilon}{4}r^2 + \text{h.o.t.} \right),
\end{aligned}$$

and in particular, the radial normal form given by setting  $\rho = |\zeta|$  becomes

$$\begin{aligned}
\dot{\rho} &= \rho \left( -\frac{7}{16}\rho^2 + \frac{b-2\epsilon}{2}r^2 \right) + \text{h.o.t.}, \\
\dot{r} &= r \left( \frac{1}{8}\rho^2 + \frac{\epsilon}{4}r^2 \right) + \text{h.o.t.}
\end{aligned} \tag{2.6}$$

By putting  $P = \frac{7}{8}\rho^2$  and  $R = \frac{e}{2}r^2$ , one has

$$\begin{aligned}\dot{P} &= P(-P + \eta R) + \text{h.o.t.}, \\ \dot{R} &= R\left(\frac{2}{7}P + R\right) + \text{h.o.t.}\end{aligned}$$

where  $\eta = \frac{2b}{e} - 4$ . Note that  $P$  is always positive and  $e$  and  $R$  have the same sign.

We shall make further blow-up of the above equations by

$$P = U\bar{X}, \quad R = U\bar{Y},$$

in the charts  $\bar{X} = 1$  and  $\bar{Y} = 1$ . In  $\bar{X} = 1$  chart, the blown-up vector field is

$$\dot{U} = U^2(-1 + \eta\bar{Y}) + \dots, \quad \dot{\bar{Y}} = U\bar{Y}\left(\frac{9}{7} + (1 - \eta)\bar{Y}\right) + \dots,$$

and hence the singularities on the blow-up locus  $U = 0$  are one at  $\bar{Y} = 0$  which is a hyperbolic saddle, repelling in  $\bar{Y}$ -direction and attracting in  $U$ -direction, and another at  $\bar{Y} = \frac{9}{7(\eta-1)}$  where the linearization matrix has eigenvalues  $-\frac{9}{7}$  and  $\frac{2\eta+7}{7(\eta-1)}$ . Similarly, in  $\bar{Y} = 1$  chart, the blown-up vector field is

$$\dot{U} = U^2(1 + \frac{2}{7}\bar{X}) + \dots, \quad \dot{\bar{X}} = U\bar{X}\left((\eta - 1) - \frac{9}{7}\bar{X}\right) + \dots,$$

and the singularities on the blow-up locus  $U = 0$  are one at  $\bar{X} = 0$  which is a hyperbolic saddle unless  $\eta = 1$  and another at  $\bar{X} = \frac{7(\eta-1)}{9}$  where the linearization matrix has eigenvalues  $1 - \eta$  and  $\frac{2\eta+7}{9}$ .

Combining the above information, we obtain the structure of the blown-up vector field as indicated in Figure 2 depending on the value of  $\eta$ .

Recall that  $R = \frac{e}{2}r^2$  has the same sign as  $e$ , where  $e = \pm 1$ . If  $e = +1$  then  $\eta = 2b - 4$  and hence  $\eta = 1$  implies  $b = 5/2$  and  $\eta = -7/2$  does  $b = 1/4$ , whereas if  $e = -1$  then  $\eta = -2b - 4$  and hence  $\eta = 1$  implies  $b = -5/2$  and  $\eta = -7/2$  does  $b = -1/4$ . Therefore, all the singularities above in the blown-up locus are hyperbolic as long as  $(b, e) \neq \pm(\frac{5}{2}, 1), \pm(\frac{1}{4}, 1)$ .

Figure 3 exhibits the blown-down vector fields which are obtained from the above blow-up calculations.

This in particular shows that there is no elliptic sector in the singularity, which is crucial when one proves the 2-determinacy of the singularity as in what follows.



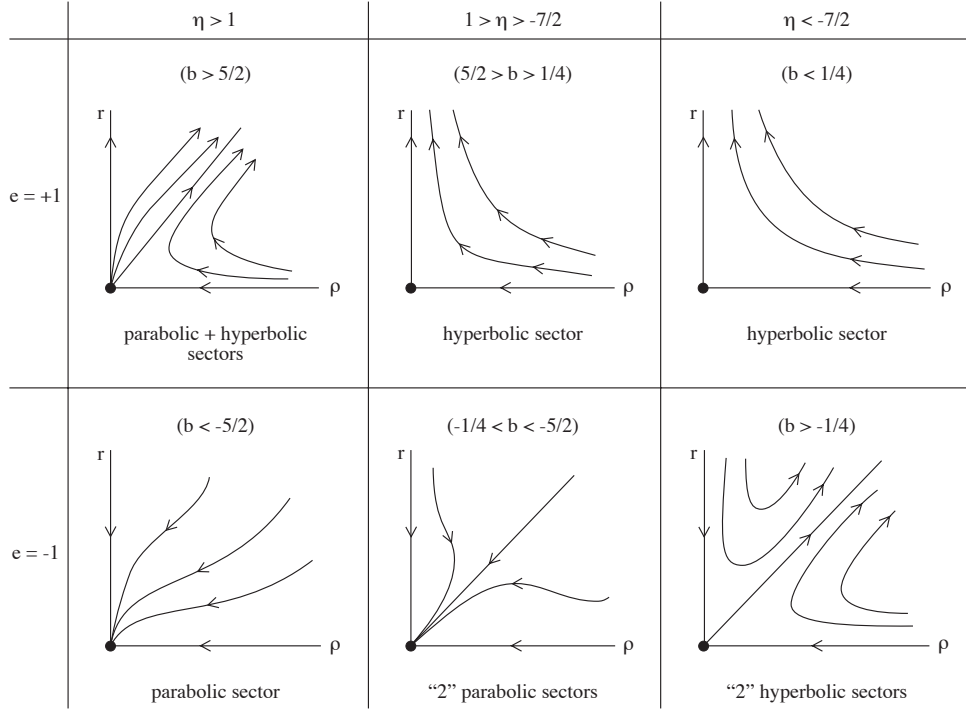


Figure 2: Further blowing-up. (a)  $\eta > 1$ ; (b)  $1 > \eta > -7/2$ ; (c)  $\eta < -7/2$ .

**Remark 2.6** We could have blown up directly the expression (2.6). It would also have led to a desingularization with hyperbolic singularities.

To study such a three dimensional vector field near  $(0, 0, 1)$ , we perform a normal form calculation which leads, up to a  $C^\infty$  coordinate change respecting the  $\bar{z}$ -axis, to an expression:

$$\lambda(1 + f(\bar{z}, \bar{x}^2 + \bar{y}^2)) \left( \bar{x} \frac{\partial}{\partial \bar{y}} - \bar{y} \frac{\partial}{\partial \bar{x}} \right) + g(\bar{z}, \bar{x}^2 + \bar{y}^2) \left( \bar{x} \frac{\partial}{\partial \bar{x}} + \bar{y} \frac{\partial}{\partial \bar{y}} \right) + h(\bar{z}, \bar{x}^2 + \bar{y}^2) \frac{\partial}{\partial \bar{z}} + Y_1 \frac{\partial}{\partial \bar{x}} + Y_2 \frac{\partial}{\partial \bar{y}} + Y_3 \frac{\partial}{\partial \bar{z}}$$

with

$$f(0, 0) = g(0, 0) = h(0, 0) = \frac{\partial h}{\partial \bar{z}}(0, 0) = 0$$

and

$$j_\infty Y_1(O) = j_\infty Y_2(O) = j_\infty Y_3(O) = 0.$$

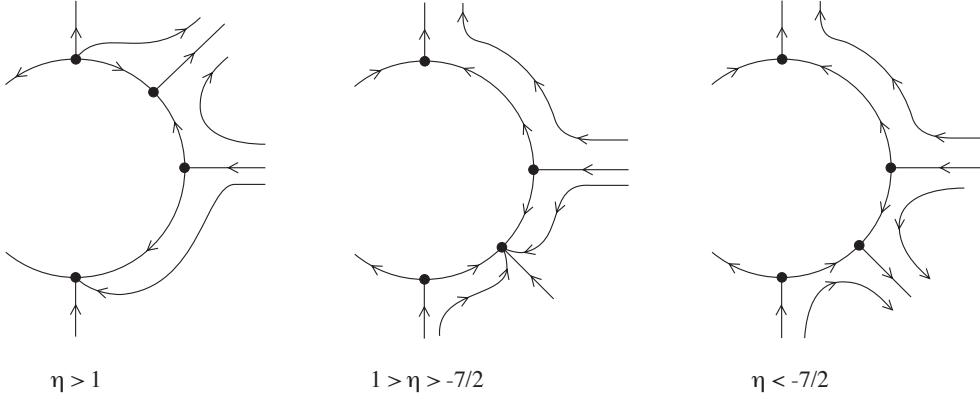


Figure 3: Structure of the singularity.

Moreover, by construction, the  $\bar{z}$ -axis is invariant.

After multiplying with  $2\pi\{\lambda(1 + f(\bar{z}, \bar{x}^2 + \bar{y}^2))\}^{-1}$ , we obtain a similar expression as above but with  $f \equiv 0$  and  $\lambda = 2\pi$ . We can now consider the first return map with respect to the section  $\{\bar{y} = 0, \bar{x} > 0\}$ . It is well-defined and it can be extended to a  $C^\infty$  diffeomorphism  $\varphi$ , defined on  $H = \{\bar{y} = 0, \bar{x} \geq 0\}$ . To avoid confusion, let us change the notation on  $H$  from  $(\bar{x}, \bar{z})$  to  $(\rho, r)$ . Then the construction permits us to show that at  $(\rho, r) = (0, 0)$ , the diffeomorphism  $\varphi$  is infinitely tangent to the time one map of the vector field

$$g(r, \rho^2)\rho \frac{\partial}{\partial \rho} + h(r, \rho^2) \frac{\partial}{\partial r}.$$

In studying the vector field, the Poincaré map  $\varphi$  calculated with respect to the positive quadrant  $\{\bar{y} = 0, \bar{x} \geq 0, \bar{z} \geq 0\}$  is infinitely tangent, at  $(0, 0)$ , to the time one mapping of a vector field having an expression:

$$\begin{aligned} \dot{\rho} &= \rho \left( -\frac{7}{16}\rho^2 + \frac{b-2e}{2}r^2 \right) + \dots, \\ \dot{r} &= r \left( \frac{1}{8}\rho^2 + \frac{e}{4}r^2 \right) + \dots. \end{aligned} \tag{2.7}$$

The usual blow-up operations to study such a singularity lead to a desingularization only exhibiting hyperbolic singularities, under the conditions  $(b, e) \neq \pm(\frac{5}{2}, 1), \pm(\frac{1}{4}, 1)$ . Moreover only parabolic and hy-

parabolic sectors show up and no elliptic sectors. As such, the theorems from [8] permit us to conclude that the diffeomorphism  $\varphi$  is  $C^\infty$ -conjugate to the time one map of the vector field (2.7), with a conjugacy that is infinitely tangent to the identity at  $(\rho, r) = (0, 0)$ . Of course the construction shows that the conjugacy respects both  $\{\rho = 0\}$  and  $\{r = 0\}$ .

Suspending this diffeomorphism by the orbits of vector field gives us the topological structure of the vector field near the singularity  $(0, 0, 0)$ . See [4].

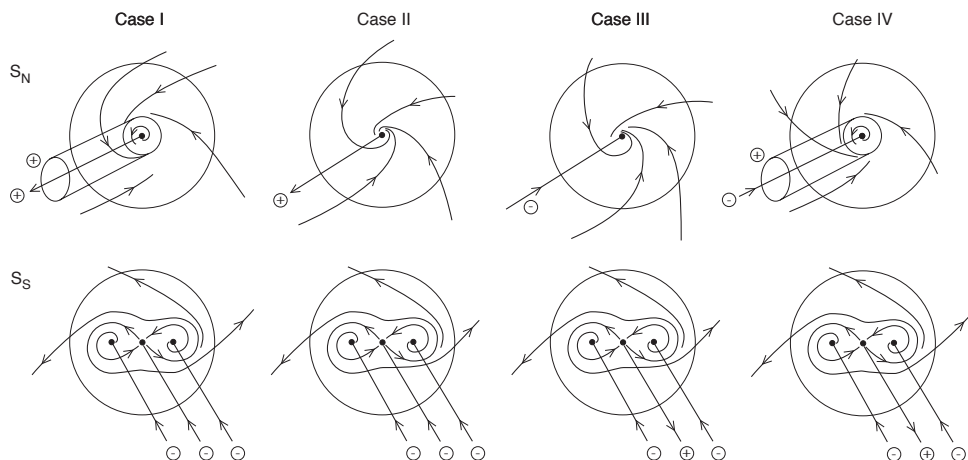


Figure 4: Topological equivalence classes of the 2-jets.

Representing the different equivalence classes is not simple. Exactly like [6], we prefer to use so-called Morse-Smale models. With similar techniques as used in [6], one can prove that the different equivalence classes are  $C^0$ -equivalent to vector fields having a (homogeneous) blow-up as presented in Figure 4. Four more phase portraits can be obtained by time-reversal. The blown-up vector fields presented are defined in  $S^2 \times [0, \varepsilon)$  for some  $\varepsilon > 0$ ; they are of Morse-Smale type in  $S^2 \times \{0\}$  and moreover their singularities and closed orbits are also hyperbolic in  $S^2 \times [0, \varepsilon)$ .

In all four cases the field on  $S^2 \times \{0\}$  has exactly four singularities; in two cases there is one closed orbit, in the other two cases there are

none. We made separate pictures of the northern and southern hemispheres,  $S_N$  and  $S_S$ , and represent the (hyperbolic) “radial” behavior by lines or cylinders equipped with  $\oplus$  and  $\ominus$ .

In Figure 5, we make a tentative picture of how the singularities look like in  $\mathbb{R}^3$ . The plane in these pictures is drawn to give an idea of the movement transverse to the three characteristic orbits below the singularity.

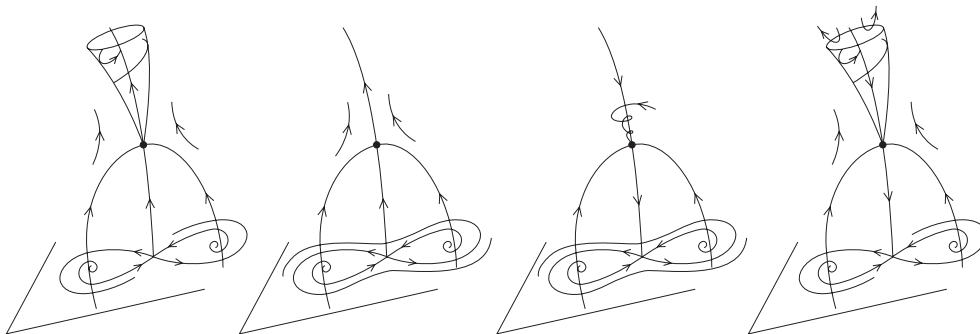


Figure 5: Three dimensional pictures for structure of singularities.

### 3 Unfolding of the singularity

A 3-parameter generic unfolding of the singularity as in (2.1) with  $e = 1$  can, by normal form calculation, always be expressed by the following form:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \lambda x + \mu y - xz + (b + \alpha(\lambda, \mu, \nu))yz \\ &\quad + (c + \beta(\lambda, \mu, \nu))x^3 + f(x, y, z, \lambda, \mu, \nu), \\ \dot{z} &= \nu + x^2 + z^2 + O(|x, y, z|^3), \end{aligned}$$

with  $\alpha(0, 0, 0) = \beta(0, 0, 0) = 0$ , and  $f = O(|x, y, z|^3)$  with no  $x^3$ -term.

In this section, we show that this system possesses a heteroclinic cycle for appropriate parameter values. This heteroclinic cycle gives

rise to infinitely many horseshoes analogously to the Shil'nikov theorem.

### 3.1 Rescaling and Limit system

We introduce the new parameter  $\varepsilon$  and rescale by

$$\begin{aligned}x &= r^3 \bar{x}, & y &= r^5 \bar{y}, & z &= r^4 \bar{z}, & \varepsilon &= ru, \\ \lambda &= \varepsilon^4 \bar{\lambda}, & \mu &= \varepsilon^2 \bar{\mu}, & \nu &= \varepsilon^8 \bar{\nu};\end{aligned}$$

we can take  $\bar{\lambda}^2 + \bar{\mu}^2 + \bar{\nu}^2 = 1$ , or  $(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \in \partial B$  with  $B = \{|\bar{\lambda}| \leq 1, |\bar{\mu}| \leq 1, |\bar{\nu}| \leq 1\}$ . Note that this choice of power is taken in order to bring singularities in a finite but non-zero position.

The rescaled family is given by

$$\begin{aligned}r \dot{\bar{x}} + 3\bar{x} \dot{r} &= r^3 \bar{y}, \\ r \dot{\bar{y}} + 5\bar{y} \dot{r} &= r^3 (\bar{\lambda} u^4 \bar{x} + \bar{\mu} u^2 \bar{y} - \bar{x} \bar{z} + b r^2 \bar{y} \bar{z} + c r^2 \bar{x}^3 + O(r^4)), \\ r \dot{\bar{z}} + 4\bar{z} \dot{r} &= r^3 (\bar{\nu} r^2 u^8 + \bar{x}^2 + r^2 \bar{z}^2 + O(r^4)), \\ u \dot{r} &= -r \dot{u},\end{aligned}$$

and hence the family rescaling given by  $u = 1$  takes the form

$$\begin{aligned}\dot{\bar{x}} &= \bar{y}, \\ \dot{\bar{y}} &= \bar{\lambda} \bar{x} + \bar{\mu} \bar{y} - \bar{x} \bar{z} + O(r^2) O(\bar{x}, \bar{y}), \\ \dot{\bar{z}} &= \bar{x}^2 + r^2 \bar{\nu} + r^2 \bar{z}^2 + O(r^4) O(\bar{x}, \bar{y}, \bar{z}).\end{aligned}\tag{3.1}$$

In order to have downward flow on the  $z$ -axis near 0, we will keep  $\bar{\nu} < 0$ .

For  $r \neq 0$ , singularities on the  $\bar{z}$ -axis are given by  $\bar{z}_{\pm} = \pm \sqrt{-\bar{\nu}} + O(r^2)$  with  $\bar{\nu} < 0$ . Choose  $\bar{\nu} = -1$  without loss of generality. In what follows we will always keep  $\bar{\mu} < 0$  and  $\bar{\lambda} \in (-1, 1)$ . Then the nature of the singularities near  $(0, 0, \pm 1)$  with  $r > 0$  are a saddle-focus with attracting focus transverse to the  $\bar{z}$ -axis near  $(0, 0, +1)$  and a saddle with hyperbolic saddle behavior transverse to the  $\bar{z}$ -axis near  $(0, 0, -1)$ .

Our goal is to show the existence of a heteroclinic orbit besides the  $\bar{z}$ -axis which connects singularities  $(0, 0, \bar{z}_{\pm})$  near  $(0, 0, -1)$  and  $(0, 0, +1)$  for certain parameter values. Together with the orbit on the  $\bar{z}$ -axis, this would form a heteroclinic cycle connecting these two singularities.

**Theorem 3.1** *There exists a smooth curve  $\bar{\lambda} = h(\bar{\mu})$  for sufficiently small  $\bar{\mu} < 0$  with  $h(\bar{\mu}) \in (-1, 1)$  such that for any  $(\bar{\lambda}_0, \bar{\mu}_0) = (h(\bar{\mu}_0), \bar{\mu}_0)$  and for any sufficiently small neighborhood  $\Lambda$  of  $(\bar{\lambda}_0, \bar{\mu}_0)$ , there exist  $r_0 > 0$  and a smooth function  $H(\bar{\mu}, r)$  defined for  $r \in [0, r_0]$  and for  $\bar{\mu} < 0$  with  $H(\bar{\mu}, 0) = h(\bar{\mu})$  for which the following statements are equivalent:*

- (1) *there exists a connecting orbit from  $(0, 0, \bar{z}_-)$  to  $(0, 0, \bar{z}_+)$  with  $(\bar{\lambda}, \bar{\mu}, r) \in \Lambda \times [0, r_0]$ ;*
- (2)  $\bar{\lambda} = H(\bar{\mu}, r)$ .

**Remark 3.2**  $\bar{\lambda}$  is a transverse parameter for the connecting orbit. The above theorem follows from a similar result for the limit system which is given by setting  $r = 0$  to (3.1). Indeed smooth dependence of the stable and unstable manifolds on parameters and  $\lambda$  being the transverse parameter imply the claim.

### 3.2 Reduction to a problem near infinity on some Poincaré-Lyapunov ball

We first study the limit system given by  $r = 0$  when  $\bar{\mu} < 0$  and  $\bar{\lambda} \in (-1, 1)$  as follows.

$$\begin{aligned}\dot{\bar{x}} &= \bar{y}, \\ \dot{\bar{y}} &= \bar{\lambda}\bar{x} + \bar{\mu}\bar{y} - \bar{x}\bar{z}, \\ \dot{\bar{z}} &= \bar{x}^2.\end{aligned}$$

Note that this limit system has a line of singularities on the  $\bar{z}$ -axis. By the change of coordinates:

$$\bar{x} = |\bar{\mu}|^{3/2}X, \quad \bar{y} = |\bar{\mu}|^{5/2}Y, \quad \bar{z} = |\bar{\mu}|^2Z, \quad \frac{d}{d\tau} = |\bar{\mu}|^{-1}\frac{d}{dt},$$

the rescaled system takes the form

$$\begin{aligned}\dot{X} &= Y, \\ \dot{Y} &= CX - Y - XZ, \\ \dot{Z} &= X^2,\end{aligned}\tag{3.2}$$

with  $C = \frac{\bar{\lambda}}{\bar{\mu}^2}$  and where  $\dot{\phantom{x}}$  stands for time-derivatives with respect to the rescaled time coordinate. With this time rescaling, the singularities  $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \pm 1)$  of the limit system change to  $(X, Y, Z) = (0, 0, \pm A)$  where  $A = 1/\bar{\mu}^2$ . Translation:

$$x = X, \quad y = Y, \quad z = Z - C$$

then brings the above system to

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -y - xz, \\ \dot{z} &= x^2, \end{aligned} \tag{3.3}$$

which no longer depends on parameters. The singularities corresponding to  $(X, Y, Z) = (0, 0, \pm A)$  are  $(0, 0, \bar{B})$  and  $(0, 0, -B)$  where

$$\bar{B} = A - C = \frac{1 - \bar{\lambda}}{\bar{\mu}^2}, \quad B = A + C = \frac{1 + \bar{\lambda}}{\bar{\mu}^2}.$$

Remember that the  $z$ -axis is the line of singularities. The behavior normal to the  $z$ -axis of a singularity  $(0, 0, z_0)$  of (3.3) is a stable focus if  $z_0 > \frac{1}{4}$ , a stable node if  $0 < z_0 < \frac{1}{4}$ , and a hyperbolic saddle if  $z_0 < 0$ .

Theorem 3.1 will reveal to be a consequence of the next proposition.

**Proposition 3.3** *For sufficiently large  $B$ , there is a smooth function  $\bar{B} = f(B)$  with  $f'(B) > 0$  and  $f(B) > B$  such that the system (3.3) has a connecting orbit from  $(0, 0, -B)$  to  $(0, 0, \bar{B})$ . Moreover  $f(B) \approx B^{7/5}$  for  $B \rightarrow +\infty$ .*

To prove the proposition, we will first present a number of constructions and intermediate results. First we compactify  $\mathbb{R}^3$  to a so-called Poincaré-Lyapunov ball by taking

$$x = \frac{u}{s^3}, \quad y = \frac{v}{s^5}, \quad z = \frac{w}{s^4}, \quad \text{with } u^2 + v^2 + w^2 = 1.$$

With this change together with the multiplication of  $s^2$  to the system, we get an analytic vector field on  $S^2 \times I$  where  $I$  is a neighborhood of 0 which corresponds to  $\infty$  in the previous coordinates. This analytic

vector field on  $S^2 \times \{0\}$  is exactly the blown-up vector field of the degenerate singularity studied in Section 2 on the blown-up locus. We have already seen that there is a connecting orbit from the south pole to the north pole in the blown-up vector field.

Near the south pole, it holds that

- the  $s$ -axis consists of singularities and is normally hyperbolic;
- the stable and unstable manifolds of the  $s$ -axis are regularly foliated by the stable and unstable manifolds of each of the singularities;
- within the unstable manifold of the  $s$ -axis, the projection from any “transversal” to the  $s$ -axis along the unstable leaves is regular, and hence has a regular inverse;
- such a “transversal” can be chosen in a sufficiently small neighborhood of the north pole by following the (known) connecting orbit in  $S^2 \times \{0\}$ .

We then study the vector field near the north pole. For this purpose, we may take the  $w = +1$  chart, in which the vector field is given by:

$$\begin{aligned}\dot{s} &= -\frac{1}{4}su^2, \\ \dot{u} &= v - \frac{3}{4}u^3, \\ \dot{v} &= -u - \frac{5}{4}u^2v - s^2v.\end{aligned}\tag{3.4}$$

The vector field along the  $s$ -axis is rotational in this case. Let us take a compact line segment  $I$  of singularities along which the linearization has the eigenvalues  $0$  and  $a(s) \pm ib(s)$  with  $b(s) \neq 0$ . We work in  $C^\infty$  coordinates  $(x, y, s)$  such that the line segment is represented by  $\{x = y = 0, 0 \leq s \leq 1\}$ . Observe that the  $C^\infty$  coordinates  $(x, y, s)$  can be chosen in such a way to have the eigenspace of the eigenvalues  $a(s) \pm ib(s)$  given by  $\{s = \text{constant}\}$ . Moreover, we may ask the 1-jet along the line segment  $I$  to have the expression:

$$b(s) \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) + a(s) \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right).\tag{3.5}$$

In an appendix, we will prove the following theorem.

**Theorem 3.4** *Let  $X$  be a smooth vector field on  $\mathbb{R}^3$  defined in a neighborhood of the line segment  $I$  with all the conditions as described*



above. Let  $(x, y, s)$  denote the  $C^\infty$  coordinates as above, including the expression (3.5) for the 1-jet of  $X$  along  $I$ .

Then there exists a smooth diffeomorphism  $\varphi$  defined in a neighborhood of  $I$  with the 1-jet being the identity at each point of  $I$ , such that  $\varphi_*X$  has an expression

$$\begin{aligned} & (b(s) + f(x^2 + y^2, s)) \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \\ & + (a(s) + g(x^2 + y^2, s)) \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \\ & + h(x^2 + y^2, s) \frac{\partial}{\partial s} + Z(x, y, s), \end{aligned} \quad (3.6)$$

with  $f, g, h, Z \in C^\infty$ ,  $f(0, s) = g(0, s) = h(0, s) = 0$ , and  $j_\infty Z(0, 0, s) = 0$  for all  $s \in I$ .

We bring the expression (3.4) into a normal form as above. If we multiply the expression (3.6) by  $\frac{2\pi}{b(s) + f(x^2 + y^2, s)}$ , we get a similar expression as in (3.6) with  $b(s) = 2\pi$  and  $f(x^2 + y^2, s) \equiv 0$ . Because of this expression, it is clear that the Poincaré map of it with respect to  $\{y = 0, x \geq 0\}$  is a smooth mapping which is infinitely tangent along  $I$  to the time one mapping of the reduced vector field

$$(a(s) + g(r^2, s))r \frac{\partial}{\partial r} + h(r^2, s) \frac{\partial}{\partial s}.$$

The corresponding planar expression for the normal form of (3.4) is given by

$$\begin{aligned} \dot{r} &= -2\pi r \left( \frac{1}{2}s^2 + \frac{7}{16}r^2 + O(|s, r|^3) \right), \\ \dot{s} &= -2\pi r^2 s \left( \frac{1}{8} + O(|s, r|) \right). \end{aligned}$$

**Lemma 3.5** *Let  $X$  be the vector field given in (3.4). Then in a sufficiently small neighborhood  $V$  of  $(x, y, s) = (0, 0, 0)$  the stable foliation of  $X$  along  $\{x = y = 0\}$  for  $s > 0$  is smooth and extends in a continuous way to a foliation  $\mathcal{F}_N$  on  $V$  by adding the leaf  $V \cap \{s = 0\}$ . Moreover, if we take  $\kappa$  to be a segment transversely cutting  $\{s = 0\}$  at some point  $(x_0, y_0) \neq (0, 0)$  and we let  $k$  be a regular parameter on  $\kappa$  with  $k = 0$  representing  $(x_0, y_0)$  and  $\{k > 0\}$  representing  $\kappa \cap \{s > 0\}$ ,*

then the holonomy from  $\kappa$  to  $\{x = y = 0\}$  along the leaves of the foliation  $\mathcal{F}_N$  is given by a Mourtada form

$$s = ck^{7/5}(1 + \varphi(k))$$

for some positive constant  $c$  and some function  $\varphi$  which is smooth for  $k > 0$  and satisfies

$$\lim_{k \rightarrow 0} k^n \cdot \frac{d^n \varphi}{dk^n} = 0, \quad \forall n \in \mathbb{N}.$$

*Proof.* Near  $(x, y, s) = (0, 0, 0)$  we have a 3-dimensional vector field  $X$  having the  $s$ -axis as a line of singularities. It has a Poincaré map  $f$  with respect to  $\{y = 0, x \geq 0\}$ , which, along  $\{x = 0\}$ , is infinitely tangent to the time one mapping of the following vector field:

$$\begin{aligned} \dot{r} &= -2\pi r \left( \frac{1}{2}s^2 + \frac{7}{16}r^2 + O(|s, r|^3) \right), \\ \dot{s} &= -2\pi r^2 s \left( \frac{1}{8} + O(|s, r|) \right). \end{aligned} \quad (3.7)$$

In this expression we write  $r$  instead of  $x$  to stress the restriction to a 2-dimensional situation.

Let us first study the reduced vector field (3.7) by making a blow-up using  $r = u\bar{r}, s = u\bar{s}, \tau = u^2 t$ . We make the calculation using the directional blow-up.

Taking  $r = u, s = uw$ , we get

$$\begin{aligned} \dot{u} &= -2\pi u \left( \frac{7}{16} + \frac{1}{2}w^2 + O(u) \right), \\ \dot{w} &= 2\pi w \left( \frac{5}{16} + \frac{1}{2}w^2 + O(u) \right). \end{aligned}$$

There is only one singularity on  $u = 0$  situated at  $w = 0$ , which represents a hyperbolic saddle with hyperbolic ratio  $\frac{7}{5}$ . The Dulac map with respect to any segments transverse to respectively the stable and unstable manifolds can be written as

$$\xi = c\eta^{7/5}(1 + \varphi(\eta)),$$

where  $\xi$  and  $\eta$  are regular parameter with respect to these transversal segments with the origin being the intersection of the stable and unstable manifolds respectively, and the positive parts corresponding to

the part that interests us. In this expression, the function  $\varphi$  is smooth for  $\eta > 0$  and satisfies

$$\lim_{\eta \rightarrow 0} \eta^n \cdot \frac{d^n \varphi}{d\eta^n} = 0, \quad \forall n \in \mathbb{N}.$$

This expression is called the Mourtada normal form for this transition and the proof of its validity has been given in [14].

Taking  $r = uv, s = u$  for other directional blow up, we get

$$\begin{aligned} \dot{u} &= -2\pi uv^2 \left( \frac{1}{8} + O(u) \right), \\ \dot{v} &= -2\pi v \left( \frac{1}{2} + \frac{5}{16}v^2 + O(u) \right). \end{aligned}$$

For  $\{u = 0\}$  we find one singularity at  $(0, 0)$  at which the vector field is equal to  $v$  times a regular vector field having  $\{u = 0\}$  as an orbit. Therefore we know the phase portrait near  $(u, v) = (0, 0)$  as well.

Combining both directional blow-ups we obtain the complete phase portrait of (3.7) near  $(r, s) = (0, 0)$ . Now in reality, we do not want to study the vector field (3.7), but in fact the Poincaré map  $f$  which is infinitely near the time one map of (3.7) along  $\{r = 0\}$ . As we are especially interested in the structure of  $f$  near  $(r, s) = (0, 0)$ , we proceed as in [8] and apply the blow-up  $r = u\bar{r}, s = u\bar{s}$  to  $f$ , as we did to (3.7), however without applying the division by  $u^2$ .

As such we get a smooth diffeomorphism  $\hat{f}$ , which along the blow up locus  $\{u = 0\}$ , is infinitely near the time one mapping of  $\hat{Y} = u^2\bar{Y}$  where  $\bar{Y}$  is the blown-up vector field of (3.7). As in the second directional blow-up calculated above for (3.7), the diffeomorphism  $\hat{f}$  is, along  $\{u = 0\} \cup \{v = 0\}$ , infinitely tangent to the time one map of  $u^2Z$  where  $Z$  is the vector field given by

$$\begin{aligned} \dot{u} &= -2\pi uv^2 \left( \frac{1}{8} + O(u) \right), \\ \dot{v} &= -2\pi v \left( \frac{1}{2} + \frac{5}{16}v^2 + O(u) \right). \end{aligned}$$

In [8], it has been proven that under these conditions  $\hat{f}$  is smoothly conjugate to the time one map of  $\hat{Y}$ , and that the smooth conjugacy is infinitely tangent to the identity along  $\{u = 0\} \cup \{v = 0\}$ . In fact this result is not purely local, but can be extended along  $\{v = 0\}$  as far as the normal form theorem 3.4 applies. It can also be extended along  $\{u = 0\}$  on any compact segment  $\{u = 0, 0 \leq v \leq v_0\}$ . Moreover, since

in the first directional blow-up, we only encounter a hyperbolic saddle (after division by  $u^2$ ), results in [8] permit to extend the conjugacy between  $\hat{f}$  and the time one map of  $\hat{Y}$  to a neighborhood of the saddle, and in this way, we obtain that in fact  $f$  and the time one map of (3.7) are smoothly conjugate in a neighborhood of  $(r, s) = (0, 0)$ . By construction the conjugacy is infinitely tangent to the identity along  $\{r = 0\}$  and hence also has that property at  $(r, s) = (0, 0)$ .

A concluding remark of this construction is that near  $(x, y, s) = (0, 0, 0)$  the vector field  $X$  possesses an invariant codimension 1 foliation  $\mathcal{F}_N$  which is smooth on  $\{s > 0\}$  and whose leaves cut  $\{x = y = 0\}$  transversely. In fact on  $\{s > 0\}$ , it gives us the stable foliation which is known to be smooth in this degenerate case. Important to know are the differentiability properties of this foliation when we let  $s \rightarrow 0$ . There is however more to say: In the first directional blow-up,  $\hat{f}$  is, along  $\{u = 0\} \cup \{v = 0\}$ , infinitely tangent to the time one map of  $\hat{Y}$  which has the invariant foliation given by the orbits of regular vector field  $\frac{1}{v}Z$ . Thus the foliation  $\mathcal{F}_N$  has to be smooth at least in domains in the  $(x, y, s)$ -space which are suspensions by means of the  $X$ -orbits of sectors  $C \subset \{y = 0, x \geq 0\}$ , where  $C$  is the blow up to a neighborhood  $\hat{C}$  of  $\{u = 0, 0 \leq v \leq v_0\}$  in the  $(u, v)$ -plane.

For simplicity in exposition, let us from now describe as much as possible everything in terms of  $\hat{f}$ , the blowing up of the diffeomorphism  $f$ , knowing that we need to suspend by means of the  $X$ -orbits in order to get the foliation  $\mathcal{F}_N$ . Holonomy of  $\mathcal{F}_N$  inside the suspension of  $\hat{C}$  is regular, meaning that the coordinate  $s$  can be used on any segment transverse to the blown-up locus  $\{u = 0\}$  in the second directional blow up. In this way,  $\xi = s$  can be used as a regular parameter on the transverse section considered in the first directional blow up near the saddle point at  $(u, w) = (0, 0)$ . Because of the presence of this saddle point in the blow up of  $f$ , the holonomy of  $\mathcal{F}_N$  between the two transverse sections is no longer regular but is expressed by a Mourtada form as above, with the exponent  $\frac{5}{7}$  instead of  $\frac{7}{5}$ . The same is of course true for any holonomy map of  $\mathcal{F}_N$  going from a segment that cuts transversely the  $X$ -suspension of  $\{u = 0\}$  defined in the second directional blow up, to another segment that cuts transversely the  $X$ -suspension of  $\{w = 0\}$  defined in the first directional blow-up.

For the purpose we need, it means that whenever we take a segment  $\kappa$  in the original coordinates  $(x, y, s)$ , that cuts transversely  $\{s = 0\}$  at  $(x_0, y_0) \neq (0, 0)$  and we let  $k$  be a regular parameter on  $\kappa$  with

$k = 0$  representing  $(x_0, y_0)$  and  $\{k > 0\}$  representing  $\kappa \cap \{s > 0\}$ , the holonomy from  $\kappa$  to  $\{x = y = 0\}$  along the foliation  $\mathcal{F}_N$  is given by a Mourtada form

$$s = ck^{7/5}(1 + \varphi(k))$$

for some positive constant  $c$  and some function  $\varphi$  having the desired property.  $\square$

We are now ready to finish the proof of Proposition 3.3.

*Proof of Proposition 3.3.* Take a neighborhood  $U_N$  of the northpole which is foliated by  $\mathcal{F}_N$ . Take a transverse section  $\sigma$  in  $U_N$  which is regularly parametrized by  $s = 1/B^{1/4}$  where  $s$  has to be considered here as a variable near the south pole. The passage from  $\sigma$  to the  $s$ -axis (in coordinate near the north pole) is given by

$$s \mapsto cs^{7/5}(1 + \varphi(s)) = \bar{s} = \frac{1}{\bar{B}^{1/4}}.$$

Note that this form is valid as well in the original coordinates  $(u, v, s)$  before the normal form calculation.

Observe that there exists a unique connecting orbit from any sufficiently small  $s$  to an  $\bar{s}$  which is uniquely determined by the above formula. We clearly have

$$\frac{d\bar{s}}{ds} = \frac{7}{5}cs^{2/5}(1 + \psi(s)) > 0$$

for  $0 < s \ll 1$ , and hence  $\frac{d\bar{B}}{dB} > 0$  for sufficiently large  $B$ , finishing the proof of the proposition.  $\square$

We are now also able to prove Theorem 3.1 as a consequence of Proposition 3.3.

*Proof of Theorem 3.1.* For the notation we go back to the proof of Proposition 3.3. Since  $\frac{7}{5} > 1$ , we have  $\bar{s} < s$  and hence  $\bar{B} > B$ . In particular if  $B \rightarrow \infty$ ,  $\bar{B} \rightarrow \infty$  as well. Therefore from  $A = (B + \bar{B})/2$  and  $C = (B - \bar{B})/2$ , it follows that

$$\bar{\mu} = \bar{\mu}_B = -\frac{1}{A^{1/2}} \rightarrow 0.$$

as  $B \rightarrow \infty$ . We need to find, for  $\bar{\mu} < 0$  sufficiently small, a smooth function  $\bar{\lambda} = h(\bar{\mu})$  satisfying  $g(h(\bar{\mu}), \bar{\mu}) = 0$ , where

$$g(\bar{\lambda}, \bar{\mu}) = \frac{1 - \bar{\lambda}}{\bar{\mu}^2} - f\left(\frac{1 + \bar{\lambda}}{\bar{\mu}^2}\right).$$

Clearly  $g(0, \bar{\mu}) = |\bar{\mu}|^{-14/5}(|\bar{\mu}|^{4/5} - |\bar{\mu}|^{14/5}f(1/|\bar{\mu}|^2)) < 0$ , because of the asymptotic properties of  $f$ .

If we take  $\bar{\lambda} = -1 + D\bar{\mu}^2$  for some  $D > 0$  sufficiently large, then

$$g(-1 + D\bar{\mu}^2, \bar{\mu}) = \frac{2 - D\bar{\mu}^2}{\bar{\mu}^2} - f(D) = \frac{1}{\bar{\mu}^2}(2 - (D + f(D))\bar{\mu}^2) > 0,$$

and hence, fixing  $D > 0$  sufficiently large and keeping  $\bar{\mu} < 0$  small, we are sure to find some  $h(\bar{\mu}) \in (-1 + D\bar{\mu}^2, 0)$  with  $g(h(\bar{\mu}), \bar{\mu}) = 0$ . In fact, the smaller we take  $\bar{\mu}$ , the larger we may take  $D$ . Since

$$\frac{\partial g}{\partial \bar{\lambda}} = -\frac{1}{\bar{\mu}^2} \left(1 + \frac{\partial f}{\partial B} \left(\frac{1 + \bar{\lambda}}{\bar{\mu}^2}\right)\right) < 0,$$

this function  $h$  has to be unique and  $C^\infty$  because of the implicit function theorem.

It is easy to show that  $\lim_{\bar{\mu} \rightarrow 0} h(\bar{\mu}) = -1$ . Indeed, from its derivation, we see that

$$h(\bar{\mu}) = -1 + \bar{\mu}^2 f^{-1}\left(\frac{1 - h(\bar{\mu})}{\bar{\mu}^2}\right).$$

The asymptotic properties of  $f$  and the fact that  $h(\bar{\mu}) < 0$  induce the result.

Note that  $\bar{\lambda} = \bar{\lambda}(\bar{\mu})$  is a smooth function and that  $\bar{\lambda}$  is a transverse parameter. The latter follows from the fact that  $\frac{d\bar{s}}{ds} > 0$  for  $0 < s \ll 1$ .

Now we fix  $\bar{\mu}_0 < 0$  sufficiently close to 0 and choose  $C_0$  for (3.2) such that we have a unique connection from  $-B$  to  $\bar{B}$ . Going back to the system  $X_{C_0}$ , we have a connection from  $-A = -B + C_0$  to  $\bar{A} = \bar{B} + C_0$ .

Let  $T_C$  denote the translation  $(x, y, z) \mapsto (x, y, z + C)$ . Let  $(\bar{\mu}_0, \bar{\lambda}_0)$ , and corresponding  $(A_0, C_0)$ , be such that the system  $X_{C_0}$  possesses a connection from  $-A_0$  to  $A_0$  (all with  $\bar{\mu}_0 < 0$  sufficiently small). Let us now fix  $\bar{\mu}_0$  (and hence fix  $A_0$ ) and change  $C \approx C_0$ . To follow

the unstable manifold at  $(0, 0, -A_0)$  for system  $X_C$  we use again the translation to reduce the study to  $X_0$ . Define  $g(A_0, C)$  as

$$g(A_0, C) = T_C \circ f \circ T_{-C}(-A_0) = f(A_0 + C) + C.$$

Note that in system  $X_0$  the unstable manifold of  $(0, 0, -B)$  tends to  $(0, 0, \bar{B})$  with  $\bar{B} = f(B)$  where  $\frac{df}{dB} > 0$  holds. In the above we have taken  $B = A_0 + C$ . This implies that in system  $X_C$  the unstable manifold of  $(0, 0, -A_0)$  goes to  $(0, 0, g(A_0, C))$ . Now,

$$\frac{\partial g}{\partial C}(A_0, C) = 1 + \frac{\partial f}{\partial B}(A_0 + C) > 0.$$

□

**Remark 3.6** Since we found the heteroclinic orbit from  $(0, 0, -B)$  to  $(0, 0, \bar{B})$  as a perturbation of the one in the sphere at infinity of the Poincaré-Lyapunov ball, it has the so-called strong inclination property ([3]), namely, the stable manifold of  $(0, 0, \bar{B})$  is transverse to one (and hence any) invariant manifold of  $(0, 0, -B)$  which is tangent to the eigenspaces associated to the unstable and the weak stable eigenvalues, the latter being the  $z$ -axis.

## 4 Dynamics from the heteroclinic cycle

In this section, we briefly discuss the dynamics in the unfolding based on the existence of a heteroclinic cycle proven in the previous section. The cycle connecting the singularities  $P_{\pm} = (0, 0, \bar{z}_{\pm})$  where  $\bar{z}_{\pm} \approx \pm 1$  exists in the equation (3.1) with  $\bar{\nu} = -1$  and  $\bar{\lambda} = H(\bar{\mu}, r)$ ,  $\bar{\mu} < 0$  for some function  $H(\bar{\mu}, r)$  given in Theorem 3.1. The singularity  $P_+ = (0, 0, 1)$  is a saddle-focus with eigenvalues  $-\alpha \pm i\omega$  and  $\beta$ , whereas the singularity  $P_- = (0, 0, -1)$  is a real saddle with eigenvalues  $-\beta, -\gamma, \delta$ , where

$$\alpha \gg \beta > 0, \quad \text{and} \quad \gamma > \delta \gg \beta > 0,$$

for sufficiently small  $r$ . From the form of the linear part, it follows that  $-2\alpha = -\gamma + \delta = \bar{\mu} < 0$ .

**Lemma 4.1**

$$\delta \gg \alpha \gg \beta > 0$$

for sufficiently small  $r > 0$ . Moreover  $\alpha/\delta \rightarrow 0$  as  $\bar{\mu} \rightarrow 0$ .

*Proof.* From the proof of Theorem 3.1, we know that  $\lim_{\bar{\mu} \rightarrow 0} h(\bar{\mu}) = -1$ , while

$$h(\bar{\mu}) \geq -1 + D\bar{\mu}^2,$$

for some  $D > 0$ . In fact one can take  $D$  as large as wanted, if one takes  $\bar{\mu} < 0$  sufficiently close to 0.

Now for parameter values  $(\bar{\lambda}, \bar{\mu}) = (h(\bar{\mu}), \bar{\mu})$  it follows that

$$\begin{aligned} \frac{\delta}{\alpha} &= -1 - \frac{1}{\bar{\mu}} \left( \bar{\mu}^2 + 4(1 + h(\bar{\mu})) \right)^{1/2} \\ &= -1 + \left( 1 + 4 \left( \frac{1 + h(\bar{\mu})}{\bar{\mu}^2} \right) \right)^{1/2} \\ &\geq -1 + (1 + 4D)^{1/2}, \end{aligned}$$

from which the claim follows.  $\square$

Using this information, we can derive the Poincaré map along the heteroclinic cycle. Taking cross sections  $\Sigma_{in}^{\pm}, \Sigma_{out}^{\pm}$  as in Figure 6, it is given by the composition of four maps  $\Pi_{loc}^+, \Pi_{loc}^-, \Pi_{far}^+, \Pi_{far}^-$ . The maps  $\Pi_{loc}^{\sigma}$  ( $\sigma = \pm$ ) are the local transition maps from  $\Sigma_{in}^{\sigma}$  to  $\Sigma_{out}^{\sigma}$ , respectively, and take the form

$$\begin{aligned} \Pi_{loc}^+(\theta, z) &= \left( Az^{\alpha/\beta} \cos\left(B + \theta - \frac{\omega}{\beta} \log z\right), \right. \\ &\quad \left. Az^{\alpha/\beta} \sin\left(B + \theta - \frac{\omega}{\beta} \log z\right) \right), \end{aligned} \quad (4.1)$$

$$\Pi_{loc}^-(x, y) = (Cx^{\gamma/\delta}y, Dx^{\beta/\delta}), \quad (4.2)$$

with  $A, B, C, D$  depending on the choice of cross sections, if the vector fields are locally linearizable, where  $(x, y, z)$  and  $(r, \theta, z)$  are local Cartesian and cylindrical coordinates around  $P_{\pm}$ , each of which are corresponding to the eigendirections, in particular  $z$  being the eigendirection associated to the eigenvalue  $\beta$  and  $-\beta$ . If not linearizable, the maps  $\Pi_{loc}^{\pm}$  still have essentially the same expression to their leading order, due to the established results given by [19], [3], see also [12], [18]. The maps  $\Pi_{far}^{\pm}$  are regular transitions from  $\Sigma_{out}^{\pm}$  to  $\Sigma_{in}^{\mp}$ , respectively,



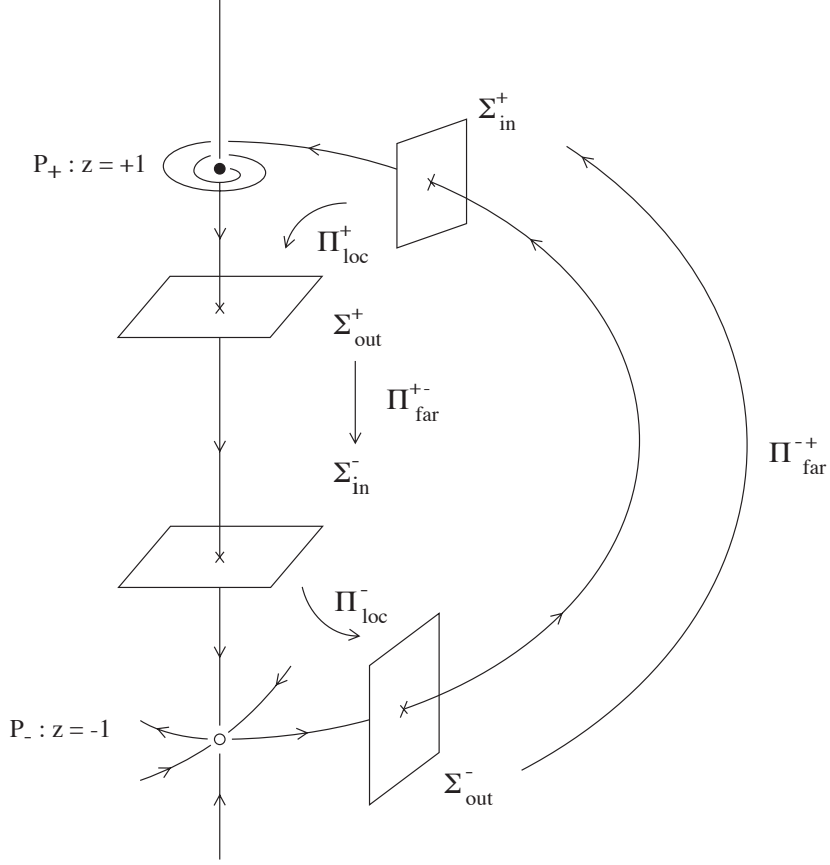


Figure 6: Cross sections for the heteroclinic cycle.

along the heteroclinic orbits away from the singularities. To their leading order, we just consider them as affine maps. Combining them together, we may express the Poincaré map  $\Pi = \Pi_{far}^{+-} \circ \Pi_{loc}^+ \circ \Pi_{far}^{-+} \circ \Pi_{loc}^-$ .

We take a rectangle  $R_+ = \{|x| < x_0, |y| < y_0\}$  in the cross section  $\Sigma_{in}^-$  for small enough  $x_0, y_0 > 0$ . It is mapped under  $\Pi_{loc}^-$  to a cusp shaped region and then mapped by  $\Pi_{far}^{-+}$  into  $\Sigma_{in}^+$  diffeomorphically. Due to the strong inclination property in Remark 3.6, the cusp shaped image in  $\Sigma_{in}^+$  lies transversely to the stable manifold of  $P_+$  as in Figure 6, and hence mapped further under  $\Pi_{loc}^+$  to a spiral-shaped region which is mapped back diffeomorphically into  $\Sigma_{in}^-$  under  $\Pi_{far}^{+-}$ . Because of the  $\mathbb{Z}_2$ -symmetry, the other part of the rectangle  $R_-$  is also mapped to a spiral shaped region  $\Pi(R_-)$ , symmetric to  $\Pi(R_+)$

under the reflection with respect to the origin in  $\Sigma_{in}^-$ .

This situation is exactly the same as one finds in the study of dynamics around the Shil'nikov type homoclinic orbit to a saddle-focus singularity, where the eigenvalue condition  $\delta > \alpha > 0$  here plays an analogous role as the so-called Shil'nikov eigenvalue condition does. Therefore one can derive the same conclusion as the Shil'nikov theorem that there exist infinitely many horseshoes for the Poincaré map  $\Pi$ . In fact, it is shown in Tresser [20] that this type of heteroclinic cycle with the eigenvalue condition  $\delta > \alpha > 0$  and the strong inclination property has infinitely many horseshoes, in exactly the same manner as the Shil'nikov theorem.

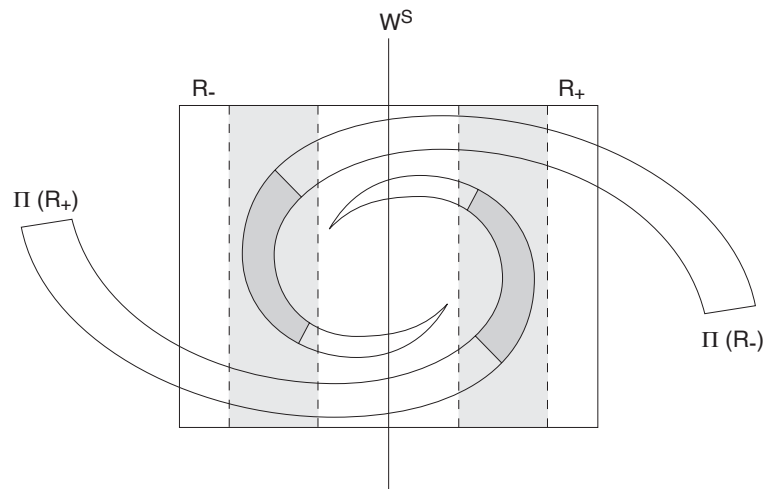


Figure 7: Possibility of Hénon-like attractors bifurcating from the heteroclinic cycle.

If the vector field is perturbed and hence the Poincaré map is deformed accordingly, one may also find parameter values in which the image of the rectangles  $R_{\pm}$  under the perturbed Poincaré map becomes as in Figure 7 or Figure 8. The former case corresponds to the situation where the suspension of a Hénon-like strange attractor may exist whereas the latter does where a Lorenz-type attractor may exist. In fact this observation may be more confirmed by a heuristic argument as follows:

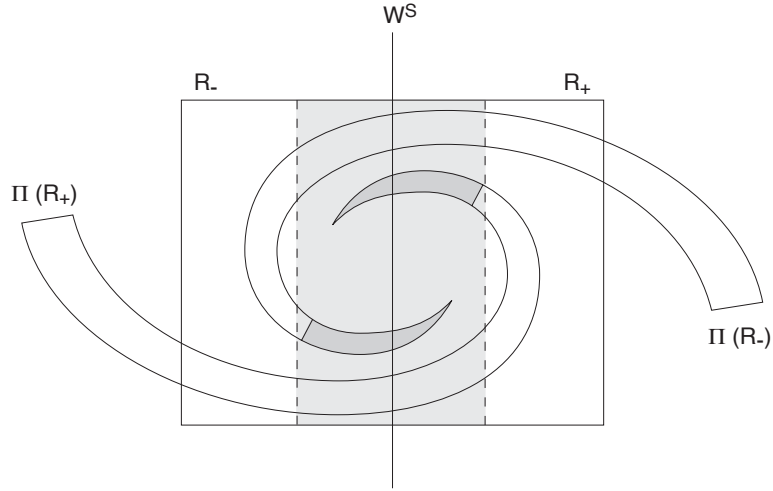


Figure 8: Possibility of Lorenz-like attractors bifurcating from the heteroclinic cycle.

Since we have strong contraction in the strong stable direction near  $P_-$ , we may be able to capture the essential feature of dynamics by only looking at the one-dimensional dynamics complementary to the strong stable direction. This one dimensional dynamics will be obtained by composing the following two maps:

$$\begin{aligned} x &\mapsto x^{\beta/\delta} + \eta = z, \\ z &\mapsto z^{\alpha/\beta} \cos(\log z) = \varphi_{\eta}(x), \end{aligned}$$

where  $\eta$  is a parameter which unfolds the heteroclinic connection from  $P_-$  to  $P_+$ . Notice that these maps are derived from the forms of  $\Pi_{loc}^{\pm}$ , but they are simplified by fixing the constants  $A, B, C, D$ , etc. appropriately.

For the one-dimensional map  $\varphi_{\eta}(x)$ , it is easy to show the following two facts:

- (1) There exists a sequence of parameters  $\{\eta_n\}$  which converge to zero as  $n \rightarrow \infty$  such that the corresponding map  $\varphi_{\eta_n}$  has a fixed point  $x_n$  at which  $\varphi'_{\eta_n}(x_n) = 0$  holds. Moreover,  $x_n \rightarrow 0$  and  $\varphi''_{\eta_n}(x_n) \rightarrow -\infty$  as  $n \rightarrow \infty$ .

- (2) There exists a sequence of parameters  $\{\tilde{\eta}_n\}$  which converge to zero as  $n \rightarrow \infty$  such that the corresponding map  $\varphi_{\tilde{\eta}_n}$  has the asymptotics

$$\varphi_{\tilde{\eta}_n}(x) \approx K_n x^{\beta/\delta}$$

as  $x \rightarrow 0$ . Moreover  $K_n \rightarrow 0$  as  $n \rightarrow \infty$ .

We consider that these facts support the existence of Hénon-like and Lorenz-like attractors in unfoldings of the heteroclinic cycle. Indeed, from the first fact, one can find, by slightly perturbing  $\varphi_{\eta_n}$ , a unimodal map in a small interval near  $x_n$  which is mapped onto itself, and this unimodal map may be thought of as a one-dimensional analogue of the Hénon-like attractors ([13]). Similarly, from the second fact, one can perturb  $\varphi_{\tilde{\eta}_n}$  slightly to have a map with discontinuity defined on a small interval around 0 which resembles the one-dimensional map of the geometric Lorenz attractor ([10]).

We believe that there also exist Hénon-like and Lorenz-like attractors in the two-dimensional Poincaré maps for unfoldings of the heteroclinic cycle because the maps are given by adding strong contraction normal to the direction where the dynamics is essentially that of  $\varphi_\eta$ . Of course it requires more detailed analysis in order to rigorously verify the existence of such attractors. We do not carry out this task here because there are rather well-established techniques for this purpose thanks to the results of [2], [13], [15] for Hénon-like attractors, and [10], [1], [17], [16], [7] for Lorenz-like attractors.

## A Proof of Theorem 3.4 on a normal form for vector field on $\mathbb{R}^3$ having a line of singularities

Here we give a proof of Theorem 3.4.

*Proof.* The proof proceeds in two steps. First we make a formal calculation along  $I$  to bring the Taylor expansion of  $X$  with respect to  $(x, y)$  and whose coefficients are smooth functions in  $s$ , to an expression as in (3.6), using near-identity coordinate changes along  $I$ . This part is similar to the traditional normal form theorem at one point, as treated in [9]. Second, we finish the proof by showing the existence of

a smooth diffeomorphism having the required condition and realizing the formal requirements in the first step.

(1st Step): For the first part of the proof we use the complex notation  $z = x + iy$ ,  $\bar{z} = x - iy$  for which

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

and hence, the 1-jet of the form (3.5) changes to

$$L(s) = b(s)i \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right) + a(s) \left( z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} \right).$$

Exactly as in the usual formal normal form calculation (see, e.g. [9]), we consider the Lie-bracket operation (or adjoint operation)

$$\begin{aligned} \left[ L(s), z^k \bar{z}^l \frac{\partial}{\partial z} \right] &= ((k-l-1)b(s)i + (k+l-1)a(s)) z^k \bar{z}^l \frac{\partial}{\partial z} \\ \left[ L(s), z^k \bar{z}^l \frac{\partial}{\partial \bar{z}} \right] &= ((k-l+1)b(s)i + (k+l+1)a(s)) z^k \bar{z}^l \frac{\partial}{\partial \bar{z}} \\ \left[ L(s), z^k \bar{z}^l \frac{\partial}{\partial s} \right] &= ((k-l)b(s)i + (k+l)a(s)) z^k \bar{z}^l \frac{\partial}{\partial z} \\ &\quad - c'(s) z^k \bar{z}^l \left( z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} \right). \end{aligned}$$

If we want to remove a term  $(a_{kl}(s) + ib_{kl}(s)) z^k \bar{z}^l \frac{\partial}{\partial z}$ , we first show that it is in the image of the adjoint operator given above by  $L(s)$ , and this is equivalent to solving the linear equation

$$\begin{pmatrix} (k-l-1)b(s) & (k+l-1)a(s) \\ (k+l-1)a(s) & -(k-l-1)b(s) \end{pmatrix} \begin{pmatrix} \alpha_{kl}(s) \\ \beta_{kl}(s) \end{pmatrix} = \begin{pmatrix} a_{kl}(s) \\ b_{kl}(s) \end{pmatrix}.$$

The determinant of the coefficient matrix is equal to

$$-(k-l-1)^2 b(s)^2 - (k+l-1)^2 a(s)^2$$

which is strictly negative in case  $k \neq l+1$ , and hence this linear equation has a solution  $(\alpha_{kl}(s), \beta_{kl}(s))$  which is smooth in  $s$ . Of course, for  $k = l+1$  and  $a(s) \neq 0$ , the equation can also be solved, but we do not do this because we admit  $a(s)$  to be zero at some values of  $s$ .

In a completely similar way, we can show that the terms with  $z^k \bar{z}^l \frac{\partial}{\partial \bar{z}}$  for  $l \neq k+1$  are in the image of the adjoint operator.

Now we proceed like in the proof of [9] for the traditional normal form theorem by using successive coordinate changes, depending on  $m \in \mathbb{N}$ , of the form

$$(z, \bar{z}, s) \mapsto (z + P(z, \bar{z}, s), \bar{z} + \bar{P}(z, \bar{z}, s), s + Q(z, \bar{z}, s))$$

where

$$\begin{aligned} P(z, \bar{z}, s) &= \sum_{i=0}^m p_i(s) z^i \bar{z}^{m-i}, \\ Q(z, \bar{z}, s) &= \sum_{i=0}^m q_i(s) z^i \bar{z}^{m-i}, \end{aligned}$$

with  $(p_i(s), q_i(s))$  being complex valued smooth functions in  $s$  defined on  $I$  and to be determined by the usual procedure as in [9]. Note that concerning the terms with  $z^k \bar{z}^l \frac{\partial}{\partial s}$  for  $k \neq l$ , a similar calculation as above shows that they are in the image of the adjoint operator modulo terms of degree  $m + 1$  in  $(z, \bar{z})$  when we do the formal calculation up to order  $m$ . Let us denote the coordinate transformation by  $Id + R$  and let  $v = (z, \bar{z}, s)$ . Then the transformed vector field results in

$$(Id + DR(v))\dot{v} = A(s)(v + R(v)) + g_2(v) + \cdots + g_{m-1}(v) + f_m(v) + o(|z, \bar{z}|^m),$$

where  $g_i(v)$  denotes a homogeneous polynomial of degree  $i$  in  $(z, \bar{z})$ , with coefficients being a smooth function of  $s$ . By induction, we may assume that the  $g_i(v)$  already has the required expression. The function  $f_m(v)$  represents a homogeneous polynomial of degree  $m$  in  $(z, \bar{z})$  with smooth functions in  $s$  as the coefficients, and the rest represents a smooth function of  $(z, \bar{z}, s)$  being flat of order  $m$  with respect to  $(z, \bar{z})$ .  $A(s)$  is a matrix representation of  $L(s)$  in the basis.

This expression can also be written as

$$\begin{aligned} \dot{v} &= (Id + DR(v))^{-1} \{ A(s)(v + R(v)) + g_2(v) \\ &\quad + \cdots + g_{m-1}(v) + f_m(v) + o(|z, \bar{z}|^m) \} \\ &= A(s)(v + R(v)) + g_2(v) + \cdots + g_{m-1}(v) \\ &\quad + f_m(v) + A(s)R(v) - DR(v)A(s)v + o(|z, \bar{z}|^m). \end{aligned}$$

The  $(m - 1)$ -jet with respect to  $(z, \bar{z})$  remains unchanged, while for the terms of order  $m$ , we get

$$f_m(v) - \text{ad}_m(L(s))R(v),$$

showing that the Lie-bracket operation is the exact equation to solve in order to find the adequate  $(P(z, \bar{z}, s), Q(z, \bar{z}, s))$ . As already calculated before, the adjoint operator is not completely operating at a homogeneous level in  $(z, \bar{z})$ , but introduces also terms of higher degree. This does not of course give problem to the induction procedure.

(2nd Step): To finish the proof, we can look at infinite succession of coordinate changes as introduced above for increasing degree  $m$ . Because of the construction, this infinite succession definitely leads to a well-defined  $\infty$ -jet in  $(z, \bar{z})$  and hence also in  $(x, y)$  with smooth functions of  $s$  as the coefficients. This formal expression in  $(x, y)$  can now be extended to a smooth mapping  $\varphi$  defined in a neighborhood of  $I$  if we apply the version of the Borel theorem given in [11]. The condition on the 1-jet of  $\varphi$  along  $I$  implies that  $\varphi$  is a diffeomorphism in the neighborhood of  $I$ . This completes the proof of Theorem 3.4.  $\square$

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