An example of an infinitely renormalizable cubic polynomial and its combinatorial class

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The unicritical family and the Multibrot set

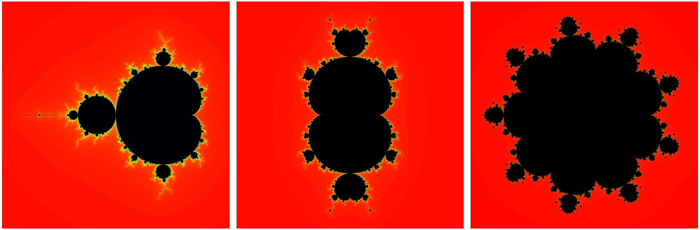
For $d \geq 2$, consider the ***unicritical family***

$$f_c(z) = z^d + c, \quad c \in \mathbb{C}.$$

Its ***connectedness locus***

$$\mathcal{M}_d = \{c \in \mathbb{C};\ K(f_c) \text{ is connected}\}$$

is called the ***Multibrot set*** of degree d.
We often identify $c \in \mathcal{M}_d$ with the corresponding map f_c.



Hyperbolicity

- A map $f_c \in \mathcal{M}_d$ is ***hyperbolic*** $\overset{\text{def}}{\iff}$ f_c has an attracting cycle.
- A ***hyperbolic component*** in $\mathcal{M}_d :=$ a connected component of the set of hyperbolic maps in $\mathcal{M}_d$ (open set).
- The ***period*** of a hyperbolic component $\mathcal{H} :=$ the period of the (unique) attracting cycle for $f_c \in \mathcal{H}$ (independent of the choice of f_c).
- $\mathcal{H}$: ***satellite*** $\overset{\text{def}}{\iff}$ $\mathcal{H}$ has a common boundary point with another hyperbolic component $\mathcal{H}'$ of lower period.
- $\mathcal{H}$: ***primitive*** $\overset{\text{def}}{\iff}$ not satellite.
- $\mathcal{H}_0$: the main hyperbolic component
  ($0 \in \mathcal{H}_0$, period=1)

Self-similarity

Theorem 1

For every hyperbolic component $\mathcal{H} \in \operatorname{int} \mathcal{M}_d$ of period p, there exists a "baby Multibrot set" centered at $\mathcal{H}$; more precisely, there exist

- $M(\mathcal{H}) \subset \mathcal{M}_d$ and
- $\exists \chi_{\mathcal{H}} : M(\mathcal{H}) \to \mathcal{M}_d$: a homeomorphism

such that

- $\mathcal{H} \subset M(\mathcal{H})$ and $\chi_{\mathcal{H}}(\mathcal{H}) = \mathcal{H}_0$.
- For every $c \in M(\mathcal{H})$ (except the root for satellite $\mathcal{H}$), f_c is ***renormalizable*** of period p; i.e., there exists a polynomial-like restriction $f_c^p : U_c' \to U_c$ of degree d with connected filled Julia set.
- The renormalization $f_c^P : U_c' \to U_c$ is hybrid equivalent to $f_{\chi_{\mathcal{H}}(c)}$.

Straightenings and tunings

The map $\chi_{\mathcal{H}} : M(\mathcal{H}) \to \mathcal{M}_d$ is called the ***straightening map***, and the inverse operation is called ***tuning***:

$$\mathcal{M}_d \ni c' \mapsto c = \mathcal{H} * c' := \chi_{\mathcal{H}}^{-1}(c).$$

Roughly speaking, the filled Julia set of f_c can be obtained by replacing the closure of each Fatou component for $K(f_{c_0})$ for $c_0 \in \mathcal{H}$ by the filled Julia set of $K(f_{c'})$.

Satellites

For each hyperbolic component $\mathcal{H}$ of period p, We can associate the ***multiplier map***

$$\lambda_p : \mathcal{H}_0 \ni c \mapsto \lambda(f_c) := (f_c^p)'(x_c) \in \mathbb{D} := \{|z| < 1\},$$

where x_c is an attracting periodic point for f_c.

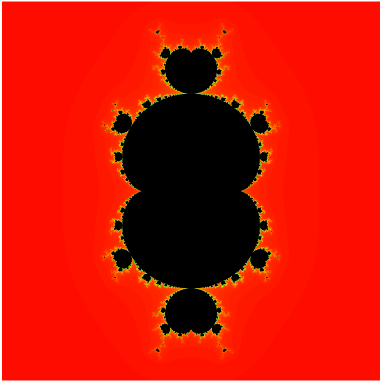
The multiplier map is a branched covering of degree $d - 1$, branched only at the ***center*** $\lambda_p^{-1}(0)$ and extends continuously to the closure $\overline{\mathcal{H}} \to \overline{\mathbb{D}}$.

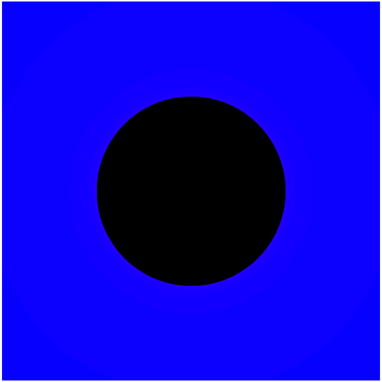
For each

$$c \in \partial\mathcal{H} \text{ with } \lambda_p(c) = e^{2\pi i m/q}, \quad (m/q \in \mathbb{Q}/\mathbb{Z},\ m \neq 0),$$

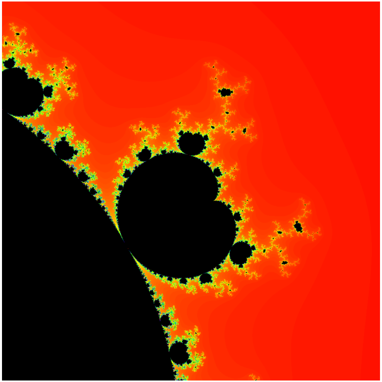
there exists a unique hyperbolic component $\mathcal{H}' \neq \mathcal{H}$ such that $c \in \partial\mathcal{H}'$. We say $\mathcal{H}'$ is a ***satellite attached*** to $\mathcal{H}$ with ***internal angle*** m'/q.
There are $d - 1$ satellites of internal angle m'/p.

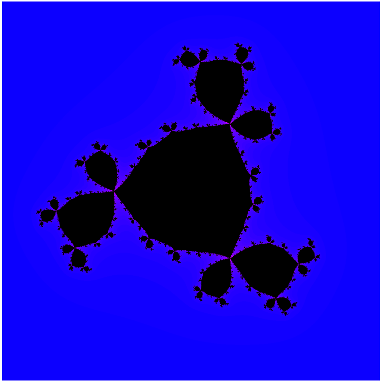
Satellites: Cubic Multibrot set



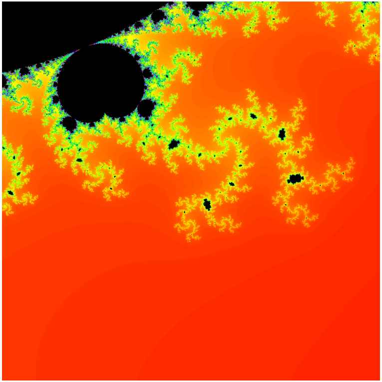


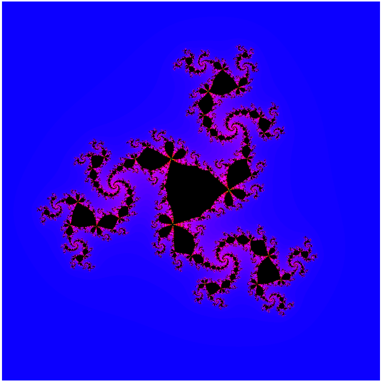
Satellites: 1/3-satellite $\mathcal{H}_3$



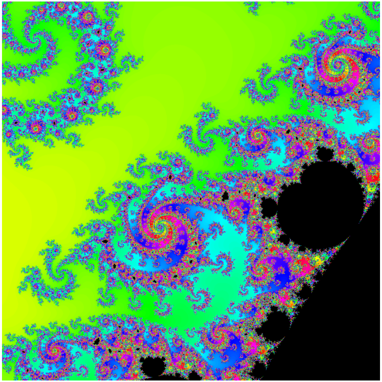
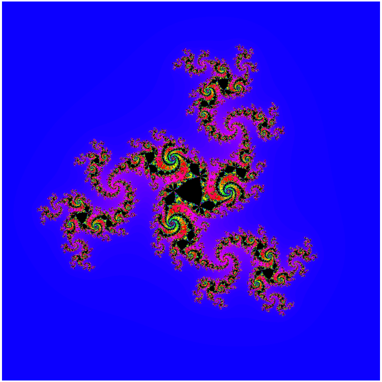


Satellites: 1/4-satellite of 1/3-satellite $\mathcal{H}_3 * \mathcal{H}_4$





Satellites: $\mathcal{H}_3 * \mathcal{H}_4 * \mathcal{H}_5$

Dynamics in a satellite hyperbolic component

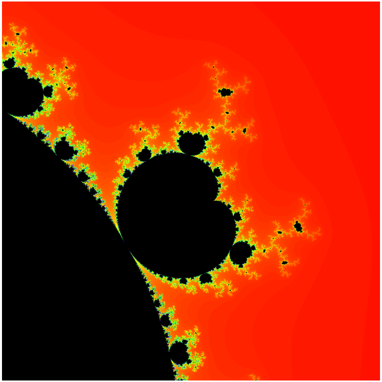
Let $\mathcal{H}_q$ be the satellite attached to $\mathcal{H}_0$ with internal angle $1/q$, closest to the positive real axis ($q > 1$).

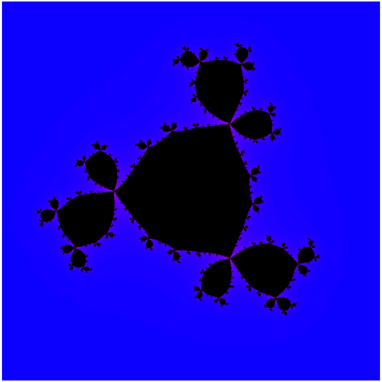
For simplicity, we only consider hyperbolic components obtained by tuning of $\mathcal{H}_q$'s.

Let $c \in M(\mathcal{H}_q)$. Then f_c has a fixed point x_1 of rotation number $1/q$. The external ray of angle $1/(2^q - 1)$ lands at x_1.
Let $K_1 \subset K(f_c)$ be the filled Julia set of the renormalization $f_c^q : U_c' \to U_c$.
Then $x_1 \in K_1$.

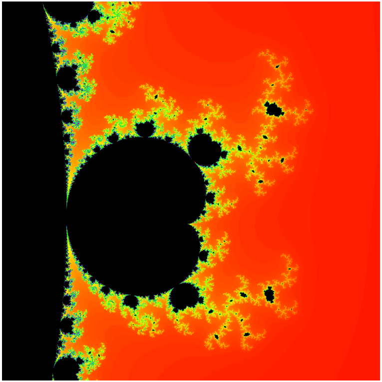
By the Yoccoz inequality, if q is sufficiently large, then x_1 is arbitrarily close to another fixed point x_0, which is the landing point of $R_{f_c}(0)$.

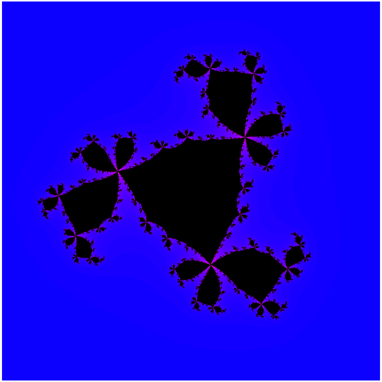
Satellites: 1/3-satellite $\mathcal{H}_3$



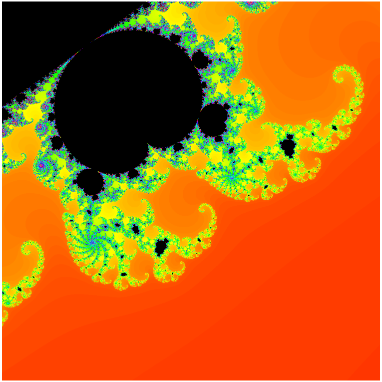


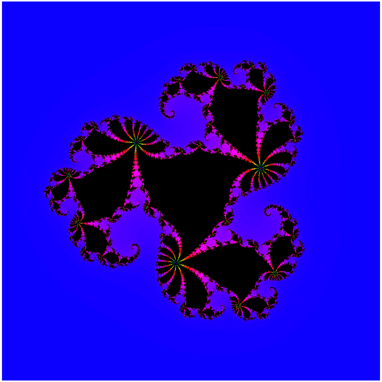
Satellites: 1/4-satellite $\mathcal{H}_4$



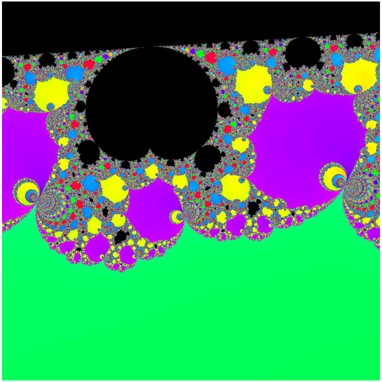


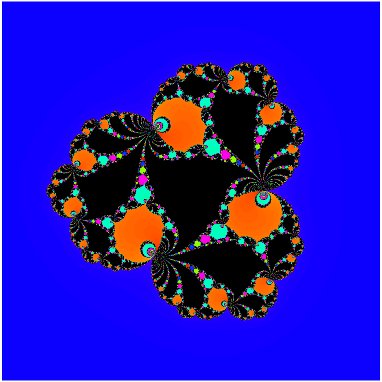
Satellites: 1/10-satellite $\mathcal{H}_{10}$



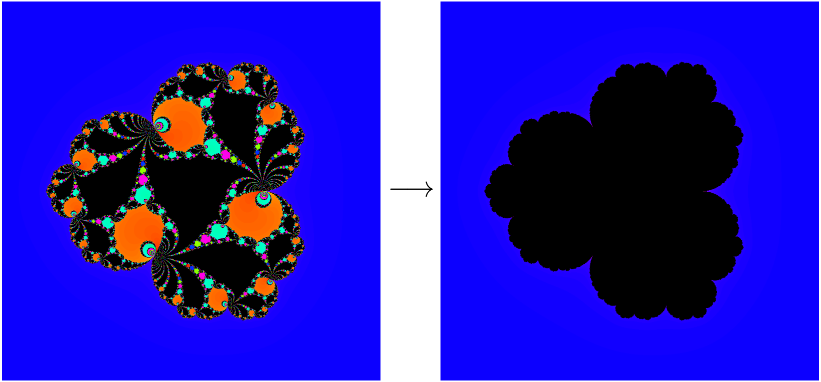


Satellites: 1/100-satellite $\mathcal{H}_{100}$





“$1/\infty$-satellite”



As $q \to \infty$, $M(\mathcal{H}_q)$ converges to $c = \frac{2}{\sqrt{3}}$, for which f_c has a parabolic fixed point.

For a finite sequence $(q_1, \ldots, q_n)$ with $q_k \geq 2$, let

$$\begin{aligned}\mathcal{H}_{(q_1, q_2, \ldots, q_n)} &:= \mathcal{H}_{q_1} * \mathcal{H}_{q_2} * \cdots * \mathcal{H}_{q_n} \\ &= \chi_{\mathcal{H}_{q_1}}^{-1} \circ \cdots \circ \chi_{\mathcal{H}_{q_{n-1}}}^{-1} (\mathcal{H}_{q_n}).\end{aligned}$$

For $c \in M(\mathcal{H}_{(q_1, \ldots, q_n)})$ and $1 \leq k \leq n$, let K_k be the filled Julia set of k-th (simple) renormalization for f_c of period

$$p_k := q_1 \ldots q_k$$

and let $x_k \in K_k$ be the periodic point of period p_{k-1}.

For a small $\varepsilon > 0$, if the sequence $(q_1, \ldots, q_n)$ glows sufficiently fast, then

$$|x_k - x_{k-1}| < \frac{\varepsilon}{2^k}, \qquad |x_0 - x_n| < \varepsilon.$$

Fibers and combinatorial classes

Following Milnor, Sørensen, Pérez-Marco

Consider an infinite sequence $\underline{\boldsymbol{q}} = (q_1, q_2, \ldots, q_n, \ldots)$ and let $\underline{\boldsymbol{q}}_n = (q_1, \ldots, q_n)$. Let us consider the ***fiber***

$$M_{\underline{\boldsymbol{q}}} = \bigcap_n M(\mathcal{H}_{\underline{\boldsymbol{q}}_n}).$$

in $\mathcal{M}_d$ associated to $\underline{\boldsymbol{q}}$.

By the above argument, we have the following:

Theorem 2

If q_n tends to infinity sufficiently fast as $n \to \infty$, then $\bigcap_n K_n$ is not a singleton for $c \in M_{\underline{\boldsymbol{q}}}$.
In particular, $K(f_c)$ is not locally connected.

The set $\bigcap_n K_n$ is the ***fiber*** in $K(f_c)$ containing 0.

Fibers and local connectivity

More generally, fibers are defined as follows:

For $K = \mathcal{M}_d$ or $K(f_c)$ ($c \in \mathcal{M}_d$), we say $z, z' \in K$ are ***separated*** if there exists two external rays of rational angles landing at the same point such that z and z' lie in different complementary component of the union of the rays and their common landing point.

A ***fiber*** is the maximal set of points which are not separated from each other.

Conjecture

The Multibrot set $\mathcal{M}_d$ is locally connected.

In particular, every infinitely renormalizable fiber in $\mathcal{M}_d$ is conjecturally trivial (a singleton).

The cubic connectedness locus

Now consider the cubic family

$$f_{a,b}(z) = z^3 - 3a^2 z + b, \quad (a,b) \in \mathbb{C}^2.$$

Let

$$\mathcal{C}_3 = \{(a,b);\ K(f_{a,b}) \text{ is connected}\}$$

be the ***cubic connectedness locus***.

We identify the slice $\{a = 0\}$ with the cubic unicritical family, so

$$\mathcal{C}_3 \cap \{a = 0\} = \mathcal{M}_3, \qquad f_c(z) = f_{0,c}(z) = z^3 + c.$$

Rational lamination and combinatorial renormalization

For $(a, b) \in \mathcal{C}_3$, let $\lambda(f_{a,b})$ be the ***rational lamination*** of $f_{a,b}$. Namely, it is an equivalence relation $\mathbb{Q}/\mathbb{Z}$ and t and s are equivalent if $R_{f_{a,b}}(t)$ and $R_{f_{a,b}}(s)$ land at the same point.

For $\underline{\boldsymbol{q}} = (q_1, q_2, \dots)$, let

$$\lambda(\mathcal{H}_{\underline{\boldsymbol{q}}_n}) := \lambda(f_c),$$

which is independent of the choice of $c \in \mathcal{H}_{\underline{\boldsymbol{q}}_n}$.

Let

$$\mathcal{C}(\mathcal{H}_{\underline{\boldsymbol{q}}_n}) := \{(a, b) \in \mathcal{C}_3;\ \lambda(f_{a,b}) \supset \lambda(\mathcal{H}_{\underline{\boldsymbol{q}}_n})\}.$$

be the set of ***combinatorially renormalizable*** parameters with combinatorics defined by $\underline{\boldsymbol{q}}_n$.

Fact

$$\mathcal{C}(\mathcal{H}_{\underline{\boldsymbol{q}}_n}) \cap \{a = 0\} = M(\mathcal{H}_{\underline{\boldsymbol{q}}_n}).$$

Main result

Let

$$\mathcal{C}_{\underline{\boldsymbol{q}}} = \bigcap_{n=1}^{\infty} \mathcal{C}(\mathcal{H}_{\underline{\boldsymbol{q}}_n}).$$

Theorem 3

If $q_1, q_2, \dots$ are sufficiently large and tends to infinity sufficiently fast as $n \to \infty$, then there exists $(a, b) \in \mathcal{C}_{\underline{\boldsymbol{q}}}$ such that

- $f_{a,b}$ has two distinct critical points ω and ω'.
- ω, ω' lie in the same fiber $\bigcap_n K_n$, where K_n is the filled Julia set of n-th renormalization $f_{a,b}^{p_n} : U'_n \to U_n$.
- ω is recurrent, but ω' is not.

Cor 4

$\mathcal{C}_{\underline{\boldsymbol{q}}}$ is non-trivial. Moreover, it contains a continuum.

(cf. Henriksen: Non-trivial fiber for infinitely renormalizable combinatorics of capture type.)

Construction

For $q \geq 2$, let $g_q \in \mathcal{C}(\mathcal{H}_q)$ be such that the fixed point x_1 of rotation number $1/q$ is parabolic and there exists a critical point ω' satisfies

$$g_q(\omega') = x_1.$$

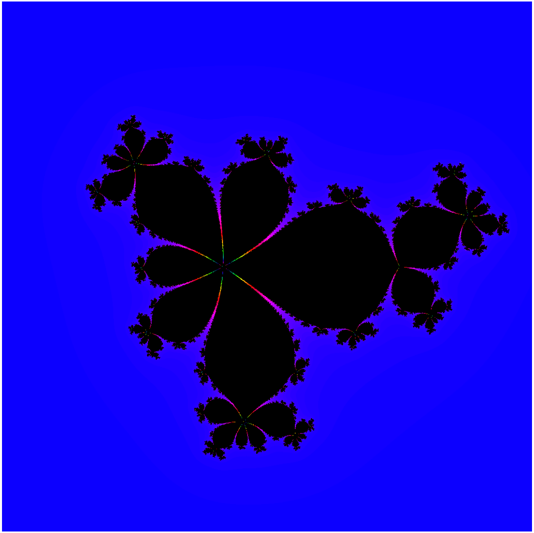
Lemma 5

$g_\infty := \lim_{q \to \infty} g_q$ is affinely conjugate to $z(z+1)^2$.

(Julia sets for g_q w/ attr. per pt)

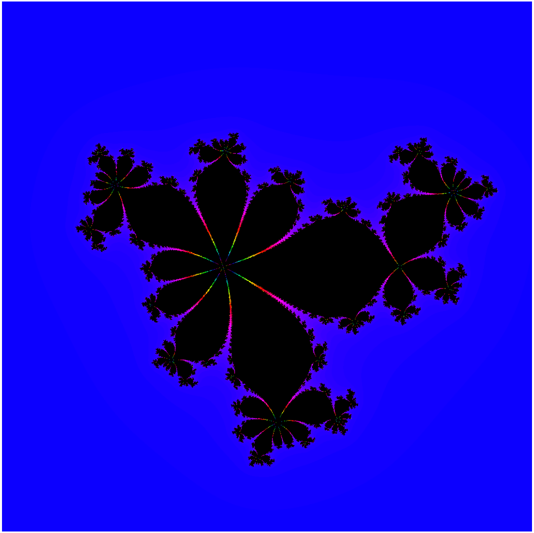
The Julia set of g_q, $q = 3$

(perturbed slightly to have an attracting cycle)



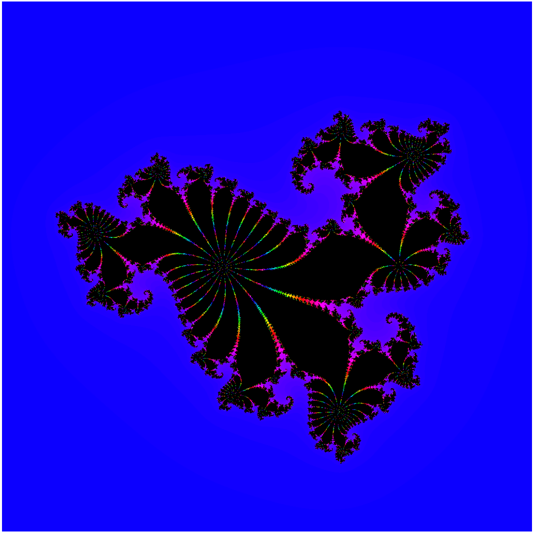
The Julia set of g_q, $q = 4$

(perturbed slightly to have an attracting cycle)

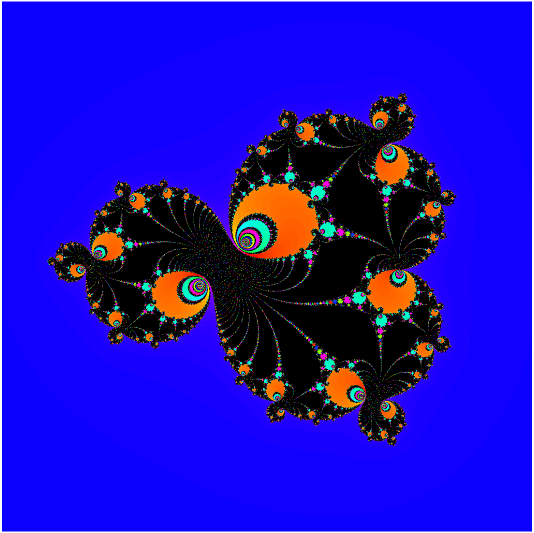


The Julia set of g_q, $q = 10$

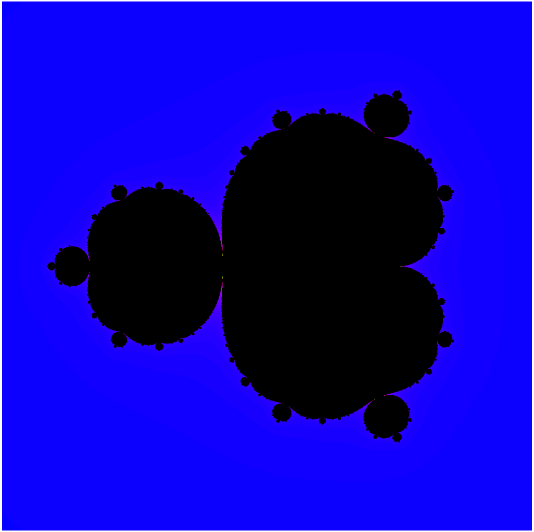
(perturbed slightly to have an attracting cycle)



The Julia set of g_q, $q = 100$
(perturbed slightly to have an attracting cycle)



The Julia set of $g_\infty = \lim g_q$,



Construction (cont'd)

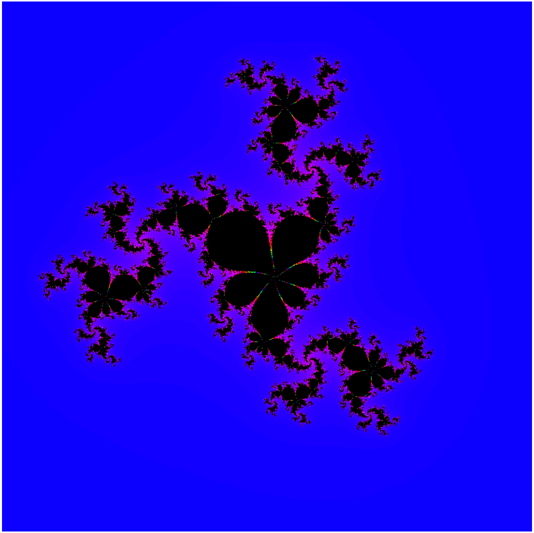
For $\underline{\boldsymbol{q}} = (q_1, q_2, \dots)$, and $n \geq 2$, let $g_{\underline{\boldsymbol{q}}_n} \in \mathcal{C}(\mathcal{H}_{\underline{\boldsymbol{q}}_n})$ be such that $(n-1)$-st renormalization of $g_{\underline{\boldsymbol{q}}_n}$ is hybrid equivalent to g_{q_n}.

Lemma 6

1. The periodic point x_n of period $p_{n-1} (= q_1 \dots q_{n-1})$ of rotation number $1/q_n$ in the small filled Julia set K_{n-1} is parabolic.
2. There exists a critical point $\omega' \in K_{n-1}$ such that
$$g_{\underline{\boldsymbol{q}}_n}(\omega') = g_{\underline{\boldsymbol{q}}_n}(x_n).$$
3. Fix $q_1, \dots, q_{n-1}$. Then
$$\lim_{q_n \to \infty} g_{\underline{\boldsymbol{q}}_n} = g_{\underline{\boldsymbol{q}}_{n-1}}.$$

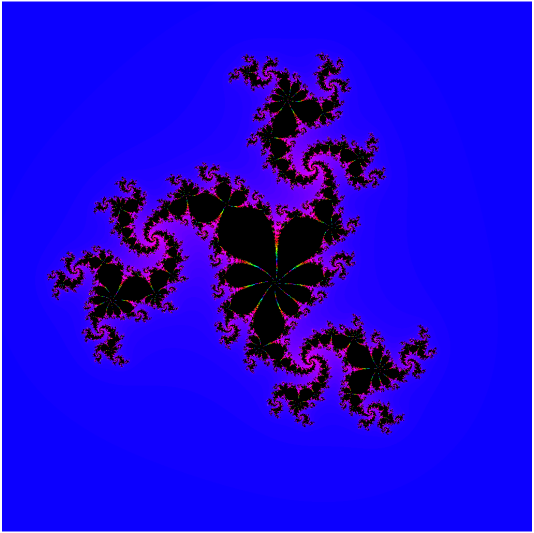
The Julia set of $g_{3,3}$

(perturbed slightly to have an attracting cycle)

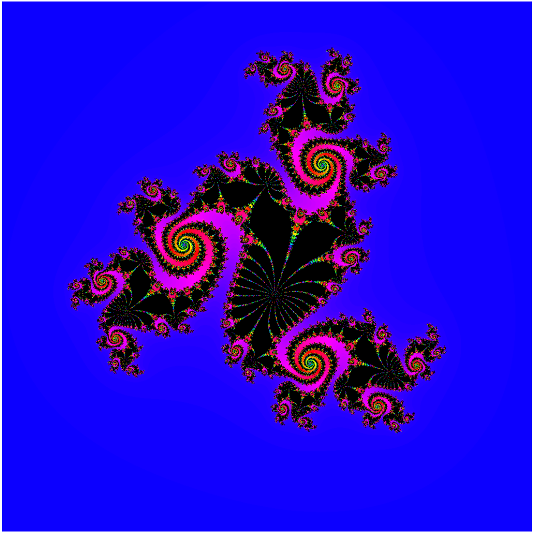


The Julia set of $g_{3,4}$

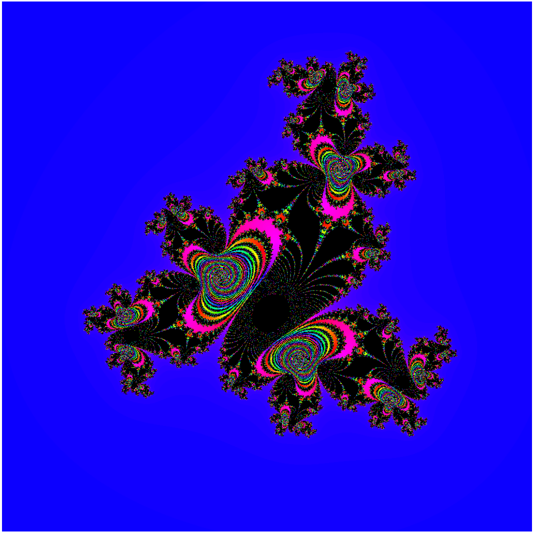
(perturbed slightly to have an attracting cycle)



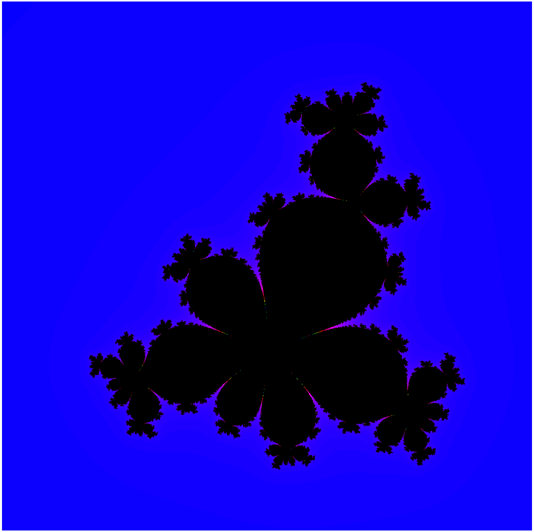
The Julia set of $g_{3,10}$
(perturbed slightly to have an attracting cycle)



The Julia set of $g_{3,100}$
(perturbed slightly to have an attracting cycle)



The Julia set of $g_3 = \lim g_{3,q}$



Construction (cont'd)

Consider a subsequential limit g of $\{g_{\underline{\boldsymbol{q}}_n}\}_{n\in\mathbb{N}}$.

Lemma 7

1. $g \in \mathcal{C}_{\underline{\boldsymbol{q}}}$.
2. g is infinitely renormalizable.
3. The critical points $\omega, \omega' \in \bigcap_n K_n$.

Furthermore, if $q_n \to \infty$ sufficiently fast, then

4. $\omega \neq \omega'$, hence the fiber $\bigcap_n K_n$ is non-trivial.
5. g is infinitely renormalizable in the sense of near-parabolic renormalization (I-Shishikura).
6. The domain of definition of each near-parabolic renormalization contains ω but not ω'.

Non-trivial fiber in the cubic connectedness locus

The fiber $\mathcal{C}_{\underline{\boldsymbol{q}}}$ is non-trivial because a non-unicritical map g and a unicritical map in $M_{\underline{\boldsymbol{q}}}$ both lie in $\mathcal{C}_{\underline{\boldsymbol{q}}}$.
Furthermore, for each n, there exists a path $\gamma_{\underline{\boldsymbol{q}}_n}$ connecting $g_{\underline{\boldsymbol{q}}_n}$ and $\mathcal{H}_{\underline{\boldsymbol{q}}_n}$ in $\mathcal{C}(\mathcal{H}_{\underline{\boldsymbol{q}}_n})$.

Theorem 8

Let $n \geq 3$. For any convergent sequence in $\mathcal{C}(\mathcal{H}_{\underline{\boldsymbol{q}}_n})$, the limit lies in $\mathcal{C}(\mathcal{H}_{\underline{\boldsymbol{q}}_{n-2}})$.
In particular, for any convergent sequence $\{f_n\}$ with $f_n \in \mathcal{C}(\mathcal{H}_{\underline{\boldsymbol{q}}_n})$, the limit lies in $\mathcal{C}_{\underline{\boldsymbol{q}}}$ and is infinitely renormalizable..

Therefore, the derived set

$$\bigcap_N \overline{\bigcup_{n \geq N} \gamma_n}$$

is a continuum contained in $\mathcal{C}_{\underline{\boldsymbol{q}}}$.