SCHRÖDINGER OPERATORS ON THE WIENER SPACE

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Abstract. We discuss a Schrödinger operator on the Wiener space of the form $L - V$, L being the Ornstein-Uhlenbeck operator and V is a potential function. We determine the domain of $L-V$ and show the spectral gap under the assumption of exponential integrability of the negative part of V.

1. Introduction

We consider a Schrödinger operator $\mathfrak{A} = L - V$ on an abstract Wiener space (B, H, μ). Here L is the Ornstein-Uhlenbeck operator and V is a scalar potential. Our goal is to determine the domain of $L - V$. To be precise, we will show that $\mathrm{Dom}(\mathfrak{A}) = \mathrm{Dom}(L) \cap \mathrm{Dom}(V)$ under a suitable condition.

This kind of problem was considered in the context of scalar field of quantum field theory. Essential self-adjointness was discussed by Segal [5], B. Simon [10] and others. Issues of determining the domain and the spectral gap was considered by Glimm and Jaffe [1], Simon and Høegh-Krohn [11], etc. Their methods depend on the hypercontractivity but we use logarithcic Sobolev inequality, which is known to be equivalent with the hypercontractivity. To determine the operator domain we use the intertwining properties of operators.

The organization of the paper is as follows. In §2, we discuss the essential self-adjointness of a Schrödinger operator. We use the perturbation theory and the logarithmic Sobolev inequality. We also give a refinement of the logarithmic Sobolev inequality in terms of the generator itself. In §3, we determine the domain of the Schrödinger operator. The intertwining property of operators plays an important role. Last, in §4, we discuss the spectral gap of the operator.

2. Essential self-adjointness

We first fix notations. Let (B, H, μ) be an abstract Wiener space, i.e., B is a Banach space, H is a Hilbert space imbedded in B, and μ is the Wiener measure with the characteristic function

$$\int_B e^{i\langle x,\varphi\rangle}\mu(dx) = \exp\Big\{-\frac{1}{2}|\varphi|^2_{H^*}\Big\}, \quad \varphi \in B^* \subset H^*. \tag{2.1}$$

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Let $\mathcal{F}C_0^\infty$ be the set of all functions $f\colon B \to \mathbb{R}$ such that there exist $n \in \mathbb{N}$, $F \in C_0^\infty(\mathbb{R}^n)$ and $\varphi_1, \dots, \varphi_n \in B^*$ with

$$f(x) = F(\langle x, \varphi_1\rangle, \dots, \langle x, \varphi_n\rangle). \tag{2.2}$$

We suppose that a Schrödinger operator $L - V$ is defined on $L^2(\mu)$, where L is the Ornstein-Uhlenbeck operator, and V is a scalar potential. We first consider when $L - V$ is essentially self-adjoin on $\mathcal{F}C_0^\infty$. To do this, we use the theory of positive generalized function (see, e.g., [3]). We denote the L^2-norm by $\|\ \|_2$ and the operator norm by $\|\ \|_{\mathrm{op}}$. We divide V into the positive part $V_+ := \max\{V, 0\}$ and the negative part $V_- := \max\{-V, 0\}$. We have the following

Proposition 2.1. *We assume that $V_+ \in L^{2+}$. For the negative part, we assume that there exist $0 < a < 1$, $b > 0$ such that*

$$\|V_- f\|_2 \le a\|Lf\|_2 + b\|f\|_2. \tag{2.3}$$

Then $-L + V$ is essentially self-adjoint on $\mathcal{F}C_0^\infty$.

Proof. From (2.3), we have

$$\|V_-(c - L)^{-1} f\|_2 \le a\|L(c - L)^{-1} f\|_2 + b\|(c - L)^{-1} f\|_2.$$

Further, by using the spectral decomposition, we have $\|L(c - L)^{-1}\|_{\mathrm{op}} \le 1$, $\|(c - L)^{-1}\|_{\mathrm{op}} \le 1/c$ and thereby

$$\|V_-(c - L)^{-1} f\|_2 \le \Big(a + \frac{b}{c}\Big)\|f\|_2.$$

Take c to be large so that $a + \frac{b}{c} < 1$. Since L is symmetric, we also have $\|(c - L)^{-1} V_-\|_{\mathrm{op}} < 1$. From the assumption $V_+ \in L^{2+}$, we can take p, $p' > 1$ so that $\frac{1}{p} + \frac{1}{p'} + \frac{1}{2} = 1$ and $V_+ \in L^p$.

$c - L + V$ is clearly well-defined on $\mathcal{F}C_0^\infty$. To prove the essential self-adjointness, it suffices to show that

$$\operatorname{Ker}(c - L + V)^* = \{0\}. \tag{2.4}$$

Now we take any $g \in \operatorname{Ker}(c - L + V)^*$. Then we have, for all $f \in \mathcal{F}C_0^\infty$,

$$\langle (c - L + V) f, g\rangle = 0. \tag{2.5}$$

Let q' be a conjugate exponent of p'. Then $Vg \in L^{q'}$. By noting that $\mathcal{F}C_0^\infty$ is dense in $\mathcal{F}_{2,p'}$, we can see that (2.5) holds for all $f \in \mathcal{F}_{2,p'}$. Let $\{T_t\}$ be a semigroup generated by $L - c$. Since $\mathcal{F}_{2,p'}$ is invariant under T_t, we have, for all $f \in \mathcal{F}_{2,p'}$,

$$\langle T_t f, g\rangle - \langle f, g\rangle = \int_0^t \frac{d}{ds}\langle T_s f, g\rangle ds = \int_0^t \langle (L - c) T_s f, g\rangle ds = \int_0^t \langle T_s f, Vg\rangle ds.$$

Recalling that $\mathcal{F}_{2,p'}$ is dense in $L^{p'}$ and using the continuity, we have

$$\langle f, T_t g\rangle - \langle f, g\rangle = \int_0^t \langle T_s f, Vg\rangle ds, \qquad \forall f \in L^{p'}.$$

In particular, taking $u \in (\mathcal{F}_{2,p'})_+$, $f = u \operatorname{sgn} g \in L^{p'}$, it follows that

$$\langle u \operatorname{sgn} g, T_t g\rangle - \langle u, |g|\rangle = \int_0^t \langle T_s(u \operatorname{sgn} g), Vg\rangle ds.$$

Since the semigroup $\{T_t\}$ preserves the positivity, we have

$$\langle u, T_t|g|\rangle \geq \langle u, \operatorname{sgn} g\, T_t g\rangle$$

and therefore

$$\langle u, T_t|g|\rangle - \langle u, |g|\rangle \geq \int_0^t \langle T_s(u \operatorname{sgn} g), Vg\rangle ds.$$

Dividing the both hand by t and letting $t \to 0$,

$$\langle u, (L-c)|g|\rangle \geq \langle u \operatorname{sgn} g, Vg\rangle = \langle u, V|g|\rangle \geq -\langle u, V_-|g|\rangle.$$

Here $(L-c)|g|$ is regarded a generalized function as an element of $\mathcal{F}_{-2,q'}$. Since the identity above holds for all $u \in (\mathcal{F}_{2,p'})_+$, it follows that $(L-c)|g| + V_-|g| \in (\mathcal{F}_{-2,q'})_+$.

On the other hand, for $f \in \mathcal{F}_{2,2}$, we have

$$|\langle V_-|g|, f\rangle| = |\langle |g|, V_- f\rangle| \leq \|g\|_2 \|V_- f\|_2 \leq \|g\|_2 (a\|Lf\|_2 + b\|f\|_2).$$

Then we can see that $V_-|g| \in \mathcal{F}_{-2,2}$. The same is true for $(L-c)|g|$ and hence we have $(L-c)|g| + V_-|g| \in \mathcal{F}_{-2,2}$. Now, using [3, Proposition 3.5], we can get $(L-c)|g| + V_-|g| \in (\mathcal{F}_{-2,2})_+$, which means

$$(c-L)|g| \leq V_-|g| \quad \text{in } \mathcal{F}_{-2,2}.$$

Since $(c-L)^{-1}$ preserves the positivity, we have

$$|g| \leq (c-L)^{-1} V_-|g|$$

and further

$$\|g\|_2 \leq \|(c-L)^{-1} V_-|g|\|_2 \leq \|(c-L)^{-1} V_-\|_{\mathrm{op}} \|g\|_2.$$

Since c was chosen so that $\|(c-L)^{-1}V_-\|_{\mathrm{op}} < 1$, we have $g = 0$. This shows (2.4) and the proof is complete. □

We now have to give a sufficient condition for (2.3). To do this, we make use of the logarithmic Sobolev inequality. It is usually written in terms of bilinear form, but the generator itself is involved in our case. So we have to rewrite it a bit.

In general setting, the (defective) logarithmic Sobolev inequality for a Dirichlet form $\mathcal{E}$ is written as

$$\int_B |f|^2 \log(|f|/\|f\|_2)\, d\mu \leq \alpha \mathcal{E}(f,f) + \beta \|f\|_2^2. \tag{2.6}$$

Here (B,μ) is a general measure space. We also denote the associated generator by L (not specify to the Ornstein-Uhlenbeck operator. For the Dirichlet form $\mathcal{E}$, we assume the local property and the existence of square field operator, i.e.,

$$\mathcal{E}(f,g) = \int_B \Gamma(f,g)\, d\mu \tag{2.7}$$

and Γ has the derivation property. In the case of an abstract Wiener space, $\Gamma(f,f) = |\nabla f|^2$, ∇ being a gradient operator. We have the following theorem.

Theorem 2.2. *Assume that the logarithmic Sobolev* (2.6) *holds. Then, for any* $\varepsilon > 0$*, there exist positive constants* K_1*,* K_2 *such that*

$$\int_B f^2 \log_+^2 f \, d\mu \le \alpha^2(1+\varepsilon)\|Lf\|_2^2 + K_1 + K_2\|f\|_2^6. \tag{2.8}$$

Proof. Let k be a function on $\mathbb{R}$ such that k is a concave and even function of class C^2 with $k(0) = 0$ and

$$k(x) = x \log^{1/2} x, \qquad x \ge e.$$

Since $k'(x) = \log^{1/2} x + \frac{1}{2\log^{1/2} x}$, we have

$$\begin{aligned} \log k(x) &= \log x + \log\log^{1/2} x, \\ k'(x)^2 &= \log x + \frac{1}{4\log x} + 1. \end{aligned}$$

Hence

$$k'(x)^2 \le \log k(x) \tag{2.9}$$

for sufficiently large x. We also note that

$$k(x)k'(x) = x\Big(\frac{1}{2} + \log|x|\Big) \tag{2.10}$$

for $|x| \ge e$. By the logarithmic Sobolev inequality (2.6), we have

$$\int_B g^2 \log|g| \, d\mu \le -\alpha \int_B gLg \, d\mu + \beta\|g\|_2^2 + \|g\|_2^2 \log\|g\|_2.$$

Set $g = k(f)$. Then, by the local property of $\mathcal{E}$,

$$Lg = k'(f)Lf + k''(f)\Gamma(f,f).$$

Hence

$$\begin{aligned} &\int_B k(f)^2 \log k(f) \, d\mu \\ &\quad \le -\alpha \int_B k(f)\{k'(f)Lf + k''(f)\Gamma(f,f)\} \, d\mu + \beta\|k(f)\|_2^2 + \|k(f)\|_2^2 \log\|k(f)\|_2 \\ &\quad \le \alpha \int_B k(f)|k'(f)||Lf| \, d\mu + \beta\|k(f)\|_2^2 + \|k(f)\|_2^2 \log\|k(f)\|_2. \end{aligned}$$

Now, for large x, by using (2.9),

$$k(x)^2 \log k(x) \ge k(x)^2 k'(x)^2.$$

For small x, by taking $C_1 > 0$ large enough, we have

$$k(x)^2 \log k(x) \ge k(x)^2 k'(x)^2 - C_1.$$

Combining them gives

$$\begin{aligned}
&\int_B k(f)^2 k'(f)^2\,d\mu - C_1 \\
&\quad\le \int_B k(f)^2 \log k(f)\,d\mu \\
&\quad\le \alpha\int_B k(f)|k'(f)||Lf|\,d\mu + \beta\|k(f)\|_2^2 + \|k(f)\|_2^2\log\|k(f)\|_2 \\
&\quad\le \frac{1}{2}\int_B k(f)^2|k'(f)|^2\,d\mu + \frac{\alpha^2}{2}\int_B |Lf|^2\,d\mu \\
&\qquad + \beta\|k(f)\|_2^2 + \|k(f)\|_2^2\log\|k(f)\|_2.
\end{aligned}$$

Here, in the last line, we used the inequality $ab \le \frac{1}{2}(a^2+b^2)$. Thus

$$\int_B k(f)^2 k'(f)^2\,d\mu \le 2C_1 + \alpha^2\int_B |Lf|^2\,d\mu + 2\beta\|k(f)\|_2^2 + 2\|k(f)\|_2^2\log\|k(f)\|_2.$$

Recall that $k(x)k'(x) = x(\frac{1}{2} + \log|x|)$ for large x. Then, by changing constants, we have

$$\begin{aligned}
&\int_B f^2\Big(\frac{1}{2} + \log|f|\Big)^2\,d\mu \\
&\quad\le C_2 + \alpha^2\int_B |Lf|^2\,d\mu + 2\beta\|k(f)\|_2^2 + 2\|k(f)\|_2^2\log\|k(f)\|_2.
\end{aligned}$$

Again, by changing constant, we have

$$\int_B f^2\log_+^2|f|\,d\mu \le C_3 + \alpha^2\int_B |Lf|^2\,d\mu + 2\beta\|k(f)\|_2^2 + 2\|k(f)\|_2^2\log\|k(f)\|_2. \tag{2.11}$$

The essential part has been done. Therefore, it remains to estimate $\|k(f)\|_2^2$ and $\|k(f)\|_2^2\log\|k(f)\|_2$. We first need to compute $\|k(f)\|_2^2$.

$$\|k(f)\|_2^2 \le E[f^2\log_+|f|] + C_4.$$

Here we denote the integral with respect to μ by $E[\ \ ]$. We use this convention from now on. We may assume $C_4 \ge 1$. If $r \ge 1$, then $\log(x+r) \le \log_+ x + \log 2r$ $(x \ge 0)$. Therefore

$$\begin{aligned}
\|k(f)\|_2^2\log\|k(f)\|_2 &\le E[f^2\log_+|f| + C_4]\log E[f^2\log_+|f| + C_4] \\
&\le E[f^2\log_+|f| + C_4]\{\log E[f^2\log_+|f|] + \log(2C_4)\} \\
&\le 2E[f^2\log_+|f|]\log_+ E[f^2\log_+|f|] + C_5.
\end{aligned}$$

By the Schwarz inequality,

$$E[f^2\log_+|f|] = E[f\cdot f\log_+|f|] \le \{E[f^2]E[f^2\log_+^2|f|]\}^{1/2}.$$

Combining this with the inequality $x\log_+ x \le x^{3/2}$ leads to

$$\begin{aligned}
&E[f^2\log_+|f|]\log_+ E[f^2\log_+|f|]\\
&\quad\le \{E[f^2]E[f^2\log_+^2|f|]\}^{1/2}\log_+\{E[f^2]E[f^2\log_+^2|f|]\}^{1/2}\\
&\quad\le \{E[f^2]E[f^2\log_+^2|f|]\}^{3/4}\\
&\quad= E[f^2]^{3/4}E[f^2\log_+^2|f|]^{3/4}.
\end{aligned}$$

Further, by using the inequality

$$xy \le \frac{x^4}{4} + \frac{y^{4/3}}{\frac{4}{3}},$$

we have, for $\delta > 0$,

$$E[f^2\log_+|f|]\log_+ E[f^2\log_+|f|] \le \frac{E[f^2]^3}{4\delta^4} + \frac{\delta^{4/3}E[f^2\log_+^2|f|]}{\frac{4}{3}}.$$

Then, for any $\varepsilon > 0$, there exist constants C_6, C_7 such that

$$\|k(f)\|_2^2\log\|k(f)\|_2 \le \varepsilon E[f^2\log_+^2|f|] + C_6 + C_7E[f^2]^3.$$

We can estimate $\|k(f)\|_2^2$ in a similar way. By combining this with (2.11), we can get the desired inequality. $\square$

Set $\Phi(x) = x\log_+^2 x$. Then

$$\phi(x) = \Phi'(x) = \log_+^2 x + 2\log_+ x.$$

Further

$$\phi(e^{\sqrt{x+1}-1}) = x,$$

which yields that the inverse function ψ of ϕ is given by

$$\psi(x) = e^{\sqrt{x+1}-1}.$$

From this, we can easily see that $\psi(x) \le e^{\sqrt{x}}$. Let Ψ be an integral of ψ. Φ is called a complimentary function. Φ can be estimated as

$$\begin{aligned}
\Psi(x) = \int_0^x \psi(y)dy \le \int_0^x e^{\sqrt{y}}dy &= \int_0^x 2\sqrt{y}(e^{\sqrt{y}})'dy\\
&= 2\sqrt{x}e^{\sqrt{x}} - \int_0^x \frac{1}{\sqrt{y}}e^{\sqrt{y}}dy \le 2\sqrt{x}e^{\sqrt{x}}.
\end{aligned}$$

Thus, by the Hausdorff-Young inequality, we have

$$xy \le \Phi(x) + \Psi(y) \le x\log_+^2 x + 2\sqrt{y}e^{\sqrt{y}} \tag{2.12}$$

Now we are ready to give a sufficient condition for the inequality (2.3) by means of the logarithmic Sobolev inequality.

Theorem 2.3. *Assume the same conditions in Theorem 2.2. Suppose that a non-negative function v satisfies*

$$e^v \in L^{2\alpha+} = \bigcup_{p>2\alpha} L^p. \tag{2.13}$$

Then, there exist positive constant $a < 1$ and b such that

$$\|vf\|_2 \le a\|Lf\|_2 + b\|f\|_2. \tag{2.14}$$

Proof. Take $\varepsilon > 0$ so that $e^v \in L^{2(1+2\varepsilon)\alpha}$. From the inequality (2.12), we have

$$\begin{aligned} 4(1+\varepsilon)^2\alpha^2\|vf\|_2^2 &= E[4(1+\varepsilon)^2\alpha^2 v^2 f^2] \\ &\le E[f^2 \log_+^2 f^2 + 4(1+\varepsilon)\alpha v e^{2(1+\varepsilon)\alpha v}] \\ &\le 4E[f^2 \log_+^2 f] + 4(1+\varepsilon)\alpha E[v e^{2(1+\varepsilon)\alpha v}]. \end{aligned}$$

Now the inequality (2.7) leads to

$$4(1+\varepsilon)^2\alpha^2\|vf\|_2^2 \le 4\alpha^2(1+\varepsilon)\|Lf\|_2^2 + 4K_1 + 4K_2\|f\|_2^6 + 4(1+\varepsilon)\alpha E[v e^{2(1+\varepsilon)\alpha v}].$$

By taking $f/\|f\|_2$ instead of f, we have

$$4(1+\varepsilon)^2\alpha^2\frac{\|vf\|_2^2}{\|f\|_2^2} \le 4\alpha^2(1+\varepsilon)\frac{\|Lf\|_2^2}{\|f\|_2^2} + 4K_1 + 4K_2 + 4(1+\varepsilon)\alpha E[v e^{2(1+\varepsilon)\alpha v}].$$

Thus

$$\|vf\|_2^2 \le \frac{1}{1+\varepsilon}\|Lf\|_2^2 + \left\{\frac{K_1+K_2}{(1+\varepsilon)^2\alpha^2} + \frac{1}{(1+\varepsilon)\alpha}E[v e^{2(1+\varepsilon)\alpha v}]\right\}\|f\|_2^2.$$

Taking the sqaure and using the inequality $\sqrt{x+y} \le \sqrt{x} + \sqrt{y}$, we have

$$\|vf\|_2 \le \frac{1}{\sqrt{1+\varepsilon}}\|Lf\|_2 + \left\{\frac{K_1+K_2}{(1+\varepsilon)^2\alpha^2} + \frac{1}{(1+\varepsilon)\alpha}E[v e^{2(1+\varepsilon)\alpha v}]\right\}^{1/2}\|f\|_2,$$

which is the desired result. □

On an abstract Wiener space, it is known that the inequality (2.6) holds for $\alpha = 1$, $\beta = 0$ (Gross' inequality):

$$\int_B |f|^2 \log(|f|/\|f\|_2)\, d\mu \le \int_B |\nabla f|^2\, d\mu. \tag{2.15}$$

Now, using the previous result, we have

Theorem 2.4. *For a Schrödinger operator $L - V$ on an abstract Wiener space (B, H, μ), assume that V_+, $e^{V_-} \in L^{2+}$ where $V_+ := \max\{V, 0\}$, $V_- := \max\{-V, 0\}$. Then $L - V$ is essentially self-adjoint on $\mathcal{F}C_0^\infty$.*

3. The domain of a Schrödinger operator

In this section, we consider an issue of the domain of a Schrödinger operator of the form $\mathfrak{A} = L - V + W$. In §2, we have decomposed the potential into positive part and negative part. Here we decompose the potential as follows:

(A.1) $V \ge 1$ and $V \in L^{2+}$.
(A.2) W is non-positive and there exists a constant $0 < \alpha < 1$ such that $e^W \in L^{2/\alpha}$.

Though the way of decomposing the potential is different from the one in previous section, it is not hard to see that we can apply the result of §2 and so we have that $\mathfrak{A} = L - V + W$ is essentially self-adjoint on $\mathcal{F}C_0^\infty$. We fix the constant α in (ii) throughout this section. We only remark that α can be chosen as close as 1.

To determine the domain, we use the intertwining property of operators; to be precise, we make use of an operator A satisfying

$$\sqrt{V}\mathfrak{A} = A\sqrt{V}. \tag{3.1}$$

Let us give a definition of an operator A. Assuming the differentiability, we define a vector field b by

$$b = \frac{\nabla V}{2V} = \frac{1}{2}\nabla \log V. \tag{3.2}$$

We further define a bilinear form as

$$\mathcal{E}_A(f,g) = (\nabla f, \nabla g) + (b \cdot \nabla f, g) - (f, b \cdot \nabla g) + ((V - W - |b|^2)f, g). \tag{3.3}$$

Note that this bilinear form is not symmetric. By the following formal computation

$$(f, b \cdot \nabla g) = (\nabla^*(fb), g) = (f\nabla^* b - b \cdot \nabla f, g),$$

an associated generator is given by

$$A = L - 2b \cdot \nabla + (\nabla^* b - V + W + |b|^2). \tag{3.4}$$

We remark that this is not a rigorous expression because we do not assume the differentiability of b and so $\nabla^* b$ is not well-defined. (3.4) is merely a formal expression. A is defined through the bilinear form (3.3).

We now give an sufficient condition so that $\mathcal{E}_A$ defines a closed bilinear form whose symmetric part is bounded from below. The problem is what kind of regularity of b should be imposed. We consider the symmetric part and skew-symmetric part separately. The symmetric part is given by

$$\hat{\mathcal{E}}_A(f,g) = (\nabla f, \nabla g) + ((V - W - |b|^2)f, g)$$

and the skew-symmetric part is

$$\check{\mathcal{E}}_A(f,g) = (b \cdot \nabla f, g) - (f, b \cdot \nabla g).$$

We denote a bilinear form added by λ times the inner product of L^2 by

$$\hat{\mathcal{E}}_{A-\lambda}(f,g) = (\nabla f, \nabla g) + ((V - W - |b|^2)f, g) + \lambda(f,g).$$

We first consider the symmetric part $(\nabla f, \nabla g) + ((V - W - |b|^2)f, g)$. We use a general theory of perturbation: the KLMN theorem for bilinear form (see, e.g., Reed-Simon [4, Theorem X.17]). We regard $W + |b|^2$ as a perturbation of a bilinear form associated to $L - V$:

$$\mathcal{E}_{L-V}(f,g) = (\nabla f, \nabla g) + (Vf, g).$$

The domain of this bilinear form is $\mathrm{Dom}(\nabla) \cap \mathrm{Dom}(\sqrt{V})$. It is well-known that this form is a Dirichlet form.

Though there are many ways to give sufficient conditions, we restrict ourselves to typical ones. One of them is

$$e^{W+|b|^2} \in L^{2/\alpha}. \tag{3.5}$$

The other is that there exists a constant $C > 0$ such that

$$|b|^2 \le \alpha V + C. \tag{3.6}$$

Then we have the following

Proposition 3.1. *In addition to (A.1), (A.2) we assume that either* (3.5) *or* (3.6) *is fulfilled. Then there exists a constant* β *such that*

$$((W + |b|^2)f, f) \le \alpha \mathcal{E}_{L-V}(f, f) + \beta(f, f) \tag{3.7}$$

and hence

$$(1-\alpha)\mathcal{E}_{L-V}(f, f) \le \mathcal{E}_A(f, f) + \beta(f, f) \le (1+\alpha)\mathcal{E}_{L-V}(f, f) + \beta(f, f). \tag{3.8}$$

Therefore $\hat{\mathcal{E}}_A$ *with the domain* $\mathrm{Dom}(\mathcal{E}_{L-V}) = \mathrm{Dom}(\nabla) \cap \mathrm{Dom}(\sqrt{V})$ *is a closed bilinear form that is bounded from below.*

Proof. First we assume (3.5). By the logarithmic Sobolev inequality (2.15) and the Housdorff-Young $xy \le x \log x - x + e^y$, we have

$$\begin{aligned}
(2/\alpha)((W + |b|^2)f, f) &= (2/\alpha)E[(W + |b|^2)f^2] \\
&\le E[f^2 \log f^2] - E[f^2] + E[e^{2(W+|b|^2)/\alpha}] \\
&\le 2E[|\nabla f|^2] + 2\|f\|_2^2 \log \|f\|_2^2 - \|f\|_2^2 + E[e^{2(W+|b|^2)/\alpha}].
\end{aligned}$$

Multiplying the both hand by $\alpha/2$, we have

$$((W + |b|^2)f, f) \le \alpha E[|\nabla f|^2] + \alpha\|f\|_2^2 \log \|f\|_2^2 - \frac{\alpha}{2}\|f\|_2^2 + \frac{\alpha}{2}E[e^{2(W+|b|^2)/\alpha}].$$

Take $f/\|f\|_2$ instead of f. Then we get

$$\begin{aligned}
((W + |b|^2)f, f) &\le \alpha E[|\nabla f|^2] + \frac{\alpha}{2}E[e^{2(W+|b|^2)/\alpha} - 1]\|f\|_2^2 \\
&\le \alpha \mathcal{E}_{L-V}(f, f) + \frac{\alpha}{2}E[e^{2(W+|b|^2)/\alpha} - 1]\|f\|_2^2.
\end{aligned}$$

Thus we have obtained (3.7) and the rest is easy form the KLMN theorem.

Second we assume (3.6). By the logarithmic Sobolev inequality and the assumption (A.2), we have

$$(Wf, f) \le \alpha E[|\nabla f|^2] + \frac{\alpha}{2}E[e^{2W/\alpha} - 1]\|f\|_2^2.$$

Further, by (3.6),

$$(|b|^2 f, f) \le \alpha E[V|f|^2] + C\|f\|_2^2.$$

By combining them, we can deduce

$$((W + |b|^2)f, f) \le \alpha \mathcal{E}_{L-V}(f, f) + \Big(\frac{\alpha}{2}E[e^{2W/\alpha} - 1] + C\Big)\|f\|_2^2.$$

The rest is the same as before. □

We can give a sufficient condition which is a mixture of (3.5) and (3.6), but we do not go into details.

Proposition 3.2. $\mathcal{F}C_0^\infty$ *is dense in* $\mathrm{Dom}(\hat{\mathcal{E}}_A)$.

Proof. By Proposition 3.1, $\mathrm{Dom}(\hat{\mathcal{E}}_A)$ is the same as the domain of the bilinear form associated with $L-V$. Recall that $L-V$ is essentially self-adjoint in $\mathcal{F}C_0^\infty$, i.e., $\mathcal{F}C_0^\infty$ is dense in $\mathrm{Dom}(L-V)$. Hence $\mathcal{F}C_0^\infty$ is dense in the domain of the bilinear form. This completes the proof. □

Let us proceed to the estimate of skew-symmetric part. Assume the inequality (3.7). We impose one of the following conditions on b:

$$e^{|b|^2} \in L^{0+} \tag{3.9}$$

or there exists a constant $C > 0$ such that

$$|b|^2 \leq CV. \tag{3.10}$$

Proposition 3.3. *In addition to* (3.7)*, we assume either* (3.9) *or* (3.10)*. Then, for sufficiently large λ, there exists a constant $K > 0$ such that*

$$|\check{\mathcal{E}}_A(f,g)| \leq K\hat{\mathcal{E}}_{A-\lambda}(f,f)^{1/2}\hat{\mathcal{E}}_{A-\lambda}(g,g)^{1/2}. \tag{3.11}$$

Therefore $\mathcal{E}_A$ satisfies the sector condition.

Proof. We first consider the case that (3.9) is satisfied. We use the logarithmic Sobolev inequality. We only need to estimate $(b\cdot\nabla f, g)$. Take $\varepsilon > 0$ so that $e^{\varepsilon|b|^2} \in L^1$. Then, by the Schwarz inequality,

$$|(b\cdot\nabla f, g)| \leq E[\frac{1}{\varepsilon}|\nabla f|^2]^{1/2}E[\varepsilon|b|^2g^2]^{1/2}.$$

Further, by using the Housdorff-Young $xy \leq x\log x - x + e^y$, we have

$$\begin{aligned}E[\varepsilon|b|^2g^2] &\leq E[g^2\log g^2] - E[g^2] + E[e^{\varepsilon|b|^2}]\\ &\leq 2E[|\nabla g|^2] + \|g\|_2^2\log\|g\|_2^2 - \|g\|_2^2 + E[e^{\varepsilon|b|^2}].\end{aligned}$$

Now replacing g with $g/\|g\|_2$, we get

$$E[\varepsilon|b|^2g^2] \leq 2E[|\nabla g|^2] + E[e^{\varepsilon|b|^2}-1]\|g\|_2^2.$$

Hence

$$|(b\cdot\nabla f, g)| \leq \frac{1}{\sqrt{\varepsilon}}E[|\nabla f|^2]^{1/2}\{2E[|\nabla g|^2] + E[e^{\varepsilon|b|^2}-1]\|g\|_2^2\}^{1/2}.$$

The right hand side can be estimated the norm in $\hat{\mathcal{E}}_A$ by (3.8) and so can obtain the desired result.

We can also prove it when (3.10) is fulfilled. In fact, it is enough to notice that in the proof above $E[|b|^2g^2]$ can be dominated by $E[CVg^2]$. □

In the sequel, we always assume the conditions (A.1), (A.2). Further we assume either (3.5) or (3.6) so that $\mathcal{E}_A$ is well-defined as a closed bilinear form. Establishing the $\mathcal{E}_A$, we proceed to an issue of intertwining property. Our next task is to show the following intertwining property:

$$\sqrt{V}\mathfrak{A} = A\sqrt{V}.$$

But A in (3.4) is merely a formal expression, we prove this in the following form:

$$\mathcal{E}_{\mathfrak{A}}(f, \sqrt{V}g) = \mathcal{E}_A(\sqrt{V}f, g). \tag{3.12}$$

Proposition 3.4. (3.12) *holds for* $f,g \in \mathcal{F}C_0^\infty$. *Moreover, we have, for* $f \in \mathrm{Dom}(\mathfrak{A})$, $g \in \mathrm{Dom}(A^*)$,

$$(\mathfrak{A}f, \sqrt{V}g) = (\sqrt{V}f, A^*g). \tag{3.13}$$

Proof. We take any $f,g \in \mathcal{F}C_0^\infty$. Then

$$[\nabla, \sqrt{V}]f = \nabla(\sqrt{V}f) - \sqrt{V}\nabla f = (\nabla\sqrt{V})f = \frac{\nabla V}{2\sqrt{V}}f = \sqrt{V}fb,$$

which means

$$[\nabla, \sqrt{V}]f = \sqrt{V}fb. \tag{3.14}$$

Using this, we will investigate the commutation relation between L and $\sqrt{V}$.

$$\begin{aligned}
&(\nabla f, \nabla(\sqrt{V}g)) \\
&\quad = (\nabla f, [\nabla, \sqrt{V}]g) + (\nabla f, \sqrt{V}\nabla g) \\
&\quad = (\nabla f, \sqrt{V}gb) + (\sqrt{V}\nabla f, \nabla g) \\
&\quad = (\sqrt{V}\nabla f, gb) + (\sqrt{V}\nabla f, \nabla g) \\
&\quad = ([\sqrt{V}, \nabla]f, gb) + (\nabla(\sqrt{V}f), gb) + ([\sqrt{V}, \nabla]f, \nabla g) + (\nabla(\sqrt{V}f), \nabla g) \\
&\quad = -(\sqrt{V}fb, gb) + (b \cdot \nabla(\sqrt{V}f), g) - (\sqrt{V}fb, \nabla g) + (\nabla(\sqrt{V}f), \nabla g) \\
&\quad = -(|b|^2\sqrt{V}f, g) + (b \cdot \nabla(\sqrt{V}f), g) - (\sqrt{V}f, b \cdot \nabla g) + (\nabla(\sqrt{V}f), \nabla g).
\end{aligned}$$

It is also clear that $\sqrt{V}$ and $V - W$ are commutative. Let $\mathcal{E}_{\mathfrak{A}}$ be a bilinear form associated with $\mathfrak{A}$:

$$\mathcal{E}_{\mathfrak{A}}(f, g) = (\nabla f, \nabla g) + ((V - W)f, g). \tag{3.15}$$

Then the commutation relation between $\sqrt{V}$ and $\mathfrak{A}$ is obtained as

$$\begin{aligned}
\mathcal{E}_{\mathfrak{A}}(f, \sqrt{V}g) &= (\nabla(\sqrt{V}f), \nabla g) + (b \cdot \nabla(\sqrt{V}f), g) \\
&\qquad - (\sqrt{V}f, b \cdot \nabla g) - (|b|^2\sqrt{V}f, g) + ((V - W)\sqrt{V}f, g) \\
&= \mathcal{E}_A(\sqrt{V}f, g).
\end{aligned}$$

So far, f and g are assumed to be taken form $\mathcal{F}C_0^\infty$. We will show that, by taking limit, this holds for $f \in \mathrm{Dom}(\mathfrak{A})$, $g \in \mathrm{Dom}(A^*)$. To do this, we first fix $f \in \mathcal{F}C_0^\infty$. Then we have

$$-(\mathfrak{A}f, \sqrt{V}g) = \mathcal{E}_A(\sqrt{V}f, g) \tag{3.16}$$

for $g \in \mathcal{F}C_0^\infty$. Using Proposition 3.2 and taking limit, we can show that (3.16) holds for $g \in \mathrm{Dom}(\mathcal{E}_A)$. In particular, if $g \in \mathrm{Dom}(A^*)$, then

$$(\mathfrak{A}f, \sqrt{V}g) = (\sqrt{V}f, A^*g) \quad \forall f \in \mathcal{F}C_0^\infty.$$

Now, by noting the essential self-adjointness, we can see that this identity holds for all $f \in \mathrm{Dom}(\mathfrak{A})$. This completes the proof. □

We are ready to determine the domain of the Schrödinger operator $\mathfrak{A}$. The method here is to use the intertwining property of operators, which was developed in [8, 9]. The main result in this section is as follows:

Theorem 3.5. *We assume the same assumptions as before. Then we have*

$$\mathrm{Dom}(\mathfrak{A}) = \mathrm{Dom}(L) \cap \mathrm{Dom}(V).$$

Moreover, for sufficiently large λ, there exist positive constants K_1, K_2 such that

$$K_1\|(\mathfrak{A}-\lambda)f\|_2 \le \|Lf\|_2 + \|Vf\|_2 \le K_2\|(\mathfrak{A}-\lambda)f\|_2. \tag{3.17}$$

Proof. We use the intertwining property among $(\mathfrak{A}, L^2(B))$, $(A, L^2(B))$ and $\sqrt{V}$. For $f \in \mathrm{Dom}(\mathfrak{A})$, we define a linear functional on $\mathrm{Dom}(\mathcal{E}_A)$ as

$$\Phi(g) = ((\lambda-\mathfrak{A})f, \sqrt{V}g).$$

Then, by Proposition 3.1, we have

$$|\Phi(g)| \le \|(\lambda-\mathfrak{A})f\|_2\|\sqrt{V}g\|_2 \le C\|(\lambda-\mathfrak{A})f\|_2\mathcal{E}_{A-\lambda}(g,g)^{1/2}.$$

Now, by the Lax-Milgram theorem, there exist $k \in \mathrm{Dom}(\mathcal{E}_A)$ such that

$$\Phi(g) = \mathcal{E}_{A-\lambda}(k, g).$$

We also note that the norm of k is estimated as

$$\mathcal{E}_{A-\lambda}(k,k)^{1/2} \le C\|(\lambda-\mathfrak{A})f\|_2. \tag{3.18}$$

Take any $g \in \mathrm{Dom}(A)$. By using the intertwining property, we have

$$(\sqrt{V}f, (\lambda-A^*)g) = ((\lambda-\mathfrak{A})f, \sqrt{V}g) = \Phi(g) = \mathcal{E}_{A-\lambda}(k,g) = (k, (\lambda-A^*)g).$$

Since g can run through $\mathrm{Dom}(A)$, we can get $\sqrt{V}f = k$. Now noting Proposition 3.1 and (3.18), we can deduce that

$$\|Vf\|_2 = \|\sqrt{V}k\|_2 \le \mathcal{E}_{L-V}(k,k)^{1/2} \le C_1\mathcal{E}_{A-\lambda}(k,k)^{1/2} \le C_2\|(\lambda-\mathfrak{A})f\|_2$$

We also note that, by Theorem 2.3, for $\alpha < \alpha' < 1$,

$$\|Wf\|_2 \le \alpha'\|Lf\|_2 + \beta\|f\|_2. \tag{3.19}$$

Hence, if we take $f \in \mathcal{F}C_0^\infty$, we have

$$\begin{aligned}\|Lf\|_2 &\le \|(L-V+W-\lambda)f\|_2 + \|Vf\|_2 + \|Wf\|_2 + \lambda\|f\|_2\\ &\le \|(\mathfrak{A}-\lambda)f\|_2 + C_2\|(\lambda-\mathfrak{A})f\|_2 + \alpha'\|Lf\|_2 + \beta\|f\|_2 + \lambda\|f\|_2,\end{aligned}$$

which leads to

$$(1-\alpha')\|Lf\|_2 \le (1+C_2)\|(\lambda-\mathfrak{A})f\|_2 + (\beta+\lambda)\|f\|_2.$$

This, combined with the essential self-adjointness of $\mathfrak{A}$, yields that $\mathrm{Dom}(\mathfrak{A}) \subset \mathrm{Dom}(L)$. Thus we have obtained that if $f \in \mathrm{Dom}(\mathfrak{A})$, then $f \in \mathrm{Dom}(L) \cap \mathrm{Dom}(V)$.

The reversed inclusion $\mathrm{Dom}(L) \cap \mathrm{Dom}(V) \subseteq \mathrm{Dom}(\mathfrak{A})$ can be seen as

$$\begin{aligned}\|(L-V+W-\lambda)f\|_2 &\le \|Lf\|_2 + \|Vf\|_2 + \|Wf\|_2 + \lambda\|f\|_2\\ &\le \|Lf\|_2 + \|Vf\|_2 + \alpha'\|Lf\|_2 + \beta\|f\|_2 + \lambda\|f\|_2,\end{aligned}$$

which completes the proof. □

4. Spectral gap of a Schrödinger oerator

In this section, we investigate the spectrum of a Schöodinger $\mathfrak{A} = L - V$ on an abstract Wiener space (B, H, μ). We first consider the case that V is bounded. In this case, $L - V$ is a bounded perturbation of L and so it is essentially self-adjoint on $\mathcal{F}C_0^\infty$. We denote the spectrum of $\mathfrak{A}$ by $\sigma(\mathfrak{A})$. We have the following

Theorem 4.1. $l = \sup \sigma(\mathfrak{A})$ *is a point spectrum of multiplicity one and the associated eigenfunction can be chosen to be positive. Moreover, the spectrum is discrete on* $(l-1, l]$*, i.e., it consists of point spectrums of finite multiplicity.*

Proof. The spectrum of L is well-known: $\sigma(L) = \{0, -1, -2, \dots\}$ and the maximal eigenvalue 0 is a point spectrum of multiplicity one and the associated eigenfunction is positive. Therefore, if (B, H, μ) is of finite dimension, then the spectrum of L is discrete and the resolvent operators are compact. Moreover $\mathfrak{A} = L - V$ has a compact resolvent if V is bounded and so the spectrum of $\mathfrak{A}$ is discrete.

To deal with infinite dimensional case, we use an approximation method. Let $\{\varphi_i\}_{i=1}^\infty \subseteq B^*$ be a c.o.n.s of H^*. We set $\mathcal{F}_n = \sigma(\varphi_1, \varphi_2, \dots, \varphi_n)$ and define $V_n = E[V|\mathcal{F}_n]$. Then V_n is uniformly bounded and converges to V a.s. as $n \to \infty$. Let $\{h_i\}_{i=1}^\infty$ be a dual basis of $\{\varphi_i\}$, i.e., $\{h_i\}_{i=1}^\infty$ is a c.o.n.s. of H and satisfies $\langle h_i, \varphi_j\rangle = \delta_{ij}$ where δ_{ij} is the Kronecker delta. We set $H_n = \mathrm{span}\{h_1, h_2, \dots, h_n\}$ and $B_n = \{x \in B; \langle x, \varphi_i\rangle = 0, i = 1, 2, \dots, n\}$. Then B can be decomposed as a direct sum $B = H_n \oplus B_n$. According to this decomposition, $L - V_n$ can also be decomposed as follows. On H_n, it has of the form $L_n - V_n$, and on B_n it is just the Ornstein-Uhlenbeck operator on B_n. This reveals that the spectrum of $L - V_n$ is discrete on $(\lambda(V_n) - 1, \lambda(V_n)]$ where $\lambda(V_n) = \sup \sigma(L - V_n)$.

It remains to show that $L - V_n$ converges to $L - V$ as $n \to \infty$ in the norm resolvent sense.

We denote the resolvents of $L - V_n$ and $L - V$ by $G^{(n)}$ and G, respectively, namely, let $G^{(n)} = (\lambda - L + V_n)^{-1}$ and $G = (\lambda - L + V)^{-1}$. We take λ to be large enough. Then it holds that

$$G - G^{(n)} = G^{(n)}(V - V_n)G.$$

Since λ is large, we may assume that $\|G^{(n)}\|_{\mathrm{op}} \le 1$ and so it suffices to show that $\|(V - V_n)G\|_{\mathrm{op}} \to 0$ as $n \to \infty$. By the Hausdorff-Young $xy \le x \log x - x + e^y$ and the logarithmic Sobolev inequality, we have

$$\begin{aligned}
&\|(V - V_n)Gf\|_2^2 \\
&\quad = E[(V - V_n)^2(Gf)^2] \\
&\quad = \frac{1}{N} E[N(V - V_n)^2(Gf)^2] \\
&\quad \le \frac{1}{N} E[(Gf)^2 \log(Gf)^2 - (Gf)^2 + e^{N(V-V_n)^2}] \\
&\quad \le \frac{1}{N}\{2E[|\nabla Gf|^2] + \|Gf\|_2^2 \log \|Gf\|_2^2 - \|Gf\|_2^2 + E[e^{N(V-V_n)^2}]\}.
\end{aligned}$$

Now replacing f with $f/\|Gf\|_2$, we get

$$\begin{aligned}
\|(V-V_n)Gf\|_2^2 &\le \frac{1}{N}\{2E[|\nabla Gf|^2]+E[e^{N(V-V_n)^2}-1]\|Gf\|_2^2\} \\
&\le \frac{1}{N}\{-2E[(LGf)Gf]+E[e^{N(V-V_n)^2}-1]\|f\|_2^2\} \\
&\le \frac{1}{N}\{2E[((\lambda-L+V)Gf)Gf]-\lambda E[(Gf)^2]-E[V(Gf)^2] \\
&\qquad +E[e^{N(V-V_n)^2}-1]\|f\|_2^2\} \\
&\le \frac{1}{N}\{2E[fGf]-\lambda E[(Gf)^2]+E[|V|(Gf)^2] \\
&\qquad +E[e^{N(V-V_n)^2}-1]\|f\|_2^2\} \\
&\le \frac{1}{N}\{E[f^2]+E[(Gf)^2]+E[|V|(Gf)^2] \\
&\qquad +E[e^{N(V-V_n)^2}-1]\|f\|_2^2\} \\
&\le \frac{1}{N}\{(2+\|V\|_\infty)\|f\|_2^2+E[e^{N(V-V_n)^2}-1]\|f\|_2^2\}.
\end{aligned}$$

Hence

$$\|(V-V_n)G\|_{\mathrm{op}}^2 \le \frac{1}{N}\{2+\|V\|_\infty+E[e^{N(V-V_n)^2}-1]\}.$$

Since $V-V_n$ is uniformly bounded and converges to 0 a.s., we can use the Lebesgue dominated convergence theorem and so we obtain

$$\limsup_{n\to\infty}\|(V-V_n)G\|_{\mathrm{op}}^2 \le \frac{2+\|V\|_\infty}{N}.$$

Since N is arbitrary, it follows that

$$\lim_{n\to\infty}\|(V-V_n)G\|_{\mathrm{op}}=0.$$

Thus we have shown that $G^{(n)}$ converges to G in norm sense, which we wanted. Using the Feynman-Kac formula, we can show that the multiplicity of the maximal eigenvalue is simple and the associated eigenfunction can be taken to be positive by a standard argument. This completes the proof. □

Now we proceed to the general case. We consider a operator of the form $\mathfrak{A}=L-V+W$. We assume the assumptions of the previous section to use Theorem 3.5.

Theorem 4.2. *Assume that V, W satisfy the conditions (A.1), (A.2) in §3. We also assume that either* (3.5) *or* (3.6) *is fulfilled. Then the spectrum is discrete on $(l-1,l]$, i.e., it consists of point spectrums of finite multiplicity.*

Proof. Define functions V_n, W_n that approximate V, W as follows. Let $\psi_n\colon\mathbb{R}\mapsto\mathbb{R}$ be a function satisfying

$$\psi_n(x)=\begin{cases} n+1, & x\ge n+2 \\ x, & |x|\le n \\ -n-1, & x\le -n-2 \end{cases}$$

and that $0 \le \psi_n' \le 1$. We set

$$V_n = \psi_n(V), \quad W_n = \psi_n(W).$$

As in the case of bounded potential, we set $G^{(n)} = (\lambda - L + V_n - W_n)^{-1}$, $G = (\lambda - L + V - W)^{-1}$ and note that

$$G - G^{(n)} = G(V - V_n - W + W_n)G^{(n)}.$$

We need show the norm convergence of $G - G^{(n)}$ to 0 as $n \to \infty$. By taking λ to be large enough, we may assume that $\|G^{(n)}\|_{\mathrm{op}} \le 1$, $\|G\|_{\mathrm{op}} \le 1$. Take any f, $g \in L^2$. Then

$$\begin{aligned}((G - G^{(n)})f, g) &= (G(V - V_n - W + W_n)G^{(n)}f, g)\\ &= ((V - V_n - W + W_n)G^{(n)}f, Gg)\\ &= \left(\frac{V - V_n - W + W_n}{V + W}G^{(n)}f, (V + W)Gg\right).\end{aligned}$$

We first estimate the term $(V + W)Gg$. By using (3.10) and (3.19), we have

$$\begin{aligned}\|(V + W)Gg\|_2 &\le \|VGg\|_2 + \|WGg\|_2\\ &\le K_2\|(\mathfrak{A} - \lambda)Gg\|_2 + \alpha'\|LGg\|_2 + \beta\|Gg\|_2\\ &\le K_2\|g\|_2 + K_2\alpha'\|(\mathfrak{A} - \lambda)Gg\|_2 + \beta\|Gg\|_2\\ &\le K_2(1 + \alpha')\|g\|_2 + \beta\|g\|_2.\end{aligned}$$

Further, by the Hausdorff-Young inequality, $\frac{V - V_n - W + W_n}{V + W}G^{(n)}f$ can be estimated as

$$\begin{aligned}&\left\|\frac{V - V_n - W + W_n}{V + W}G^{(n)}f\right\|_2^2\\ &= E\left[\frac{(V - V_n - W + W_n)^2}{(V + W)^2}(G^{(n)}f)^2\right]\\ &\le \frac{1}{N}E[(G^{(n)}f)^2\log(G^{(n)}f)^2 - (G^{(n)}f)^2 + e^{N(V - V_n - W + W_n)^2/(V + W)^2}]\\ &\le \frac{1}{N}\{E[|\nabla G^{(n)}f|^2] + \|G^{(n)}f\|^2\log\|G^{(n)}f\|^2\\ &\qquad - \|G^{(n)}f\|_2^2 + E[e^{N(V - V_n - W + W_n)^2/(V + W)^2}]\}.\end{aligned}$$

Replacing f with $f/\|G^{(n)}f\|_2$, we have

$$\begin{aligned}&\left\|\frac{V - V_n - W + W_n}{V + W}G^{(n)}f\right\|_2^2\\ &\le \frac{1}{N}\{2E[|\nabla G^{(n)}f|^2] + E[e^{N(V - V_n - W + W_n)^2/(V + W)^2} - 1]\|G^{(n)}f\|_2^2\}\\ &\le \frac{1}{N}\{2E[|\nabla G^{(n)}f|^2] + E[e^{N(V - V_n - W + W_n)^2/(V + W)^2} - 1]\|f\|_2^2\}.\end{aligned}$$

Further, by using (3.17) again, we get

$$\begin{aligned} E[|\nabla G^{(n)}f|^2] &= -(LG^{(n)}f, G^{(n)}f) \\ &\le \|LG^{(n)}f\|_2\|G^{(n)}f\|_2 \\ &\le K_2\|(L - V_n + W_n - \lambda)G^{(n)}f\|_2\|f\|_2 \\ &\le K_2\|f\|_2^2. \end{aligned}$$

Here, we must be careful about that (3.10) holds uniformly in n. To see this, note that constants in (3.10) depend on the integrability of e^{W_n} and the estimate of $2b_n = \nabla V_n/V_n$. As for e^{W_n}, their L^p norm can be easily estimated uniformly since $W_n \le W$. Let us consider b_n. Since $b_n = 0$ when $V \ge n+2$, and $b_n = b$ when $V \le n$, it is clear that they can be estimated uniformly. When $n \le V \le n+2$, note that

$$|b_n| = |\nabla V_n|/V_n = \phi_n'|\nabla V|/V_n \le (|\nabla V|/V)(V/V_n) \le (|\nabla V|/V)(n+2)/n.$$

Then the uniform estimate can be deduced easily.

Thus we have

$$\begin{aligned} |((G - G^{(n)})f, g)| &\le \{K_2(1+\alpha) + \beta\}\|g\|_2 \\ &\quad \times \frac{1}{N}\{2K_2 + E[e^{N(V-V_n-W+W_n)^2/(V+W)^2} - 1]\}\|f\|_2 \end{aligned}$$

and hence

$$\|G - G^{(n)}\|_{\mathrm{op}} \le \frac{1}{N}\{K_2(1+\alpha) + \beta\}\{2K_2 + E[e^{N(V-V_n-W+W_n)^2/(V+W)^2} - 1]\}.$$

Now we let $n \to \infty$ and, by the Lebesgue bounded convergence theorem, we obtain

$$\limsup_{n\to\infty} \|G - G^{(n)}\|_{\mathrm{op}} \le \frac{2}{N}\{K_2(1+\alpha) + \beta\}K_2.$$

Since N is arbitrary, we eventually get

$$\lim_{n\to\infty} \|G - G^{(n)}\|_{\mathrm{op}} = 0,$$

which is what we wanted. The rest is the same as the previous theorem. The proof is completed. □

References

1. Glimm, J. and Jaffe, A.: A $\lambda\phi^4$ quantum field theory without cutoffs II, The field operators and the approximate vacuum; *Ann. Math.* **91** (1970) 362–401.
2. Guibourg, D.: Inégalités maximales pour l'opérateur de Schrödinger; *C. R. Acad. Sci. Paris, Série I Math.* **316** (1993) 249–252.
3. Kazumi, T. and Shigekawa, I.: Measures of finite (r,p)-energy and potentials on a separable metric space; in *Séminaire de Probabilités, XXVI*, J. Azéma et al. (eds.) Lecture Notes in Math. **1526** (1992) 415–444, Berlin-Heidelberg-New York, Springer-Verlag.
4. Reed, M. and Simon, B.: *Method of modern mathematical physics, II: Fourier analysis, self-adjointness.* Academic Press, San Diego, 1975.
5. Segal, I.: Notes towards the construction of nonlinear relativistic quantum fields. III. Properties of the C^*-dynamics for a certain class of interactions; *Bull. Amer. Math. Soc.* **75** (1969) 1390–1395.
6. Shen, Z.: L^p estimates for Schrödinger operators with certain potentials; *Ann. Inst. Fourier, Grenoble* **45** (1995) 513–546.

7. Shen, Z.: Estimates in L^p for magnetic Schrödinger operators; *Indiana Univ. Math. J.* **45** (1996) 817–841.
8. Shigekawa, I.: Semigroup domination on a Riemannian manifold with boundary; *Acta Applicandae Math.* **63** (2000) 385–410.
9. Shigekawa, I.: Defective intertwining property and generator domain; *J. Funct. Anal.* (to appear).
10. Simon, B.: Essential self-adjointness of Schrd̈inger operators with positive potentials; *Math. Ann.* **201** (1973) 211–220.
11. Simon, B. and Høegh-Krohn, R.: Hypercontractive semigroups and two-dimensional self-coupled Bose fields; *J. Funct. Anal.* **9** (1972) 121–180.

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