# Surgery construction for inverse branches of renormalization operator

Hiroyuki Inou*
Department of Mathematics, Kyoto University

March 11, 2002

**Abstract**

Renormalization can be considered as an operator extracting from a given polynomial a skew map on $\mathbb{Z}_N \times \mathbb{C}$ over $k \to (k+1)$ on $\mathbb{Z}_N$ whose restriction on each fiber is a polynomial. By using a quasiconformal surgery, we construct some inverse branches of this renormalization operator. Namely, from a given $N$-polynomial with fiberwise connected Julia sets, gluing $N$-sheets of the complex plane together and construct a polynomial having a renormalization of period $N$ which is hybrid equivalent to it and whose small filled Julia sets have a repelling fixed point of the constructed polynomial.

## 1 Introduction

In this paper, we introduce a new method to construct a new polynomial from given polynomials by quasiconformal surgery. This construction, which we call "rotatory intertwining surgery", is considered as an inverse branch of renormalization.

Roughly speaking, a renormalization of period $N$ for a polynomial $f$ is a polynomial-like restriction $f^N : U \to V$ with connected filled Julia set (the precise definition is given in Section 2). In this paper, we decompose a renormalization $f^N : U \to V$ into $N$ restrictions of $f$ and consider it as an $N$-tuple of proper maps $(f : U_k \to V_{k+1})_{k \in \mathbb{Z}_N}$ such that $U_k$ is a relatively compact subset of $V_k$ and that its filled Julia set is fiberwise connected (that is, the filled Julia set on each $V_k$ is connected). Then it is uniquely (up to affine conjugacy) hybrid equivalent to some $N$-tuple of polynomials $G = (G_k)_{k \in \mathbb{Z}_N}$ which acts on $\mathbb{Z}_N \times \mathbb{C}$ by $G(k, z) = (k + 1, G_k(z))$.

On the contrary, for a given $N$-tuple of polynomials $G = (G_k)_{k \in \mathbb{Z}_N}$ with fiberwise connected filled Julia set, can we construct a polynomial $f$ having a renormalization of period $N$ hybrid equivalent to $G$?

*Partially supported by JSPS Research Fellowship for Young Scientists.

Such $f$ is not unique. For example, consider three polynomials:

$$f_1(z) = z^2 - 0.1225611... + 0.7448617...\, i,$$
$$f_2(z) = z^2 - 1.754877...,$$
$$f_3(z) = z^3 + (-0.5178286... + 0.396073...\, i)\, z - 0.3177042... - 0.5544967...\, i.$$

Then $f_k$ has a period three renormalization hybrid equivalent to $(z^2, z, z)$ for $k = 1, 2, 3$.

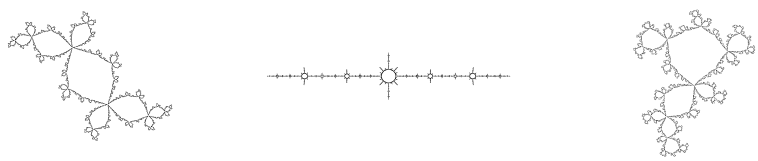

Figure 1: The Julia sets of $f_1$, $f_2$, and $f_3$.

In this paper, we only treats the case like $f_1$. More precisely, we construct a polynomial having a renormalization hybrid equivalent to given $G = (G_0, \ldots, G_{N-1})$ whose filled Julia set contains a fixed point of this polynomial, and no critical point outside the filled Julia set of the renormalization.

The idea of construction is based on the intertwining surgery [EY], but we cyclically rotate sectors by the map defined on them. And its uniqueness follows from the combinatorial property and the fact that the set of points in the Julia set whose forward orbit does not hit the Julia set of the renormalization has zero Lebesgue measure (Theorem 5.1).

We can also consider renormalization as an operator from some subset of the connectedness locus of a family of polynomials to the connected locus of another family of polynomials (or tuples of polynomials). It is known that this renormalization operator is not continuous, when the degree is greater than two [DH]. But our result implies that it is a bijection when the filled Julia set of renormalization contains a fixed point. So this can be considered as a part of the self-similarity of the connectedness locus for a family of polynomials.

**Acknowledgement.** I would like to thank Mitsuhiro Shishikura for helpful comments, especially on the proof of Theorem 5.1. I also thank Akira Kono for valuable suggestion.

## 2 $N$-polynomial maps

We first give a notion of $N$-polynomial maps. An $N$-polynomial map is simply a skew map from a union of $N$ sheets of the complex plane $\mathbb{Z}_N \times \mathbb{C}$ to itself, whose restriction of each sheet is a polynomial and mapped to the next sheet. We can easily generalize the theory on dynamics of usual polynomials to $N$-polynomial maps. In this section, we give an overview of its dynamical properties. Furthermore, we consider a renormalization of a given polynomial as an $N$-polynomial-like restriction. So we can also consider it as the operator extracting an $N$-polynomial map from a given polynomial.

**Definition.** Let $N > 0$. An *N-polynomial map* is an $N$-tuple of polynomials. An $N$-polynomial map $F = (F_0, \dots, F_{N-1})$ is considered as a map on $\mathbb{Z}_N \times \mathbb{C}$ to itself as follows:

$$F(k, z) = (k + 1, F_k(z)).$$

The *filled Julia set* $K(F)$ is the set of all points whose forward orbits by $F$ are bounded. The *Julia set* $J(F)$ is the boundary of $K(F)$. The *k-th small filled Julia set* is defined by $K_k(F) = \{ z \,|\, (k, z) \in K(F)\}$ and *k-th small Julia set* $J_k(F) = \partial K_k(F)$.

$C(F) = \{ (k, z) \,|\, F_k(z) = 0\}$ is the set of critical points of $F$ and

$$P(F) = \overline{\bigcup_{n>0} F^n(C(F))}$$

is called the *postcritical set* of $F$.

**Definition.** An *N-polynomial-like map* is an $N$-tuple of holomorphic proper maps $F = (F_k : U_k \to V_{k+1})_{k\in\mathbb{Z}_N}$ such that:

- $U_k$ and $V_k$ are topological disks in $\mathbb{C}$.
- $U_k$ is a relatively compact subset of $V_k$.

We also consider an $N$-polynomial-like map $F$ as a map between disjoint union of disks:

$$F : \bigsqcup_{k\in\mathbb{Z}_N} U_k \to \bigsqcup_{k\in\mathbb{Z}_N} V_k \qquad\qquad F|_{U_k} = F_k.$$

The *k-th small filled Julia set* $K_k(F)$ is defined by

$$K_k(F) = \left\{ z \in U_k \;\middle|\; F^n(z) \in U_{n+k} \right\}$$

and the *k-th small Julia set* $J_k(F)$ is defined by the boundary of $K_k(F)$. The *(resp. filled) Julia set* is defined by the disjoint union of the $k$-th small (*resp.* filled) Julia sets. We say the (filled) Julia set is *fiberwise connected* if $k$-th small (filled) Julia set is connected for any $k$.

For an $N$-polynomial or an $N$-polynomial-like map $F = (F_k)$, we write

$$F_k^n = F_{k+n-1} \circ \cdots \circ F_{k+1} \circ F_k,$$

so that $F^n(k, z) = (k + n, F_k^n(z))$.

Note that the degree of an $N$-polynomial map (or an $N$-polynomial-like map) $F$ is not well-defined ($\deg(F_k)$ may be different). So we define the *multi-degree* of $F$ by $\mathrm{m.deg}\, F = (\deg F_0, \dots, \deg F_{N-1})$. The degree of $F^N$ is well-defined (it is equal to $\prod \deg(F_k)$ ). In this paper, we always assume $\deg F^N > 1$.

**Definition.** Let $F = (F_k : U_k \to V_{k+1})$ and $G = (G_k : U'_k \to V'_{k+1})$ be $N$-polynomial-like maps. We say $F$ and $G$ are *hybrid equivalent* if there exist quasiconformal homeomorphisms $\phi_k$ ($k \in \mathbb{Z}_N$) between some neighborhoods of $K_k(F)$ and $K_k(G)$ such that $G_k \circ \phi_k = \phi_{k+1} \circ F_k$ and $\bar\partial\phi_k \equiv 0$ on $K_k(F)$.

**Theorem 2.1 (Straightening theorem for $N$-polynomial-like maps).** *For any $N$-polynomial-like map $F$, there exist an $N$-polynomial map $G$ of the same multi-degree as $F$ hybrid equivalent to $F$.*

*Furthermore, if the Julia set of $F$ is fiberwise connected, then $G$ is unique up to affine conjugacy.*

*Proof.* Let $d_k = \deg F_k$ and $d = \prod d_k (= \deg F^N)$. Take a $C^1$-simple closed curve $\gamma_k \subset V_k \setminus U_k$ which encloses all critical values of $F_{k-1}$ and $K_k(F)$. Then $\delta_k = F_k^{-1}(\gamma_{k+1})$ is also a simple closed curve and encloses all critical point of $F_k$ and $K_k(F)$. Let $A_k$ be the closed annulus between $\gamma_k$ and $\delta_k$. Let $D_k$ be the bounded component of $\mathbb{C} \setminus A_k$ and $E_k = \mathrm{int}(D_k \cup A_k)$ (note that $E_k = F_{k-1}(D_{k-1})$).

Fix $R_k$ with $R_k^{d_k} > R_{k+1}$ for all $k \in \mathbb{Z}_N$ and let $R'_k = R_{k+1}^{1/d_k}$. We can take $C^1$-diffeomorphisms $\phi_k : A_k \to \{R'_k \le |z| \le R_k\}$ ($k \in \mathbb{Z}_N$) with $(\phi_k(z))^{d_k} = \phi_{k+1} \circ F_k(z)$ on $\delta_k$. Let $\sigma_0$ be the standard complex structure and define an almost complex structure $\sigma$ on $\bigsqcup V_k$ by:

$$\sigma = \begin{cases} \phi_k^*\sigma_0 & \text{on } A_k, \\ (F^n)^*\sigma & \text{on } F^{-n}(\bigcup A_k) \text{ for some } n > 0, \\ \sigma_0 & \text{on } K(f). \end{cases}$$

Then $F^*\sigma = \sigma$ and since $\phi_k$ is quasiconformal and $F$ is holomorphic, $\sigma$ is of bounded dilatation ratio, so it is in fact a complex structure by the Riemann mapping theorem. Let $\psi_k : V_k \to \mathbb{C}$ be a quasiconformal map with $\psi_k^*\sigma_0 = \sigma$. Then $\tilde G_k = \psi_{k+1} \circ F_k \circ \psi_k^{-1} : \psi_k(U_k) \to \psi_{k+1}(V_{k+1})$ is holomorphic, $\psi_k \circ \phi_k^{-1} : \{R'_k \le |z| \le R_k\} \to \mathbb{C}$ is conformal, and $\tilde G = (\tilde G_0, \dots, \tilde G_{N-1})$ is an $N$-polynomial-like map hybrid equivalent to $F$.

For $k \in \mathbb{Z}_N$, consider a Riemann surface $S_k = (\psi_k(U_k) \cup \{\,|z| \ge R'_k\} \cup \infty)/(\psi_k \circ \phi_k^{-1})$. We can identify $S_k$ conformally to $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Define a map $G_k : S_k \to S_{k+1}$ by

$$G_k(z) = \begin{cases} F_k(z) & \text{if } z \in \psi_k(U_k), \\ z^{d_k} & \text{if } z \in \{\,|z| > R'_k\}. \end{cases}$$

Then $G_k$ is a polynomial of degree $d_k$. Therefore, the $N$-polynomial map $G = (G_0, \dots, G_{N-1})$ is hybrid equivalent to $F$.

When the Julia set of $F$ is fiberwise connected, uniqueness follows from the fact that the Teichmüller space of a superattractive basin without critical points other than the superattractive periodic point is trivial. □

Usually, we consider a renormalization as a polynomial-like map with connected Julia set which is a restriction of some iterate of a polynomial. But here, we consider it as an $N$-polynomial-like map;

**Definition.** A polynomial $f$ is *renormalizable* for period $N$ if there exist disks $U_k$ and $V_k$ ($k \in \mathbb{Z}_N$) such that:

- $G = (f : U_k \to V_{k+1})_{k \in \mathbb{Z}_N}$ is an $N$-polynomial-like map with fiberwise connected Julia set.
- $U_k \cap U_{k'}$ contains no critical point of $f$ if $k \neq k'$.
- When $N = 1$, $U_0$ does not contain all the critical points of $f$.

We call $G$ a *renormalization* of period $N$.

The small filled Julia sets of a renormalization are "almost disjoint" (they intersects only at a repelling periodic orbit [Mc], [In]). So we define the (*resp. filled*) *Julia set* of a renormalization by the union (not the *disjoint* union) of the small (*resp.* filled) Julia sets.

We may assume an $N$-polynomial map $F$ is monic (that is, each $F_k$ is monic). Let $\Delta = \{|z| < 1\}$. Easy calculation shows:

**Proposition 2.2 (The existence of the Böttcher coordinates).** *For a given monic $N$-polynomial map $F$, there exist conformal maps $\varphi_k : (\mathbb{C} \setminus \overline{\Delta}) \to (\mathbb{C} \setminus K_k(F))$ such that $\varphi_{k+1}(z^{\deg F_k}) = F_k \circ \varphi_k(z)$.*

In fact, we may take $\varphi_k$ the Böttcher coordinate for the monic polynomial $F_k^N = F_{k-1} \circ \cdots \circ F_{k+1} \circ F_k$.

So, we can define external rays for $F$ just as the usual polynomial case.

**Definition.** Let $F$ and $\varphi_k$ as above. The *k-th external ray* $\mathcal{R}_k(F; \theta)$ of angle $\theta$ for an $N$-polynomial map $F$ is defined by:

$$\mathcal{R}_k(F; \theta) = \left\{ \varphi_k(r \exp(2\pi i \theta)) \;\middle|\; 1 < r < \infty \right\}.$$

If the limit

$$x = \lim_{r \to 1} \varphi_k(r \exp(2\pi i \theta))$$

exists, then we say $\mathcal{R}_k(F; \theta)$ lands at $x$ and $\theta$ is the landing angle for $(k, x)$.

Let $R > 1$. We also define

$$\mathcal{R}_k(F; \theta, R) = \left\{ \varphi_k(r \exp(2\pi i \theta)) \;\middle|\; 1 < r < \infty \right\},$$
$$\mathcal{R}_k(F; \theta, R, \epsilon) = \left\{ \varphi_k(r \exp(2\pi i \eta)) \;\middle|\; 1 < r < \infty,\ \eta = \theta + \epsilon \log r \right\}.$$

If $\mathcal{R}(F; \theta)$ lands at $x$, then $\mathcal{R}(F; \theta, R, \epsilon)$ also converges to $x$. By the proposition above,

$$F(\mathcal{R}_k(F; \theta)) = \mathcal{R}_{k+1}(F; \deg(F_k) \cdot \theta),$$
$$F(\mathcal{R}_k(F; \theta, R)) = \mathcal{R}_{k+1}(F; \deg(F_k) \cdot \theta,\ R^{\deg(F_k)}),$$
$$F(\mathcal{R}_k(F; \theta, R, \epsilon)) = \mathcal{R}_{k+1}(F; \deg(F_k) \cdot \theta,\ R^{\deg(F_k)},\ \epsilon).$$

We say the ray is *periodic* if $F^n(\mathcal{R}_k(F; \theta)) = \mathcal{R}_k(F; \theta)$ for some $n > 0$. The least such $n$ is called the period of this ray. Clearly, the period of every periodic ray is divisible by $N$.

Let $x = (k, z)$ be a periodic point of $F$ with period $n$. If $x$ is repelling or parabolic, then there are finite number of rays landing at $x$ and they have same period. Let $q$ be the number of rays landing at $x$ and let $\theta_1, \dots \theta_q$ be the angle of these rays ordered counterclockwise. Since $F^n$ permutes the rays landing at $x$ and it preserves the cyclic order of them, there exist $p$ such that $F^n(\mathcal{R}_k(F; \theta_i)) = F^n(\mathcal{R}_k(F; \theta_{i+p}))$ for every $i \in \mathbb{Z}_q$. We say that the (*combinatorial*) *rotation number* of the periodic point $x$ is $p/q$.

We also consider external rays for an $N$-polynomial-like map. They are defined by the inverse images of external rays for an $N$-polynomial map hybrid equivalent to it by the hybrid conjugacy given in Proposition 2.1.

# 3 Results

Let $F$ be an $N$-polynomial map with fiberwise connected Julia set and $O = \{ (k, x_k) \mid k \in \mathbb{Z}_N\}$ be a repelling periodic orbit of period $N$ with rotation number $p_0/q_0$.

**Definition.** We say a polynomial $(g, x)$ with a marked fixed point $x$ is a *$p$-rotatory intertwining* of $(F, O)$ if:

- $g$ has a renormalization of period $N$ hybrid equivalent to $F$.
- $x$ corresponds to $O$ by the hybrid conjugacy above.
- $x$ has a rotation number $p/(Nq_0)$.
- $\deg(g) = \sum(\deg(F_k) - 1) + 1$. (Equivalently, all critical points of $g$ lie in the filled Julia set of the renormalization above.)

Note that the filled Julia set of such a polynomial is connected.

To construct a $p$-rotatory intertwining of $(F, O)$, we need some combinatorial property of the dynamics near the fixed point $x$.

**Definition.** A 4-tuple of integers $(N, p_0, q_0, p)$ is admissible if $p \equiv p_0 \mod q_0$ and $p$ and $N$ are relatively prime.

Note that the above definition also makes sense when $N$ and $q_0$ are integers, $p_0 \in \mathbb{Z}_{q_0}$ and $p \in \mathbb{Z}_{Nq_0}$. The following proposition is easy:

**Proposition 3.1.** *If a $p$-rotatory intertwining of $(F, O)$ exists, then $(N, p_0, q_0, p)$ is admissible.*

**Theorem 3.2.** *Let $F$ be an $N$-polynomial map with fiberwise connected Julia set and $O = \{(k, x_k)\}$ is a repelling periodic orbit of period $N$ with rotation number $p_0/q_0$.*

*When an integer $p$ satisfies that $(N, p_0, q_0, p)$ is admissible, then there exists a $p$-rotatory intertwining $(g, x)$ of $(F, O)$ and it is unique up to affine conjugacy.*

The following two sections are devoted to prove this theorem.

Let $\mathcal{M}_D(p/q, N)$ be the set of all pairs $(F, O)$, where $F$ is a $N$-polynomial map of multi-degree $D = (d_0, \dots, d_{N-1})$ with connected filled Julia sets and $O$ is a repelling periodic point of period $N$ with rotation number $p/q$. Let $\mathcal{M}_d(p/q) = \mathcal{M}_{(d)}(p/q, 1)$. By the theorem above, we can define the map

$$\mathcal{M}_D(p_0/q_0) \ni (F, \mathcal{O}) \mapsto (g, x) \in \mathcal{M}_d(p/q_0).$$

The following corollary is easy to prove.

**Corollary 3.3.** *When $(N, p_0, q_0, p)$ is admissible, the map $(F, \mathcal{O}) \mapsto (g, x)$ is a bijection $\mathcal{M}_D(p_0/q_0, N)$ into its image, which is a subset of $\mathcal{M}_d(p/(Nq_0))$, where $d = \prod d_k$. Its inverse is given by renormalization.*

# 4 Construction

In this section, we prove the existence part of Theorem 3.2. We use the idea of the intertwining surgery [EY].

Let $(F, \mathcal{O})$ be an $N$-polynomial map with a marked periodic point satisfying the assumption of Theorem 3.2. Fix $R > 0$ and let

$$V_k = \left\{ (k, z) \,\middle|\, |\varphi_k(z)| < R \right\} \cup K_k(F)$$

and $U_k = F_k^{-1}(V_{k+1})$. Let $\mathcal{V} = \bigsqcup V_k$ and $\mathcal{U} = \bigsqcup U_k$. Then $(F_k : U_k \to V_{k+1})$ is an $N$-polynomial-like map (we also use the word $F$ for it and simply write $F : \mathcal{U} \to \mathcal{V}$).

Let $\theta_0, \dots, \theta_{q_0-1}$ be all the external angles for $(0, x_0)$ ordered counterclockwise.

Let $\epsilon > 0$ and $0 < \delta < \epsilon/(2N)$. For $0 \le k < N$ and $l \in \mathbb{Z}_{q_0}$, consider arcs

$$\gamma_0(k + Nl) = \mathcal{R}_0\left(F; \theta_k, R, \left(\frac{k}{N} - \frac{1}{2}\right)\epsilon\right),$$
$$\gamma_0^{\pm}(k + Nl) = \mathcal{R}_0\left(F; \theta_k, R, \left(\frac{k}{N} - \frac{1}{2}\right)\epsilon \pm \delta\right).$$

When $\epsilon$ is sufficiently small, these arcs are mutually disjoint. For $j \in \mathbb{Z}_{Nq_0}$, let

$$\begin{array}{rcl} \gamma_k(j) & = & F_k(\gamma_{k-1}(j-p) \cap U_{k-1}), \\ \gamma_k^{\pm}(j) & = & F_k(\gamma_{k-1}^{\pm}(j-p) \cap U_{k-1}) \end{array} \tag{1}$$

for $k = 1, \dots, N-1$. Let $S_k(j)$ (*resp.* $L_k(j)$) be the open sector in $V_k$ between $\gamma_k(j-1)$ and $\gamma_k(j)$ (*resp.* $\gamma_k^+(j-1)$ and $\gamma_k^-(j)$).

Then, since the rotation number of $x_0$ for $F_0^N$ is $p_0/q_0$, we can easily verify $F_0^N(\gamma_0(j) \cap F_0^{-N}(V_0)) = \gamma_0(j + Np_0)$. Therefore, by the assumption that $(N, p_0, q_0, p)$ is admissible,

$$\begin{aligned} F_{N-1}(\gamma_{N-1}(j-p) \cap U_{N-1}) &= F_0^N(\gamma_0(j - Np) \cap F_0^{-N-1}(U_{N-1})) \\ &= \gamma_0(j - Np + Np_0) \\ &= \gamma_0(j). \end{aligned}$$

This equation also holds for $\gamma_k^{\pm}$ instead of $\gamma_k$. Therefore, the equation (1) holds for any $k \in \mathbb{Z}_N$.

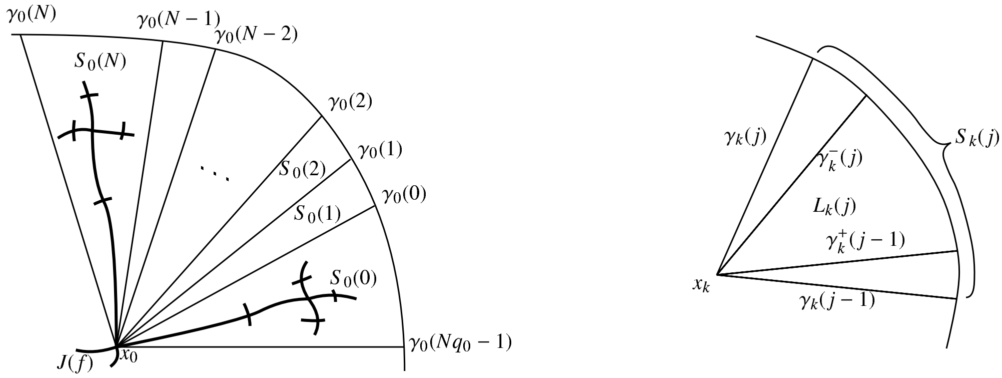


Figure 2: Sectors.

Since $O$ is repelling, $F$ is linearizable at $O$. Namely, there are a neighborhood $O_k$ of $x_k$ and a map $\psi_k : O_k \to \mathbb{C}$ for each $k$ such that $\psi_k(x_k) = 0$ and $\psi_{k+1} \circ F_k(z) = \lambda_k \cdot \psi_k(z)$ on $O'_k$, where $\lambda_k = F'_k(x_k)$ and $O'_k$ is the component of $F_k^{-1}(O_{k+1})$ containing $x_k$.

For each $j \in \mathbb{Z}_{Nq_0}$, the quotient space $(L_k(j) \cap O_k)/F_k^{Nq_0}$ is an annulus of finite modulus. So we denote the modulus of this quotient annulus by $\operatorname{mod} L_k(j)$. Since $F_k$ maps $L_k(j) \cap O'_k$ univalently to $L_{k+1}(j+p) \cap O_{k+1}$, we have $\operatorname{mod} L_k(j) = \operatorname{mod} L_{k+1}(j+p)$.

We want to identify $N$ disks $V_0, \ldots, V_{N-1}$ quasiconformally and define a quasiregular map on it. Before doing this, we deform the $N$-polynomial-like map $F : \mathcal{U} \to \mathcal{V}$ by some hybrid conjugacy.

**Lemma 4.1.** *There exists an $N$-polynomial-like map $\hat{F} = (\hat{F}_k : \hat{U}_k \to \hat{V}_{k+1})_{k \in \mathbb{Z}_N}$ hybrid equivalent to $F$ such that the sector $\hat{L}_k(j)$ which corresponds to $L_k(j)$ satisfies that*

$$\operatorname{mod} \hat{L}_k(j) = \operatorname{mod} \hat{L}_{k'}(j)$$

*for any $k, k' \in Z_N$ and $j \in Z_{Nq_0}$.*

*Proof.* For $l \in \mathbb{Z}_{q_0}$, let $A_l$ be an annulus with $\operatorname{mod} A_l = m_l = \operatorname{mod} L_0(lN)$. For $k = 1, \ldots, N-1$, take a quasiconformal homeomorphism

$$\rho_{l,k} : (L_0(lN) \cap O_0)/F_0^{Nq_0} \to A_l.$$

Since $F_0^N$ induces a conformal isomorphism $(F_0^N)^\wedge$ between $L_0(lN + pk)/F_0^{Nq_0}$ and $L_0((l+p)N + pk)/F_0^{Nq_0}$, we can choose $\rho_{l,k}$ so that

$$\rho_{l+p,k} \circ F_0^N = \rho_{l,k}. \tag{2}$$

Define a complex structure $\sigma$ on $L_0(lN + pk)$ by pulling back by $\rho_{l,k}$ and lifting the standard complex structure $\sigma_0$ on $A_l$. Note that, since $(N, p_0, q_0, p)$ is admissible,

$L_0(lN + pk)$ does not intersect the filled Julia set and we can push forward a complex structure on $(L_0(lN + pk)) \cap O_0$ to entire $L_0(lN + pk)$ by $F_0^{Nq_0}$. By (2), $\sigma$ is $F_0^N$-invariant. The modulus $\mathrm{mod}(L_0(lN + pk), \sigma)$ with respect to the complex structure $\sigma$ is equal to $m_l$.

Now we define an $F$-invariant complex structure $\hat{\sigma}$ on $\mathbb{Z}_N \times \mathbb{C}$ as follows: We identify $L_0(j)$ and $\{0\} \times L_0(j)$ and let

$$\hat{\sigma} = \begin{cases} (F^n)^*\sigma & \text{on } F^{-n}(L_0(lN + pk)) \text{ for some } k \in \{1, \dots, N-1\} \text{ and } n > -N. \\ \sigma_0 & \text{otherwise.} \end{cases}$$

Since

$$\bigcup_{l \in \mathbb{Z}_{q_0}, k=1,\dots,N-1} L_0(lN + pk)$$

is forward invariant by $F_0^N$ and $\sigma$ is $F_0^N$-invariant, $\hat{\sigma}$ is well-defined and $F$-invariant.

By the measurable Riemann mapping theorem, there exists a quasiconformal map

$$\phi : \bigsqcup V_k \to \bigsqcup \hat{V}_k$$

such that $\phi^*\sigma_0 = \sigma$. Let $\hat{F} = \phi \circ F \circ \phi^{-1}$ and $\hat{L}_k(j) = \phi(L_k(j))$. Then $\hat{F}$ is holomorphic and by the equation $m_l = m_{l+p}$ and (2), we have $\mathrm{mod}\,\hat{L}_0(j) = \mathrm{mod}\,\hat{L}_0(j + p)$ and

$$\begin{aligned} \mathrm{mod}\,\hat{L}_k(j) &= \mathrm{mod}\,\hat{L}_0(j - pk)) \\ &= \mathrm{mod}\,\hat{L}_0(j). \end{aligned}$$

□

Denote the point and the sets which correspond to $x_k$, $\gamma_k(j)$, $\gamma_k^{\pm}(j)$, $S_k(j)$, $O_k$ and $O'_k$ respectively by the hybrid conjugacy in the above lemma by $\hat{x}_k$, $\hat{\gamma}_k(j)$, $\hat{\gamma}_k^{\pm}(j)$, $\hat{S}_k(j)$, $\hat{O}_k$ and $\hat{O}'_k$.

Now we construct quasiconformal maps $\tau_k : \hat{V}_0 \to \hat{V}_k$ $(k \in \mathbb{Z}_N)$ to identify $\hat{V}_0, \dots, \hat{V}_{N-1}$ together. Let $\tau_0$ be the identity. Take $C^1$ diffeomorphisms

$$\tilde{\tau}_k : \bigcup_j \hat{\gamma}_k(j) \to \bigcup_j \hat{\gamma}_{k+1}(j)$$

which maps $\hat{\gamma}_k(j)$ to $\hat{\gamma}_{k+1}(j)$ for any $j$ as follows. First of all, take $\tilde{\tau}_0$ be a diffeomorphism such that

- It gives the conjugacy between $\hat{F}_0^N$ and $\hat{F}_1^N$. That is, $\tilde{\tau}_0 \circ \hat{F}_0^N = \hat{F}_1^N \circ \tilde{\tau}_0$.
- $(\tilde{\tau}_0^{-1} \circ \hat{F}_0)^N = \hat{F}_0^N$.

**Lemma 4.2.** *Such a diffeomorphism $\tilde{\tau}_0$ exists.*

*Proof.* Let $y_k^0(j)$ be the edge point of $\hat{\gamma}_k(j)$ other than $\hat{x}_k$. For $n > 0$, let $y_k^n(j)$ be the point of $\hat{\gamma}_k(j)$ which satisfies that $F_k^n(y_k^n(j)) = y_{k+n}^0(j + pn)$.

All rays $\hat{\gamma}_k(j)$ have the same period $cN$. The quotient space $(\bigcup_j \hat{\gamma}_k(j))/F_k^N$ is diffeomorphic to the disjoint union of circles and each component is of the form $\eta_k(j) =$

$\hat{\gamma}_k(j)/F_k^{cN}$. The points $\{y_k^n(j)\}$ corresponds to the $cN$ points $\{[y_k^n(j)]\}_{k=0,\dots,cN-1}$ (the equivalent class $[y_k^n(j)] = [y_k^{n+cN}(j)]$ in $\eta_k(j)$). $\hat{F}_k$ induces a diffeomorphism $\alpha : \eta_k(j) \to \eta_{k+1}(j+p)$ (for simplicity, we neglect indices $k$ and $j$ for $\alpha$). Then $\alpha([y_k^n(j)]) = [y_{k+1}^n - 1(j+p)]$ and $\alpha^N : \eta_k(j) \to \eta_k(j+pN)$ is identity map on $(\bigcup_j \hat{\gamma}_k(j))/F_k^N$.

Furthermore, we identify each $\eta_0(j)$ and the circle $\mathbb{R}/(cN\mathbb{Z})$ diffeomorphically so that $\{[y_0^n(j)]\}$ corresponds to $\{[n]\}$. Define $R : \mathbb{R}/(cN\mathbb{Z}) \to \mathbb{R}/(cN\mathbb{Z})$ by $R(x) = x - 1$. Since $R^{cN} = \alpha^{cN} = \mathrm{id}$, we may assume that the following diagram commutes:

$$\begin{tikzcd}
\eta_0(j) \arrow[r, "\alpha^N"] \arrow[d, equal, "\wr"'] & \eta_0(j+pN) \arrow[d, equal, "\wr"'] \\
\mathbb{R}/(cN\mathbb{Z}) \arrow[r, "R^N"'] & \mathbb{R}/(cN\mathbb{Z}).
\end{tikzcd}$$

Let $\hat{\tau} : \eta_0(j) \to \eta_1(j)$ be the diffeomorphism defined by:

$$\eta_0(j) \cong \mathbb{R}/(cN\mathbb{Z}) \xrightarrow{R^{-1}} \mathbb{R}/(cN\mathbb{Z}) \cong \eta_0(j-p) \xrightarrow{\alpha} \eta_1(j).$$

Then the following diagram commutes:

$$\begin{tikzcd}
\mathbb{R}/(cN\mathbb{Z}) \arrow[r, "R"] \arrow[d, equal, "\wr"'] & \mathbb{R}/(cN\mathbb{Z}) \arrow[r, "R"] \arrow[d, equal, "\wr"'] & {} & \cdots & {} \arrow[r, "R"] & \mathbb{R}/(cN\mathbb{Z}) \arrow[d, equal, "\wr"'] \\
\eta_0(j) \arrow[d, "\hat{\tau}"] \arrow[dr, "\alpha"] & \eta_0(j+p) \arrow[d, "\hat{\tau}"] \arrow[dr, "\alpha"] & & \cdots & {} \arrow[dr, "\alpha"] & \eta_0(j+pN) \arrow[d, "\hat{\tau}"] \\
\eta_1(j) & \eta_1(j+p) & {} & \cdots & & \eta_1(j+pN),
\end{tikzcd}$$

so,

$$(\hat{\tau}^{-1} \circ \alpha)^N = R^N = \alpha^N. \tag{3}$$

Let $\tilde{\tau}_0 : \hat{\gamma}_0(j) \to \hat{\gamma}_1(j)$ be the diffeomorphism which is a lift of $\hat{\tau}$. Then $\tilde{\tau}_0 \circ \hat{F}_0^N = \hat{F}_1^N \circ \tilde{\tau}_0$. Furthermore, since $\hat{F}_0$ is a lift of $\alpha$, (3) implies $(\tilde{\tau}^{-1} \circ \hat{F}_0)^N = \hat{F}_0^N$. □

Define $\tilde{\tau}_k$ for $k = 1, \dots, N-1$ inductively by the equation

$$\hat{F}_k \circ \tilde{\tau}_{k-1} = \tilde{\tau}_k \circ \hat{F}_{k-1}. \tag{4}$$

Then this equation is also valid for $k = 0$. Indeed,

$$\begin{aligned}
\tilde{\tau}_0 \circ \hat{F}_{N-1} \circ \hat{F}_0^{N-2} &= \tilde{\tau}_0 \circ \hat{F}_0^N = \hat{F}_1^N \circ \tilde{\tau}_0 \\
&= \hat{F}_0 \circ \hat{F}_{N-1} \circ \cdots \circ \hat{F}_1 \circ \tilde{\tau}_0 \\
&= \hat{F}_0 \circ \hat{F}_{N-1} \circ \cdots \circ \tilde{\tau}_1 \circ \hat{F}_0 \\
&\cdots \\
&= \hat{F}_0 \circ \tilde{\tau}_{N-1} \circ \hat{F}_0^{N-2}.
\end{aligned}$$

Since $\hat{F}_0^{N-1}$ maps the subarc of $\hat{\gamma}_0(j-(N-2)p)$ from $\hat{x}_k$ to $y_0^{N-2}(j-(N-2p))$ diffeomorphically to $\hat{\gamma}_{N-2}(j)$, we have $\hat{F}_0 \circ \tilde{\tau}_{N-1} = \tilde{\tau}_0 \circ \hat{F}_{N-1}$.

Similarly, we can also show that

$$\begin{aligned}\hat{F}_0^N &= (\tilde{\tau}_0^{-1} \circ \hat{F}_0)^N \\ &= \tilde{\tau}_{N-1} \circ \cdots \circ \tilde{\tau}_0 \circ \hat{F}_0^N,\end{aligned}$$

so

$$\tilde{\tau}_{N-1} \circ \cdots \circ \tilde{\tau}_0 = \mathrm{id}\,. \tag{5}$$

And it is easy to see that $\tilde{\tau}_k(y_k^n(j)) = y_{k+1}^n(j)$. Let $\tau_k = \tilde{\tau}_{k-1} \circ \cdots \circ \tilde{\tau}_0$ on $\bigcup \hat{\gamma}_k(j)$.

Let $\tau_k|_{\hat{L}_0(j)} : \hat{L}_0(j) \to \hat{L}_k(j)$ be the conformal isomorphism which sends $\hat{x}_0$ to $\hat{x}_k$, $\hat{\gamma}_0^+(j-1)$ to $\hat{\gamma}_0^+(j-1)$, and $\hat{\gamma}_0^-(j)$ to $\hat{\gamma}_0^-(j)$. Taking $\hat{L}_k(j)$ smaller (that is, taking $\epsilon$ greater) if necessary, we may assume $\tau_k|_{\hat{L}_0(j)}$ extends smoothly on $\gamma_0^+(j-1)$ and $\gamma_0^-(j)$.

The following lemma is due to Bielefeld [Bi, Lemma 6.4, 6.5].

**Lemma 4.3.** *We can extend $\tau_k$ quasiconformally to $\tau_k : \hat{V}_0 \to \hat{V}_k$ for $k = 1, \ldots, N-1$.*

*Proof.* $\tau_k$ is already defined on $\bigcup(\hat{\gamma}_k(j) \cup \overline{\hat{L}_k(j)})$. So we must define $\tau_k$ on $\bigcup(\hat{S}_k(j) \setminus \overline{\hat{L}_k(j)})$ quasiconformally.

For $k \in \mathbb{Z}_N$ and $j \in \mathbb{Z}_{Nq_0}$, let $S_k^-(j)$ (*resp.* $S_k^+(j)$) be the open sector between $\hat{\gamma}_k^-(j)$ and $\hat{\gamma}_k(j)$ (*resp.* $\hat{\gamma}_k(j)$ and $\hat{\gamma}_k^+(j)$). Then $\bigcup(\hat{S}_k(j) \setminus \overline{\hat{L}_k(j)}) = \bigcup(S_k^+(j) \cup S_k^-(j))$. So we should extend $\tau_k$ quasiconformally on $S_0^-(j)$, which maps to $S_k^-(j)$ (the case on $S_0^+(j)$ is quite similar). Furthermore, Since $\tau_k$ is smooth on $\gamma_0^\pm(j)$, we need only show the extensibility of $\tau_k$ on $\hat{O}_0$, where $F$ is linearizable. So we consider $S_0^-(j) \cap \hat{O}_0$ instead of $S_0^-(j)$.

Let $h_k(z) = \log \psi_k(z)$ on $(\overline{S_k^-(j) \cup L_k(j)}) \cap \hat{O}_k$ and $\lambda = (\prod_k \lambda_k)^{q_0}$. Then, since $\psi_k(\hat{F}_k^{Nq_0}(z)) = \lambda \psi_k(z)$,

$$h_k(\hat{F}_k^{Nq_0}(z)) = h_k(z) + \log \lambda. \tag{6}$$

Let $T_k^-(j) = h_k(\hat{S}_0^-(j) \cap \hat{O}_0)$, $M_k(j) = h_k(\hat{L}_k(j) \cap \hat{O}_0)$ and

$$\chi_k = h_k \circ \tau_k \circ h_0^{-1} : \partial^\pm T_0^-(j) \cup M_0(j) \to \partial^\pm T_k^-(j) \cup M_k(j).$$

where $\partial^+ T_k^-(j) = h_k(\hat{\gamma}_k(j))$ and $\partial^- T_k^-(j) = h_k(\hat{\gamma}_k^-(j))$ are the upper and lower boundaries of the strip $T_k^-(j)$. Then, by (4), (5) and (6), for $z \in \partial^+ T_k^-(j)$,

$$\begin{aligned}\chi_k(z + \log \lambda) &= h_k \circ \tau_k \circ h_0^{-1}(z + \log \lambda) \\ &= h_k \circ \tau_k \circ \hat{F}_0^{Nq_0}(h_0^{-1}(z)) \\ &= h_k \circ \tilde{\tau}_{k-1} \circ \cdots \circ \tilde{\tau}_0 \circ \hat{F}_0^{Nq_0} \circ h_0^{-1}(z) \\ &= h_k \circ \tilde{\tau}_{k-1} \circ \cdots \circ \tilde{\tau}_1 \circ \hat{F}_1^{Nq_0} \circ \tilde{\tau}_0 \circ h_0^{-1}(z) \\ &\cdots \\ &= h_k \circ \hat{F}_k^{Nq_0} \circ \tau_k \circ h_0^{-1}(z) \\ &= h_k(\tau_k \circ h_0^{-1}(z)) + \log \lambda \\ &= \chi_k(z) + \log \lambda.\end{aligned}$$

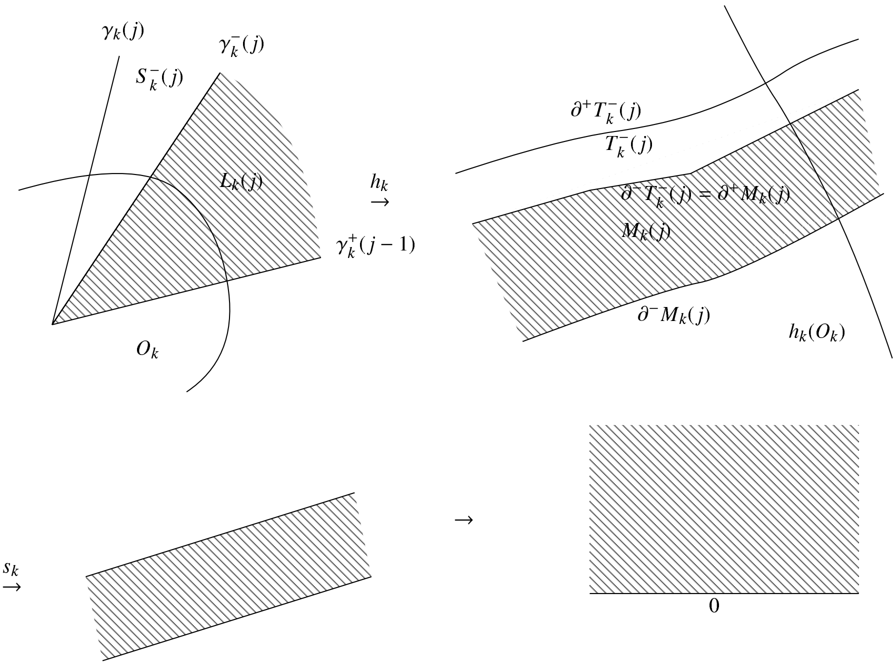


Figure 3: Conjugacy to translations and linear expansion.

We call such a diffeomorphism on curves in $\mathbb{C}$ a *near translation*. More precisely, we say a diffeomorphism on curve is a near translation if it is of the form $z + O(1)$ and its derivative is bounded away from zero and infinity.

**Claim.** $\chi_k|_{\partial^- T_0^-(j)}$ is a near translation.

Since $\tau_k|_{\hat{L}_k(j)}$ is conformal, $\chi_k|_{M_0(j)} : M_0(j) \to M_k(j)$ is conformal and it maps the upper boundary $\partial^+ M_0(j)$ $(= \partial^- T_0(j))$ to $\partial^+ M_k(j)$ $(= \partial^- T_k(j))$ diffeomorphically. Let $m = \mathrm{mod}(M_k(j)/(z \mapsto z + \log\lambda)) = \mathrm{mod}\,\hat{L}_k(j)$. Then by the assumption, $m$ is independent of $k$.

Let $H_\nu = \{ z \log\lambda \,|\, 0 < \mathrm{Im}\, z < \nu \}$. Then there exists some $\nu > 0$ such that for any $k \in \mathbb{Z}_N$, there is a conformal map $s_k$ from $M_k(j)$ into $H_\nu$ which maps the upper and lower boundary to the upper and lower boundary respectively, and which gives a conjugacy from $z \mapsto z + \log\lambda$ to itself. ($\nu$ is given by the equation $\mathrm{mod}(H_\nu/(z \mapsto z + \log\lambda)) = m$.) Since $s_k(z + \log\lambda) = s_k(z) + \log\lambda$, $s_k|_{\partial^+ M_0(j)}$ is a near translation. Let

$$\hat{\chi}_k = e \circ s_k \circ \chi_k \circ s_0^{-1} \circ e^{-1}$$
$$\text{where } e(z) = exp\left(\frac{\pi}{\nu \log\lambda} z\right).$$

Then $\hat{\chi}_k$ can be extended to some neighborhood of 0 by the reflection principle. Hence it is of the form $rz + O(z^2)$. Thus

$$\begin{aligned} s_k \circ \chi_k \circ s_k^{-1}(z) &= e^{-1} \circ \hat{\chi}_k \circ e(z) \\ &= z + O(1), \end{aligned}$$

so it is a near translation. Since the composition of near translations is also a near translation, $\chi_k$ is also a near translation.

Just as in the case of $M_k(j)$, let $\hat{t}_k$ be a conformal map from $T_k^-(j)$ into $H_{\nu'_k}$ which gives a conjugacy from $z \mapsto z + \log\lambda$ to itself. (Note that in this case, $\nu'_k$ may depends on $k$.) Let $t_k = \hat{t}_k/(\nu'_1 \log\lambda) : T_k^-(j) \to \{0 < \operatorname{Im} z < \nu_k/\nu_1\}$. Then $\tilde{\chi}_k^{\pm} = t_k \circ \chi_k \circ t_0$ restricted to the upper and lower boundary respectively are both near translations. We define $\tilde{\chi}_k : t_0(M_0(j)) \to t_k(M_k(j))$ as follows:

$$\tilde{\chi}_k(x+iy) = y(\tilde{\chi}_k^+(x+i) - i) + (1-y)\tilde{\chi}_k^-(x) + i\frac{\nu_k}{\nu_1}y.$$

(Although it may not be mapped into $t_k(M_k(j))$, it makes no problem because we only need to construct this map near the left infinity.) It is easy to check $\tilde{\chi}_k$ is a quasiconformal diffeomorphism. Therefore, $\tau_k = h_k^{-1} \circ t_k^{-1} \circ \tilde{\chi}_k \circ t_0 \circ h_0$ on $\hat{S}_0^-(j) \cap \hat{O}_0$ is a quasiconformal extension. □

Let $V = \hat{V}_0$ and

$$U = \bigcup_{j \in \mathbb{Z}_q, k=0,\dots,N-1} \tau_k^{-1}\left(\overline{\hat{S}_k(jN+kp)} \cap \hat{U}_k\right).$$

Note that since $(N, p_0, q_0, p)$ is admissible, $jN + kp$ moves all elements of $\mathbb{Z}_{Nq_0}$. Define a quasiregular map $g : U \to V$ as follows. When $z \in \hat{S}_0(jN+kp) \cap U$ for some $j \in \mathbb{Z}_{q_0}$, let

$$\tilde{g}(z) = \tau_{k+1}^{-1} \circ \hat{F}_k \circ \tau_k(z).$$

By (4), $\tilde{g}$ extends continuously on $U$.

**Lemma 4.4.**

1. *Let $E = \bigcup \hat{S}_0(j) \setminus \hat{L}_0(j)$. Then $\tilde{g}(E \cap U) \subset E$. In other words, $E$ is forward invariant by $\tilde{g}$.*
2. *$\tau_k \circ \tilde{g}^N \circ \tau_k^{-1}$ is conformal on $\hat{S}_k(jN+kp) \setminus \hat{L}_k(jN+kp)$.*

*Proof.* The first property is clear because $\hat{F}_k((\hat{S}_k(j) \setminus \hat{L}_k(j)) \cap \hat{U}_k) = \hat{S}_k(j+p) \setminus \hat{L}_k(j)$.

For $z \in \hat{S}_0(jN+kp) \setminus \hat{L}_0(jN+kp) \cap U$, $\tilde{g}(z) = \tau_{k+1}^{-1} \circ \hat{F}_k \circ \tau_k(z) \in \hat{S}_0(jN+(k+1)p) \setminus \hat{L}_0(jN+(k+1)p)$. Thus

$$\begin{aligned} \tilde{g}^N(z) &= \tilde{g}^{N-1} \circ \tau_{k+1}^{-1} \circ \hat{F}_k \circ \tau_k(z). \\ &= \tilde{g}^{N-2} \circ \tau_{k+2}^{-1} \circ \hat{F}_{k+1} \circ \tau_{k+1} \circ \tau_{k+1}^{-1} \circ \hat{F}_k \circ \tau_k(z) \\ &= \tilde{g}^{N-2} \circ \tau_{k+2}^{-1} \circ \hat{F}_k^2 \circ \tau_k(z) \\ &= \cdots \\ &= \tau_k^{-1} \circ \hat{F}_k^N \circ \tau_k. \end{aligned}$$

Therefore, $\tau_k \circ \tilde{g}_N \circ \tau_k^{-1} = \hat{F}_k^N$ and it is conformal on $\hat{S}_k(jN+kp) \setminus \hat{L}_k(jN+kp)$. □

Let $\sigma_0$ be the standard complex structure. On $\hat{S}_0(jN+kp) \setminus \hat{L}_0(jN+kp)$,

$$\begin{aligned}\sigma_0 &= (\tau_k \circ \tilde{g}^N \circ \tau_k^{-1})^*(\sigma_0) \\ &= (\tau_k^*)^{-1} \circ (\tilde{g}^N)^*(\tau_k^* \sigma_0).\end{aligned}$$

by the previous lemma. Therefore,

$$(\tilde{g}^N)^*(\tau_k^*\sigma_0) = \tau_k^*\sigma_0 \tag{7}$$

on $\hat{S}_0(jN+kp) \setminus \hat{L}_0(jN+kp)$.

So define an almost complex structure $\sigma$ on $V$ as follows:

$$\sigma = \begin{cases} (\tau_k \circ \tilde{g}^n)^*\sigma_0 & \text{on } \tilde{g}^{-n}(\hat{S}_0(Nj+kp)). \\ \sigma_0 & \text{elsewhere.} \end{cases}$$

**Lemma 4.5.** *$\sigma$ is well-defined and it is really a complex structure.*

*Proof.* On $\hat{S}_0(Nj+kp) \setminus \hat{L}_0(jN+kp)$ $(1 \le k < N)$,

$$\begin{aligned}\tilde{g}^*\sigma &= (\tau_k^{-1} \circ \hat{F}_{k-1} \circ \tau_{k-1})^*(\tau_k^*\sigma_0) \\ &= \tau_{k-1}^*(\hat{F}_{k-1}^*\sigma_0) \\ &= \tau_{k-1}^*\sigma_0 \\ &= \sigma.\end{aligned}$$

Therefore, together with (7), $\sigma$ is invariant under $\tilde{g}$ on $E$. (Note that $E$ is forward invariant by $\tilde{g}$.) Since $\sigma \neq \sigma_0$ only on $\bigcup \tilde{g}^{-n}(E)$, $\sigma$ is well-defined.

Furthermore, $\tilde{g}$ is conformal except on $\tilde{g}^{-1}(E)$. So the maximal dilatation of $\sigma$ on $V$ is equal to that of $\sigma$ on $\tilde{g}^{-1}(E)$, which is bounded. So $\sigma$ is a complex structure. □

Therefore, there exists a quasiconformal mapping $h : V \to \mathbb{C}$ such that $h^*\sigma_0 = \sigma$. Then $\hat{g} = h \circ \tilde{g} \circ h$ is a polynomial-like map, so there exists a polynomial $g$ hybrid equivalent to $\hat{g}$.

It is easy to check this $g$ is a $p$-rotatory intertwining of $F$.

# 5 Uniqueness

In this section, we show that two $p$-rotatory intertwinings $(g, x)$ and $(g', x')$ of $(F, O)$ are affinely conjugate.

## 5.1 Puzzles

Let $(g, x)$ be a $p$-rotatory intertwining of an $N$-polynomial map $(F, O)$ with marked periodic point of period $N$. Denote by $\mathcal{K}$ the filled Julia set of the renormalization $G = (g : U_k \to V_{k+1})_{k \in \mathbb{Z}_N}$ corresponding to $F$. Let $\omega_0, \dots, \omega_{Nq-1}$ be the landing angles of $x$ ordered counterclockwise.

Let $\varphi : (\mathbb{C} \setminus \overline{\Delta}) \to (\mathbb{C} \setminus K(g))$ be the Böttcher coordinate of $g$. Fix $R > 0$ and small $\epsilon > 0$ so that sectors

$$\tilde{S}_{0,j} = \left\{ \varphi(r \exp(2\pi i\theta)) \;\middle|\; 1 < r < R,\ |\theta - \omega_j| < \epsilon \log r \right\}.$$

are mutually disjoint. Let $D_0 = \varphi(\{|z| < R\}) \cup K(g)$ and $D_n = g^{-n}(D_0)$ for $n > 0$. Let $\tilde{P}_{0,j}$ be the component of $D_0 \setminus \overline{\bigcup \tilde{S}_{0,j}}$ between $\tilde{S}_{0,j-1}$ and $\tilde{S}_{0,j}$. Let $S_{0,j} = \overline{\tilde{S}_{0,j}}$, $P_{0,j} = \overline{\tilde{P}_{0,j}}$ and

$$\mathcal{P}_n = \left\{ \text{the closures of components of } g^{-n}(\tilde{P}_{0,j}) \ (j \in \mathbb{Z}_{Nq}) \right\}$$
$$\mathcal{S}_n = \left\{ \text{the closures of components of } g^{-n}(\tilde{S}_{0,j}) \ (j \in \mathbb{Z}_{Nq}) \right\}.$$

We call an element of $\mathcal{P}_n$ a *piece of depth n* and an element of $\mathcal{S}_n$ a *sector of depth n.*

Then $\mathcal{P}_n$ and $\mathcal{S}_n$ have the following properties. Let $n \geq 0$.

1. $\mathcal{P}_n \cup \mathcal{S}_n$ is a partition of $\overline{D_n}$.
2. For any $z \in K(g) \setminus \bigcup_j g^{-j}(x)$, there exists a unique piece $P_n(z)$ of depth $n$ which contains $z$. In particular, $\mathcal{P}_n$ covers $K(g)$.
3. For any $P \in \mathcal{P}_{n+1}$, there exists some $P' \in \mathcal{P}_n$ with $P \subset P'$.
4. For $P \in \mathcal{P}_{n+1}$, we have $g(P) \in \mathcal{P}_n$.
5. When $S \in \mathcal{S}_{n+1}$, either there exists some $S' \in \mathcal{S}_n$ with $S = S' \cap \overline{D_{n+1}}$, or there exists some $P \in \mathcal{P}_n$ with $S \subset \operatorname{int} P$.
6. For any $X \in \mathcal{P}_n \cup \mathcal{S}_n$, $\operatorname{int} X \cap g^{-n+1}(\mathcal{K}) \neq \emptyset$ or there exists a unique $y \in g^{-n}(x_0)$ with $y \in X$.
7. For any $P \in \mathcal{P}_n$, there exists a unique component $E$ of $g^{-n}(\mathcal{K} \setminus g^{-n}(x_0))$ with $E \subset P$. This map $P \mapsto E$ is a bijection between $\mathcal{P}_n$ and the set of components of $g^{-n}(\mathcal{K} \setminus g^{-n}(x_0))$.

The following theorem says that $\mathcal{K}$ attracts almost every point in $K(g)$.

**Theorem 5.1.** *The set $K(g) \setminus \bigcup_{n>0} g^{-n}(\mathcal{K})$ has zero Lebesgue measure.*

*Proof.* Let $z_0 \in K(g) \setminus \bigcup_{n>0} g^{-n}(\mathcal{K})$. Our proof is divided into three cases:

**Case I:** $\limsup d(\mathcal{K}, g^n(z_0)) > 0$.
Since $P(g) \subset K(g)$ and $d(P(g), g^n(z)) \to 0$ for almost every $z \in J(g)$ (see [Mc, Theorem 3.9]), the set of such $z_0$ has measure zero.

**Case II:** $\lim d(\mathcal{K}, g^n(z_0)) = 0$ and $\liminf d(x, g^n(z_0)) > 0$.
When $n$ is sufficiently large, $g^n(z_0)$ is close to $K_k(G)$ for some $k \in \mathbb{Z}_k$. Then $g^{n+1}(z_0)$ must close to $K_{k+1}(G)$ because $K_0(G), \ldots, K_{N-1}(G)$ meet only at $x$. Inductively, $g^{n+l}(z_0)$ is close to $K_{k+l}(G)$, so it lies in $U_{k+l}$. This implies $g^n(z_0)$ lies in $K_{k+1}(G) \subset \mathcal{K}$, this is a contradiction. Therefore, this case does not occur.

**Case III:** $\limsup d(\mathcal{K}, g^n(z_0)) = 0$ and $\liminf d(x, g^n(z_0)) = 0$.
We show that the Lebesgue density of $K(g)$ at $z_0$ is not equal to one. Then since the set of all such $z_0$ is contained in $K(g)$, It is of measure zero.

Let $P_{1,j} \in \mathcal{P}_1$ ($j \in \mathbb{Z}_{Nq}$) be the piece of depth 1 with $x \in P_{1,j} \subset P_{0,j}$ and let $\psi : O \to \mathbb{C}$ be the linearizing coordinate of $x$ for $g$ defined on a neighborhood $O$ of $x$. Let $\tilde{E}_j = O \cap P_{0,j}$. Take $\tilde{E}'_j$ and $\tilde{E}''_j$ so that $\tilde{E}''_j \Subset \tilde{E}'_j \Subset \tilde{E}_j$ and $\tilde{E}''_j$ contains a fundamental domain of $(K(g) \cap \tilde{E}_j)/g^N$. Let $E_j = \psi(\tilde{E}_j)$, $E'_j = \psi(\tilde{E}'_j)$ and $E''_j = \psi(\tilde{E}''_j)$.

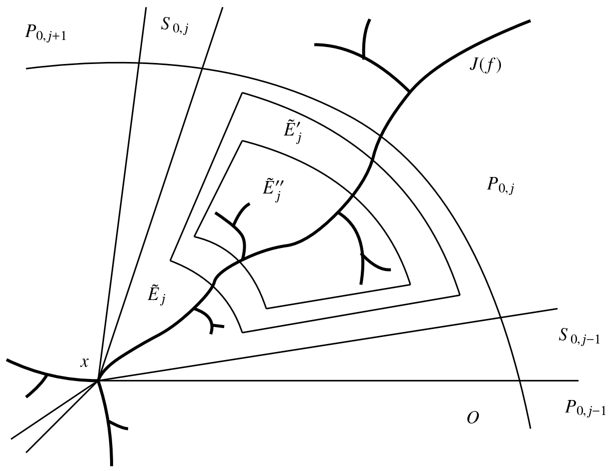


Figure 4: $E_j$, $E'_j$ and $E''_j$.

If $z \in K(g)$ is sufficiently close to $x$ (say $d(x,z) < \delta$), then there exist some $s = s(z) > 0$ and $j = j(z) \in \mathbb{Z}_N$ such that $g^s(z) \in \tilde{E}''_j$. So

$$\lambda^s \psi(z) = \psi(g^s(z)) \in E''_j,$$

where $\lambda = g'(x)$. The map $\psi^{-1}(\lambda^{-s}\cdot)$ maps $E_j$ univalently to a neighborhood $E = E(z)$ of $z$. Let $E'(z) = \psi^{-1}(\lambda^{-s}E'_j)$ and $E''(z) = \psi^{-1}(\lambda^{-s}E''_j)$. Clearly,

$$\frac{m(E'_j \cap \psi(K(f)))}{m(E'_j)} < 1.$$

where $m$ is the Lebesgue measure.

Take an $\epsilon > 0$ sufficiently small so that $d(g(z), x) < \delta$ when $z \notin \bigcup P_{1,j}$ and $d(z, \mathcal{K}) < \epsilon$. This can be done because each $P \in \mathcal{P}_1 \setminus \{P_{1,j}\}$ are attached to $\mathcal{K}$ only at some point $y \in g^{-1}(x)$.

Since $z_0 \notin g^{-n}(\mathcal{K})$, $g^n(z_0) \ni n \bigcup P_{1,j}$ for infinitely many $n$. Let $\{n_k\}_{k>0}$ be a sequence of $n > 0$ which satisfies that $g^n(z_0) \in O$ and $g^{n-1}(z_0) \notin \bigcup P_{1,j}$. We can take $n_1$ sufficiently large so that we have $d(g^n(z_0), \mathcal{K}) < \epsilon$ whenever $n \geq n_1$.

For each $k$, let
$$l_k = \min\left\{ l > 0 \;\middle|\; g^{n_k+l}(z_0) \notin \bigcup P_{1,j} \right\}.$$
Then, since $d(g^{n_k+l_k}(z_0), \mathcal{K}) < \epsilon$, $d(g^{n_k+l_k+1}(z_0), x) < \delta$. Since $g^{n_k+l_k}(z_0) \notin \bigcup P_{1,j} \supset P(g)$, $g^{n_k+l_k+1}$ is univalent on $P_{n_k+l_k+1}(z_0)$. Let $y_k = g^{n_k+l_k+1}(z_0)$ and $\tilde{\iota}_k : P_0(y_k) \to P_{n_k+l_k+1}(z_0)$ be the inverse of the univalent map. Let $j(y_k)$, $s(y_k)$, $E(y_k)$, $E'(y_k)$ and $E''(y_k)$ as before (note that $d(y_k, x) < \delta$). Then
$$y_k \in E''(y_k) \Subset E'(y_k) \Subset E(y_k) \subset P_0(y_k).$$
Define $\iota_k : E_{j(y_k)} \to P_{n_k+l_k+1}$ by
$$\iota_k(z) = \tilde{\iota}_k \circ \psi^{-1}(\lambda^{-s(y_k)}(z))$$
and let $z_k = \iota_k^{-1}(z_0)\,(= \lambda^{s(y_k)}\psi(y_k))$.

By the Koebe distortion theorem, there exist $0 < r_1 < r_2$ and $0 < C_3 < 1$ such that
$$B(z_0, r_1\iota_k'(z_k)) \subset \iota_k(E'(z_k)) \subset B(z_0, r_2\iota_k'(z_k)),$$
$$\frac{m(\iota_k(E'(y_k)) \cap K(f))}{m(\iota_k(E'(y_k)))} < C_3$$
where $B(z, r)$ is the ball of radius $r$ centered at $z$. Therefore,
$$\frac{m(B(z_0, r_2\iota_k'(z_k) \cap K(f))}{m(B(z_0, r_2\iota_k'(z_k))} < C \tag{8}$$
for some $C < 1$ independent of $l$.

Since the forward orbit of $z_0$ by $g$ does not intersect $P(f)$, $\|(g^n)'(z_0)\|$ tends to $\infty$ with respect to the hyperbolic metric on $\mathbb{C} \setminus P(f)$ (see [Mc, Theorem 3.6]). Furthermore, the piece $P_1(g^{n_k+l_k}(z_0))$ is disjoint from $P(f)$ and $\lambda^{s(y_k)}\psi \circ g$ is a univalent map from $g^{n_k+l_k}(\iota_k(E(y_k)))$ to $E_{j(y_k)}$, so the inverse of this map does not expand the hyperbolic metric on $E_{j(y_k)}$ and $\mathbb{C} \setminus P(f)$, respectively.

Therefore the differential
$$\|(\iota_k^{-1})'(z_0)\| = \|(\lambda^{s(y_k)}\psi \circ g)'(g^{n_k+l_k}(z_0))\| \cdot \|(g^{n_k+l_k})'(z_0)\|$$
with respect to the hyperbolic metric on $\mathbb{C} \setminus P(f)$ and $E_{j(y_k)}$ tends to infinity as $k \to \infty$. Since $x_k \in E''_{j(s_k)} \Subset E_{j(s_k)}$, this implies that $|\iota_k'(x_k)| \to 0$ as $l \to \infty$. By (8), the Lebesgue density of $K(f)$ at $z_0$ is at most $C < 1$. □

For a later use, we give a canonical form of the renormalization $G$. Take small $r > 0$ and $\eta > 0$. For $j \in \mathbb{Z}_{Nq}$, let $\hat{P}_{0,j}$ be the union of $B(x, r)$ and the domain in $D_0 \setminus B(x, r)$ between $\mathcal{R}(g; \omega_{j-1} - \eta, R)$ and $\mathcal{R}(g; \omega_j + \eta, R)$.

Let $Q_j$ be the component of $g^{-1}(\hat{P}_{0,j})$ which is contained in $\hat{P}_{0,j}$. Let $U_k$ and $V_k$ are disks obtained by smoothing the boundary of $\bigcup_{j\in\mathbb{Z}_q} Q_{k+Nj}$ and $\bigcup_{j\in\mathbb{Z}_q} \hat{P}_{0,k+Nj}$. Then $G = (g : U_k \to V_{k+1})$ is a renormalization hybrid equivalent to $F$.

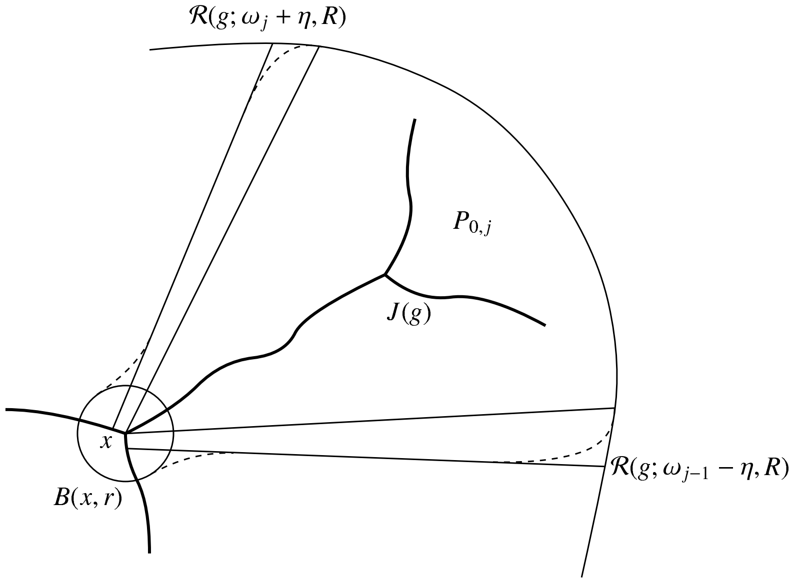


Figure 5: Thickened piece $\hat{P}_{0,j}$ and smoothing its boundary.

## 5.2 Proof of the uniqueness

Let $(g,x)$ and $(g',x')$ be two $p$-rotatory intertwinings of an $N$-polynomial map $(F,O)$ with a marked periodic point of rotation number $p_0/q$. We use the notation in section 5.1 for $g$. For $g'$, we attach a prime to each notation (*e.g.*, $\mathcal{K}', D'_n, \mathcal{P}'_n, \mathcal{S}'_n, \dots$).

In this section, we show that $g$ and $g'$ are affinely conjugate. Since $K(g)$ and $K(g')$ are connected, we need only show that $g$ and $g'$ are hybrid equivalent. To do this, we first construct a standard hybrid conjugacy between the renormalizations $G$ and $G'$. Then by pulling back it repeatedly, we construct a quasiconformal conjugacy between $g$ and $g'$. By means of Theorem 5.1, we show that it is actually a hybrid conjugacy.

**Lemma 5.2.** *There exists a quasiconformal map* $\Phi_0 : \overline{D_0} \to \overline{D'_0}$ *satisfies the following:*

- $\bar{\partial}\Phi_0 \equiv 0$ *a.e. on* $K(G)$.
- $\Phi_0 \circ g = g' \circ \Phi_0$ *on* $\bigcup(P_{1,j} \cup S_{0,j}) \cup \partial D_1$.

*Proof.* For each $k \in \mathbb{Z}_N$, take a $C^1$-diffeomorphism $\tilde{\Phi}_k : \overline{V_k \setminus U_k} \to \overline{V'_k \setminus U'_k}$ which satisfies the following:

1. $\tilde{\Phi}_k(\partial V_k) = \partial V'_k$ and $\tilde{\Phi}_k(\partial U_k) = \partial U'_k$.
2. For $j \in \mathbb{Z}_{Nq}$ with $P_{0,j} \subset V_k$ (equivalently, $j \equiv k \mod N$), we have $\tilde{\Phi}_k(\partial(P_{0,j} \setminus U_k)) = \partial(P'_{0,j} \setminus U'_k)$ and $\tilde{\Phi}_k(P_{0,j} \setminus U_k) = P'_{0,j} \setminus U'_k$.
3. For $z \in \partial U_k$, $\Phi_{k+1}(g(z)) = g'(\Phi_k(z))$.

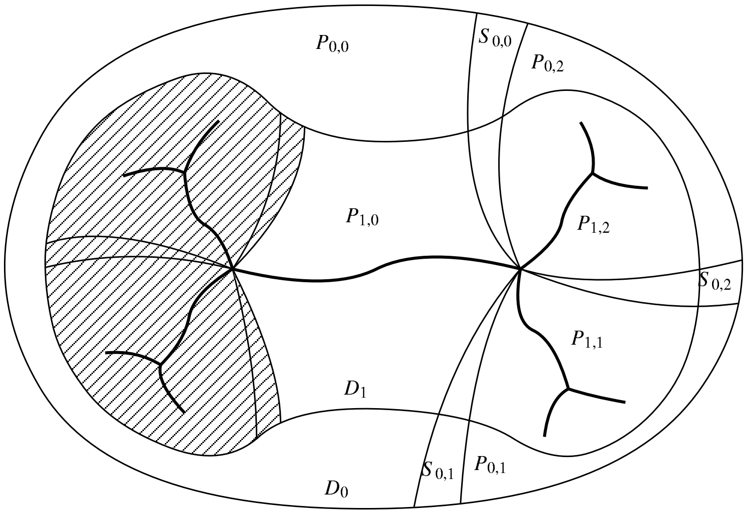


Figure 6: Puzzles (the case degree two and $p/q = 1/3$. $\Phi_0 \circ g = g' \circ \Phi_0$ on $\overline{D_1}$ except on the interior of the painted area).

As in [DH], we can extend $\tilde{\Phi}_k$ to a diffeomorphism on $\overline{V_k} \setminus K_k(G)$ to $\overline{V'_k} \setminus K_k(G')$ by the equation $\tilde{\Phi}_k(g(z)) = g'(\Phi_k(z))$. Furthermore, since $G$ and $G'$ are hybrid equivalent (they are both hybrid equivalent to $F$), this $\tilde{\Phi}_k$ extends to a hybrid conjugacy of $G$ to $G'$. (To do this, we use [DH, Proposition 6]. So we need to check $[\tilde{\Phi}_0, \psi, g^n, (g')^n] = 0$ in $\mathbb{Z}_{deg(G^n)}$ where $\psi$ is a given hybrid conjugacy of $G$ and $G'$ considered as classical polynomial-like maps. But it is trivial because of the property 2 above.)

We define $\Phi_0$ first on $\bigcup S_{0,j}$. For each $S_{0,j}$, define a quasiconformal map $\Phi_0|_{S_{0,j}} : S_{0,j} \to S'_{0,j}$ so that $\Phi_0 \circ g = g' \circ \Phi_0$ and

$$\begin{aligned} \Phi_0 &= \tilde{\Phi}_{j-1} && \text{on a neighborhood of } \mathcal{R}(g; \omega_j, R, -\epsilon), \\ \Phi_0 &= \tilde{\Phi}_j && \text{on a neighborhood of } \mathcal{R}(g; \omega_j, R, \epsilon). \end{aligned}$$

Let $\hat{\Phi}_k : \overline{V_k \setminus U_k} \to \overline{V'_k \setminus U'_k}$ be a $C^1$-diffeomorphism which satisfies the same condition as for $\tilde{\Phi}_k$ and for $k, k' \in \mathbb{Z}_N$ and $j, j' \in \mathbb{Z}_{Nq}$ with $j \equiv k \mod N$,

- $\hat{\Phi}_k = \tilde{\Phi}_k$ on $\partial(P_{0,j} \setminus U_k)$.
- $g' \circ \hat{\Phi}_k(z) = \tilde{\Phi}_{k'} \circ g(z)$ when $z$ lies in $P_{0,j} \cap \partial D_1 \cap g^{-1}(P_{0,j'})$.
- $g' \circ \hat{\Phi}_k(z) = \Phi_0 \circ g(z)$ when $z$ lies in $P_{0,j} \cap \partial D_1 \cap g^{-1}(S_{0,j})$.

As in the case of $\tilde{\Phi}_k$, we can extend $\hat{\Phi}_k$ quasiconformally to $V_k$ and obtain hybrid equivalence between $G$ and $G'$.

Now let $\Phi_0 = \hat{\Phi}_k$ on $P_{0,j}$ where $k \equiv j \mod N$. It is easy to check this $\Phi_0$ has the desired properties. □

Then we define $\Phi_n : \overline{D_0} \to \overline{D'_0}$ inductively. Suppose $\Phi_n$ is defined and satisfies:

- $\bar{\partial}\Phi_n \equiv 0$ on $g^{-n}(\mathcal{K})$.
- $\Phi_n \circ g = g' \circ \Phi_n$ on $(\overline{D_1} \setminus D_{n+1}) \cup \bigcup_j g^{-n}(P_{1,j} \cup S_{1,j})$.

(Clearly, $\Phi_0$ satisfies this condition for $n = 0$.)

To define $\Phi_{n+1}$, first let $\Phi_{n+1}|_{\overline{D_0 \setminus D_{n+1}}} = \Phi_n$. For $P \in \mathcal{P}_{n+1}$, when $\operatorname{int} P \cap g^{-n}(\mathcal{K}) \neq \emptyset$, define $\Phi_{n+1}|_P = \Phi_n$. Otherwise, by the property 6 in page 15, there exists a unique $y \in g^{-n}(x) \in P$. Let $P' \cap \mathcal{P}_{n+1}$ be the piece of depth $n+1$ which combinatorially corresponds to $P'$, *i.e.* which satisfies that $\Phi_n(g(P)) = g(P')$ and $\Phi_n(y) \in P'$. Note that when $y$ is not a critical point of $g$, such $P'$ is unique. When $y$ is a critical point, $P'$ is determined by the cyclic order of pieces and sectors at $y$ to make $\Phi_{n+1}$ continuous. Then, since $C(g) \subset \mathcal{K} \subset g^{-n}(\mathcal{K})$, $g|_P$ is conformal and so is $g'|_{P'}$. So define

$$\Phi_{n+1}|P = (g'|_{P'})^{-1} \circ \Phi_n \circ g|_P : P \to P'.$$

Similarly, for $S \in \mathcal{S}_{n+1}$, when $S \subset S'$ for some $S' \in \mathcal{S}_n$, then define $\Phi_{n+1}|_S = \Phi_n$. Otherwise, take $S' \in \mathcal{S}'_{n+1}$ combinatorially corresponds to $S$ and define

$$\Phi_{n+1}|_S = (g'|_{S'})^{-1} \circ \Phi_n \circ g|_S : S \to S'.$$

(In other words, $\Phi_{n+1}|_{\overline{D_{n+1}}}$ is defined by lifting $\Phi_n$ by the branched covering $g$ and $g'$.)

We must check $\Phi_{n+1}$ also satisfies the properties above. First, we show the continuity of $\Phi_{n+1}$. By the construction, $\Phi_{n+1}$ is continuous on and outside $\overline{D_{n+1}}$. For $z \in \partial D_{n+1}$, since $g(z) \in \partial D_n$,

$$\begin{aligned}\Phi_{n+1}(z) &= (g'|_{P'})^{-1} \circ \Phi_n \circ g|_P(z) \\ &= (g'|_{P'})^{-1} \circ g'|_{P'} \circ \Phi_n(z) \\ &= \Phi_n(z)\end{aligned}$$

by the second property above for $\Phi_n$. Hence $\Phi_{n+1}$ is continuous.

For every $X \in \mathcal{P}_{n+1} \cap \mathcal{S}_{n+1}$, $\Phi_{n+1}|_X$ is a quasiconformal homeomorphism from $X$ to the corresponding piece or sector for $g'$ and $\Phi_{n+1}|_{\overline{D_0} \setminus D_{n+1}} = \Phi_n$ is clearly quasiconformal. Hence $\Phi_{n+1}$ is a quasiconformal homeomorphism. Furthermore, by the construction, the dilatation ratio of $\Phi_{n+1}$ is equal to that of $\Phi_n$ and $\partial\Phi_{n+1} \equiv 0$ on $g^{-n-1}(\mathcal{K})$.

Clearly, $g' \circ \Phi_{n+1} = \Phi_{n+1} \circ g$ on $E_{n+1} = \bigcup_j g^{-n-1}(P_{1,j} \cup S_{1,j}) \cup (\overline{D_1} \setminus D_{n+1})$. Let $z \in D_{n+1} \setminus (E_{n+1} \cup D_{n+2})$. Then $z$ lies in some $X \in \mathcal{P}_{n+1} \cup \mathcal{S}_{n+1}$ with $\operatorname{int} X \cap g^{-n}(\mathcal{K}) = \emptyset$. Therefore,

$$\begin{aligned}g' \circ \Phi_{n+1}(z) &= g' \circ (g'|_{P'})^{-1} \circ \Phi_n \circ g(z) \\ &= \Phi_n \circ g(z).\end{aligned}$$

Since $g(z) \in D_n \setminus D_{n+1}$, we have $\Phi_n(g(z)) = \Phi_{n+1}(g(z))$ and the second property holds for $\Phi_{n+1}$.

Since all $\Phi_n$ are quasiconformal with same dilatation ratio, they form an equicontinuous family. Furthermore, $\Phi_n = \Phi_{n+1}$ except on $D_{n+1} \setminus g^{-n}(\mathcal{K})$. Hence $\Phi = \lim \Phi_n$ exists and it is quasiconformal. Also, it satisfies that $\bar{\partial}\Phi \equiv 0$ on $\bigcup_n g^{-n}(\mathcal{K})$ and that $g' \circ \Phi = \Phi \circ g$. Since $K(g) \setminus \bigcup g^{-n}(\mathcal{K})$ has zero Lebesgue measure (Theorem 5.1), $\Phi$ is a hybrid conjugacy between $g$ and $g'$.

Therefore, a $p$-rotatory intertwining of $(F, O)$ is unique up to affine conjugacy.

# References


[Bi] B. Bielefeld, *Changing the order of critical points of polynomials using quasiconformal surgery*, Thesis, Cornell, 1989.

[DH] A. Douady and J. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. sci. Éc. Norm. Sup., **18** (1985) 287-343.

[EY] A. Epstein and M. Yampolsky, *Geography of the cubic connectedness locus* I: *Intertwining surgery*, Ann. Sci. Éc. Norm. Sup. **32** (1999), 151-185.

[In] H. Inou, *Renormalization and rigidity of polynomials of higher degree*, Journal of Math, Kyoto Univ., to appear.

[Mc] C. McMullen, *Complex Dynamics and Renormalization*, Annals of Math Studies, vol. 135, 1994.

[St] N. Steinmetz, *Rational Iteration*, de Gruyter, 1993.