Vanishing theorem of the Hodge-Kodaira operator for differential forms on a convex domain of the Wiener space

Ichiro SHIGEKAWA *

Abstract

We discuss the vanishing theorem on a convex domain of the Wiener space. We show that there is no harmonic form satisfying the absolute boundary condition. Our method relies on an expression of the bilinear form associated with the Hodge-Kodaira operator.

1 Introduction

In this paper, we discuss the vanishing theorem on a domain of the Wiener space. Our object is the Hodge-Kodaira operator for differential forms. On a whole Wiener space, this problem is well-known (see [5]). In fact, the Hodge-Kodaira operator is nothing but the Ornstein-Uhlenbeck operator and all the spectrum is known and there is no harmonic p-forms for $p \geq 1$. But the Hodge-Kodaira operator on a domain has not been considered. In this paper, assuming the convexity of the domain, we prove that the spectrum of the Hodge-Kodaira operator for p-forms is contained in $(-\infty, -p]$. Here our domain has a boundary and so we have to specify the boundary condition. We take the absolute boundary condition and the relative boundary condition (the precise definition will be given later.)

The organization of the paper is as follows. We give an integration by parts formula in §2 . It is a kind of Gauss' formula. We also generalize it to differential forms.

In §3 , we discuss the Hodge-Kodaira operator for differential forms. Our domain having a boundary, we need to introduce boundary conditions. Two boundary conditions are classical in finite dimensional case. They are the

*This research was partially supported by the Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for Scientific Research (B), No. 11440045, 2001

absolute boundary condition (corresponding to the Dirichlet boundary condition) and the relative boundary condition (corresponding to the Neumann boundary condition, see, e.g., [7]). We show that the same boundary conditions can be defined in infinite dimensional case. We give an expression of the associated bilinear form. The second fundamental form naturally appears. Using this expression, we show the vanishing theorem on a convex domain. We also give an example with the relative boundary condition.

2 Integration by parts formula on a domain of the Wiener space

We consider a domain of an abstract Wiener space (B, H, μ) where B is a Banach space, H is a Hilbert space densely and continuously inbedded in B and μ is the Wiener measure with the Cameron-Martin space H. Suppose we are given a smooth non-degenerate Wiener functional F in the sense of Malliavin calculus. The domain is given as

$$M = \{x \in B; F(x) \leq 0\}. \tag{2.1}$$

M is nothing but a subset of B but we regard it as a smooth manifold with boundary. The boundary ∂M of M is naturally defined by

$$\partial M = \{x \in B; F(x) = 0\}. \tag{2.2}$$

Since F can be chosen to be quasi-continuous, we always take a quasi-continuous modification. ∂M depends on a choice of modification but it is unique up to quasi-sure equivalence. We set

$$\omega_N = -\frac{DF}{|DF|}. \tag{2.3}$$

Here D denotes the Malliavin derivative (H-derivative). ω_N is an H^*-valued function. Since H and H^* are isomorphic to each other by the Riesz theorem, we denote the isomorphism by $\sharp\colon H^* \to H$. Using this notation, we can define the inner normal vector field N on the boundary as

$$N = \omega_N^\sharp = -\frac{DF^\sharp}{|DF|}. \tag{2.4}$$

The surface measure σ on ∂M is given by

$$\sigma = |DF|\, \delta_0(F) \tag{2.5}$$

where $\delta_0(F)$ is a composite of the Dirac measure and F in the sense of Watanabe.

The Gauss' divergence formula is formulated as follows;

Proposition 2.1. *For smooth 1-form θ, it holds that*

$$\int_M D^*\theta \, d\mu = \int_{\partial M} \langle \theta, N \rangle \, d\sigma. \tag{2.6}$$

Here $\langle \ , \ \rangle$ denotes the natural coupling between H^* and H.

Proof. Take any $\phi \in C_0^\infty$ (the set of all C^∞ functions on $\mathbb{R}$ with compact support). It follows from the definition of $\delta_t(F)$ that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{d}{dt} E[\langle \theta, N \rangle |DF| \delta_t(F)] \phi(t) \, dt &= - \int_{-\infty}^{\infty} E[\langle \theta, N \rangle |DF| \delta_t(F)] \phi'(t) \, dt \\
&= -E[\langle \theta, N \rangle |DF| \phi'(F)] \\
&= E[(\theta, DF) \phi'(F)] \\
&= E[(\theta, D(\phi(F))] \\
&= E[D^*\theta \phi(F)] \\
&= \int_{-\infty}^{\infty} E[D^*\theta \delta_t(F)] \phi(t) \, dt.
\end{aligned}$$

Here we denote by E the integral with respect to μ. The identity above holds for any ϕ and hence we have

$$\frac{d}{dt} E[\langle \theta, N \rangle |DF| \, \delta_t(F)] = E[D^*\theta \, \delta_t(F)].$$

Integrating both hands from $-\infty$ to 0 with respect to t, we get

$$\begin{aligned}
E[\langle \theta, N \rangle |DF| \, \delta_0(F)] &= E[\langle \theta, N \rangle |DF| \, \delta_0(F)] - \lim_{t \to -\infty} E[\langle \theta, N \rangle |DF| \, \delta_t(F)] \\
&= \int_{-\infty}^{0} \frac{d}{dt} E[\langle \theta, N \rangle |DF| \, \delta_t(F)] dt \\
&= \int_{-\infty}^{0} E[D^*\theta \delta_t(F)] \, dt \\
&= E[D^*\theta; F \le 0],
\end{aligned}$$

which completes the proof. □

Set $\eta = f\theta$ for a scalar function f and a 1-form θ. Then it holds that

$$D^*(f\theta) = f D^*\theta - (Df, \theta).$$

Now the proposition above leads to

$$\int_M (fD^*\theta - (Df, \theta))\, d\mu = \int_{\partial M} f\langle \theta, N\rangle\, d\sigma.$$

Thus we have

$$\int_M (Df, \theta)\, d\mu = \int_M fD^*\theta d\mu - \int_{\partial M} f\langle \theta, N\rangle\, d\sigma. \tag{2.7}$$

Using this identity, we can have the integration by parts formula for tensor fields as follows. Set $H^{*\otimes p} := H^* \otimes \cdots \otimes H^*$ (p-fold). $H^{*\otimes p}$-valued function is called a tensor field of type $(0, p)$. If in addition it is alternate, it is called a differential form of order p or p-form for short. We need to introduce the covariant derivative ∇ for tensor fields, but it is nothing but the Malliavain derivative in our case: $\nabla u := Du$. For $(0, p)$-tensor u, the interior product i is defined by

$$(i(h)u)(h_1, \dots, h_{p-1}) = u(h, h_1, \dots, h_{p-1}), \quad h, h_1, \dots, h_{p-1} \in H.$$

Proposition 2.2. *Let u be a tensor of type $(0, p)$ and v be a tensor of type $(0, p+1)$. Then it holds that*

$$\int_M (\nabla u, v) d\mu = \int_M (u, \nabla^* v) d\mu - \int_{\partial M} (u, i(N)v) d\sigma. \tag{2.8}$$

Proof. It is enough to prove this in the case where

$$\begin{aligned} u &= f\omega_1 \otimes \cdots \otimes \omega_p, \\ v &= g\tilde{\omega}_0 \otimes \tilde{\omega}_1 \otimes \cdots \otimes \tilde{\omega}_p. \end{aligned}$$

Here f, g are smooth scalar functions and ω_i, $\tilde{\omega}_i$ are constant 1-forms (i.e., elements of H^*). We have

$$\begin{aligned} \int_M (\nabla u, v) d\mu &= \int_M (Df \otimes \omega_1 \otimes \cdots \otimes \omega_p, g\tilde{\omega}_0 \otimes \cdots \otimes \tilde{\omega}_p) d\mu \\ &= \int_M (Df, g\tilde{\omega}_0)(\omega_1 \otimes \cdots \otimes \omega_p, \tilde{\omega}_1 \otimes \cdots \otimes \tilde{\omega}_p) d\mu \\ &= \int_M fD^*(g\tilde{\omega}_0)(\omega_1 \otimes \cdots \otimes \omega_p, \tilde{\omega}_1 \otimes \cdots \otimes \tilde{\omega}_p) d\mu \\ &\quad - \int_{\partial M} f\langle N, g\tilde{\omega}_0\rangle(\omega_1 \otimes \cdots \otimes \omega_p, \tilde{\omega}_1 \otimes \cdots \otimes \tilde{\omega}_p) d\sigma \\ &= \int_M (f\omega_1 \otimes \cdots \otimes \omega_p, \nabla^*(g\tilde{\omega}_0 \otimes \cdots \otimes \tilde{\omega}_p) d\mu \end{aligned}$$

$$
\begin{aligned}
&- \int_{\partial M} (f\omega_1 \otimes \cdots \otimes \omega_p, i(N)(g\tilde{\omega}_0 \otimes \cdots \otimes \tilde{\omega}_p) d\sigma \\
&= \int_M (u, \nabla^* v) d\mu - \int_{\partial M} (u, i(N)v) d\sigma.
\end{aligned}
$$

This completes the proof. □

A similar formula holds for differential forms. We recall that the exterior differentiation is defined as follows: for a p-form θ,

$$d\theta(h_1, h_2, \dots, h_{p+1}) := \sum_{\tau} \operatorname{sgn} \tau \, \nabla\theta(h_{\tau(1)}, \dots, h_{\tau(p+1)})$$

where τ runs over all permutations of degree $p+1$. The dual operator d^* of d (with respect to the measure μ) coincides with ∇^*. Now we can easily get the following identity for forms.

Proposition 2.3. *Let θ be a p-form and η be a $p+1$-form. Then it holds that*

$$\int_M (d\theta, \eta) d\mu = \int_M (\theta, d^*\eta) d\mu - \int_{\partial M} (\theta, i(N)\eta) d\sigma. \tag{2.9}$$

3 Hodge-Kodaira operator on a domain of the Wiener space

In this section, we discuss the Hodge-Kodaira operator for differential forms. To do this, we use the following bilinear form:

$$\mathcal{E}^a_{(p)}(\theta, \eta) = \int_M (d\theta, d\eta)\, d\mu + \int_M (d^*\theta, d^*\eta)\, d\mu \tag{3.1}$$

with the domain

$$\operatorname{Dom}(\mathcal{E}^a_{(p)}) = \{\theta \in W_p^{\infty,\infty-} | i(N)\theta = 0 \text{ on } \partial M\}.$$

Here $W_p^{\infty,\infty-}$denotes the set of all smooth p-forms in the sense of Malliavin. Taking closure, we obtain a closed bilinear form which we also denote by $\mathcal{E}^a_{(p)}$. The associated self-adjoint operator is called the Hodge-Kodaira operator with the absolute boundary condition and will be denoted by $\Box^a_{(p)}$.

We introduce a different kind of boundary condition as follows: consider the bilinear form is given by

$$\mathcal{E}^r_{(p)}(\theta, \eta) = \int_M (d\theta, d\eta)\, d\mu + \int_M (d^*\theta, d^*\eta)\, d\mu \tag{3.2}$$

with the domain

$$\mathrm{Dom}(\mathcal{E}^r_{(p)}) = \{\theta \in W_p^{\infty,\infty-} | \theta \wedge \omega_N = 0 \text{ on } \partial M\}.$$

Here ω_N is a 1-form defined by (2.3) . We also denote its closure by $\mathcal{E}^r_{(p)}$ and the associated self-adjoint operator by $\Box^r_{(p)}$. In this case, the boundary condition is called the relative boundary condition.

We have to see the closability of bilinear forms. It follows from the following theorem.

Theorem 3.1. *Take any θ, $\eta \in \mathrm{Dom}(\mathcal{E}^a_{(p)})$ and suppose that $i(N)d\theta = 0$ on ∂M. Then it holds that*

$$\mathcal{E}^a_{(p)}(\theta, \eta) = \int_M ((dd^* + d^*d)\theta, \eta)\, d\mu. \tag{3.3}$$

Similarly, for θ, $\eta \in \mathrm{Dom}(\mathcal{E}^r_{(p)})$ with $d^*\theta \wedge \omega_N = 0$ on ∂M instead, it holds that

$$\mathcal{E}^r_{(p)}(\theta, \eta) = \int_M ((dd^* + d^*d)\theta, \eta)\, d\mu. \tag{3.4}$$

Proof. By virtue of Proposition 2.3 , we have

$$\begin{aligned}\int_M (d\theta, d\eta)\, d\mu + \int_M (d^*\theta, d^*\eta)\, d\mu = \int_M (d^*d\theta, \eta)\, d\mu - \int_{\partial M} (i(N)d\theta, \eta)\, d\sigma \\ + \int_M (dd^*\theta, \eta)\, d\mu + \int_{\partial M} (d^*\theta, i(N)\eta)\, d\sigma.\end{aligned}$$

In the case of absolute boundary condition, we can easily see that the boundary integrals above vanish and we get the desired results.

In the case of relative boundary condition, note that $(i(N)d\theta, \eta) = (d\theta, \omega_N \wedge \eta)$. We can see that the boundary integral vanishes as well. This completes the proof. □

In Theorem 3.1 , we have imposed the additional boundary condition. We show that such functions are rich enough. Let us first see the absolute boundary case. We take C^∞-function ϕ such that $\phi' \le 1$ and

$$\phi(t) = \begin{cases} -2, & t \le -3, \\ t, & -1 \le t \le 1, \\ 2, & t \ge 3. \end{cases}$$

and set $\phi_\varepsilon(t) = \varepsilon\phi(t/\varepsilon)$. p-form θ is assumed to satisfy $i(N)\theta = 0$ on ∂M. Set $\eta = d\theta(N, \cdot)$. Then, taking the same function ϕ_ε as before, we have

$$d\Big(\phi_\varepsilon(F)\frac{\eta}{|DF|}\Big) = \phi_\varepsilon'(F)dF \wedge \frac{\eta}{|DF|} + \phi_\varepsilon(F)d\Big(\frac{\eta}{|DF|}\Big)$$

$$= \phi_\varepsilon'(F)\omega_N \wedge \eta + \phi_\varepsilon(F) d\Big(\frac{\eta}{|DF|}\Big).$$

Hence

$$i(N)d\Big(\phi_\varepsilon(F)\frac{\eta}{|DF|}\Big) = \eta \quad \text{on } \partial M.$$

Thus, setting

$$\tilde{\theta}_\varepsilon = \theta - \phi_\varepsilon(F)\frac{\eta}{|DF|}, \tag{3.5}$$

$\tilde{\theta}_\varepsilon$ satisfies $i(N)\tilde{\theta}_\varepsilon = 0$ and $i(N)d\tilde{\theta}_\varepsilon = 0$ on ∂M. Moreover $\lim_{\varepsilon\to 0}\tilde{\theta}_\varepsilon = \theta$ and $\lim_{\varepsilon\to 0}\nabla\tilde{\theta}_\varepsilon = \nabla\theta$ in L^2.

For the relative boundary condition case, assume that θ satisfies $\omega_N \wedge \theta = 0$ on ∂M. Set $\eta = d^*\theta$. Then

$$\begin{aligned} d^*\Big(\phi_\varepsilon(F)\frac{\omega_N\wedge\eta}{|DF|}\Big) &= -\phi_\varepsilon'(F) i(DF^\sharp)\frac{\omega_N\wedge\eta}{|DF|} + \phi_\varepsilon(F) d^*\Big(\frac{\omega_N\wedge\eta}{|DF|}\Big) \\ &= \phi_\varepsilon'(F)\omega_N\wedge(i(N)\eta) - \phi_\varepsilon'(F)\eta + \phi_\varepsilon(F)d^*\Big(\frac{\omega_N\wedge\eta}{|DF|}\Big). \end{aligned}$$

Hence

$$\omega_N \wedge d^*\Big(\phi_\varepsilon(F)\frac{\omega_N\wedge\eta}{|DF|}\Big) = -\omega_N\wedge\eta \quad \text{on } \partial M.$$

Thus, setting

$$\tilde{\theta}_\varepsilon = \theta + \phi_\varepsilon(F)\frac{\omega_N\wedge\eta}{|DF|}, \tag{3.6}$$

$\tilde{\theta}_\varepsilon$ satisfies $\omega_N\wedge\tilde{\theta}_\varepsilon = 0$ and $\omega_N\wedge d^*\tilde{\theta}_\varepsilon = 0$ on ∂M. Moreover $\lim_{\varepsilon\to 0}\tilde{\theta}_\varepsilon = \theta$ and $\lim_{\varepsilon\to 0}\nabla\tilde{\theta}_\varepsilon = \nabla\theta$ in L^2. So we can find dense set satisfying the additional boundary condition.

Next we rewrite the bilinear forms in terms of covariant derivatives. To do this, we need the second fundamental form on ∂M. It is defined as follows. Let X, Y be vector fields tangential to ∂M. The following bilinear form α is called the second fundamental form:

$$\alpha(X,Y) = (\nabla_X Y, N). \tag{3.7}$$

α is expressed in terms of F as follows:

Proposition 3.2. *α is given as follows:*

$$\alpha(X,Y)=\frac{D^2F(X,Y)}{|DF|}. \tag{3.8}$$

In particular, α is symmetric.

Proof. Since Y is tangential to ∂M, it follows that $\langle DF, Y\rangle = 0$. Therefore

$$\begin{aligned}
0 &= \langle D\langle DF, Y\rangle, X\rangle \\
&= D^2F(X,Y) + \langle DF, \nabla_X Y\rangle \\
&= D^2F(X,Y) - (|DF|N, \nabla_X Y) \\
&= D^2F(X,Y) - |DF|\alpha(X,Y).
\end{aligned}$$

which implies (3.8) . □

Define an operator A acting on any 1-form θ by

$$A\theta = \alpha(\theta^\sharp, \cdot). \tag{3.9}$$

A can be extended to differential forms as follows

$$d\Gamma(A)(\omega_1\wedge\cdots\wedge\omega_p) := \sum_{j=1}^{p} \omega_1\wedge\cdots\wedge A\omega_j\wedge\cdots\wedge\omega_p \tag{3.10}$$

Using this notation, we have a different expression of $\mathcal{E}^a_{(p)}$.

Theorem 3.3. *It holds that, for θ, $\eta \in \mathrm{Dom}(\mathcal{E}^a_{(p)})$*

$$\mathcal{E}^a_{(p)}(\theta,\eta) = \int_M (\nabla\theta, \nabla\eta)d\mu + p\int_M (\theta,\eta)d\mu + \int_{\partial M} (d\Gamma(A)\theta, \eta)d\sigma. \tag{3.11}$$

Further, for θ, $\eta \in \mathrm{Dom}(\mathcal{E}^r_{(1)})$, it holds that

(3.12)
$$\mathcal{E}^r_{(1)}(\theta,\eta) = \int_M (\nabla\theta, \nabla\eta)d\mu + \int_M (\theta,\eta)d\mu + \int_{\partial M} d^*\omega_N \langle\theta, N\rangle\langle\eta, N\rangle\, d\sigma.$$

Proof. Take any θ, η. Recall the identity $dd^* + d^*d = -L + p$ where $L = -\nabla^*\nabla$ is the Ornstein-Uhlenbeck operator. Further we assume that $i(N)\theta = 0$, $i(N)d\theta = 0$ on ∂M. Then, by Theorem 3.1 , we have

$$\mathcal{E}^a_{(p)}(\theta,\eta) = \int_M ((d^*d + dd^*)\theta, \eta)d\mu$$

$$
\begin{aligned}
&= \int_M (\nabla^*\nabla\theta, \eta) d\mu + p \int_M (\theta, \eta) d\mu \\
&= \int_M (\nabla\theta, \nabla\eta) d\mu + p \int_M (\theta, \eta) d\mu + \int_{\partial M} (i(N)\nabla\theta, \eta) d\sigma \\
&= \int_M (\nabla\theta, \nabla\eta) d\mu + p \int_M (\theta, \eta) d\mu + \int_{\partial M} (\nabla_N\theta, \eta) d\sigma.
\end{aligned}
$$

We have to calculate $\nabla_N\theta$. First we calculate this when $p = 1$. By the definition of exterior differentiation, we have

$$
\begin{aligned}
d\theta(X, Y) &= \nabla\theta(X, Y) - \nabla\theta(Y, X) \\
&= \nabla_X\theta(Y) - \nabla_Y\theta(X).
\end{aligned}
$$

By the boundary condition $i(N)d\theta = 0$ on ∂M, it follows that

$$
\begin{aligned}
0 &= d\theta(N, Y) \\
&= \nabla_N\theta(Y) - \nabla_Y\theta(N) \\
&= \nabla_N\theta(Y) - (\nabla_Y\theta^\sharp, N) \\
&= \nabla_N\theta(Y) - \alpha(Y, \theta^\sharp).
\end{aligned}
$$

Thus we have $\nabla_N\theta = \alpha(\cdot, \theta^\sharp) = A\theta$. This proves (3.11) for $p = 1$.

So far, we have imposed the boundary condition $i(N)d\theta = 0$ on ∂M. We have to remove this restriction. Now we only assume that $i(N)\theta = 0$ on ∂M. We take $\tilde{\theta}_\varepsilon$ as in (3.5) . Then,

$$
\mathcal{E}^a_{(p)}(\tilde{\theta}_\varepsilon, \eta) = \int_M (\nabla\tilde{\theta}_\varepsilon, \nabla\eta) d\mu + p \int_M (\tilde{\theta}_\varepsilon, \eta) d\mu + \int_{\partial M} (A\tilde{\theta}_\varepsilon, \eta) d\sigma.
$$

Since $\lim_{\varepsilon\to 0} \tilde{\theta}_\varepsilon = \theta$ and $\lim_{\varepsilon\to 0} \nabla\tilde{\theta}_\varepsilon = \nabla\theta$ in L^2, letting ε go to 0, we get the desired result.

For general p, we may assume $\theta = \theta_1 \wedge \cdots \wedge \theta_p$ and θ_j satisfies the boundary condition $i(N)\theta_j = 0$, $i(N)d\theta_j = 0$ on ∂M. Then

$$
\begin{aligned}
\nabla_N\theta &= \nabla_N(\theta_1 \wedge \cdots \wedge \theta_p) \\
&= \sum_{j=1}^p \theta_1 \wedge \cdots \wedge \nabla_N\theta_j \wedge \cdots \wedge \theta_p \\
&= \sum_{j=1}^p \theta_1 \wedge \cdots \wedge A\theta_j \wedge \cdots \wedge \theta_p \\
&= d\Gamma(A)\theta.
\end{aligned}
$$

This shows (3.11) .

Let us proceed to the relative boundary condition case. In the same way as above, we may assume $\omega_N \wedge \theta = 0$, $\omega_N \wedge \eta = 0$ and $\omega_N \wedge d^*\theta = 0$ on ∂M. We need to calculate $(\nabla_N \theta, \eta)$. Set

$$\xi = \theta - \langle \theta, N \rangle \omega_N .$$

From the assumption, $\xi = 0$ holds on ∂M. Further $d^*\theta$ can be calculated as

$$\begin{aligned} 0 = d^*\theta &= d^*(\langle \theta, N \rangle \omega_N + \xi) \\ &= \langle \theta, N \rangle d^*\omega_N - (D\langle \theta, N \rangle, \omega_N) + d^*\xi. \end{aligned}$$

We note that $d^*\xi = 0$ on ∂M. In fact, recall the following identity (see [1, Proposition 2.5])

$$d^*\xi = d^*_{\partial M}\xi - \nabla\xi(N, N) - i(Qd^*Q)\xi.$$

Here $d^*_{\partial M}$ is the dual operator on ∂M with respect to σ. We do not give the explicit form of Qd^*Q, but we can see that $i(Qd^*Q)\xi = 0$ on ∂M from the assumption. Further we have, on ∂M,

$$\begin{aligned} \nabla\xi(N, N) &= \langle \nabla_N \xi, N \rangle \\ &= \nabla_N \langle \xi, N \rangle - \langle \xi, \nabla_N N \rangle \\ &= 0 \quad (\because \langle \xi, N \rangle = 0) \end{aligned}$$

and $d^*_{\partial M}\xi = 0$ since $\xi = 0$ on ∂M. Combining these identities, we have

$$\langle \theta, N \rangle d^*\omega_N = (D\langle \theta, N \rangle, \omega_N) = \nabla_N \langle \theta, N \rangle, \quad \text{on } \partial M. \tag{3.13}$$

Now we are ready to compute $(\nabla_N \theta, \eta)$. On ∂M, we have

$$\begin{aligned} (\nabla_N \theta, \eta) &= (\nabla_N \theta, \langle \eta, N \rangle \omega_N) \\ &= (\nabla_N \langle \theta, N \rangle \omega_N + \langle \theta, N \rangle \nabla_N \omega_N + \nabla_N \xi, \langle \eta, N \rangle \omega_N) \\ &= \nabla_N \langle \theta, N \rangle \langle \eta, N \rangle + \langle \theta, N \rangle \langle \eta, N \rangle (\nabla_N \omega_N, \omega_N) + \langle \eta, N \rangle (\nabla_N \xi, \omega_N) \\ &= \nabla_N \langle \theta, N \rangle \langle \eta, N \rangle + \frac{1}{2} \langle \theta, N \rangle \langle \eta, N \rangle \nabla_N (\omega_N, \omega_N) + \langle \eta, N \rangle \langle \nabla_N \xi, N \rangle \\ &= d^*\omega_N \langle \theta, N \rangle \langle \eta, N \rangle \quad (\because (3.13)) \end{aligned}$$

which shows (3.12) . This completes the proof. □

α is non-negative definite when the boundary ∂M is convex. In fact, since $\alpha = D^2F/|DF|$, the positivity of α is equivalent to the positivity of D^2F. By the expression of (3.11) , we easily obtain the following theorem.

Theorem 3.4. *Assume that α is non-negative definite, then, for $\square^a_{(p)}$ ($p \geq 1$), there is no harmonic form satisfying the absolute boundary condition. More precisely, the spectrum of $\square^a_{(p)}$ is contained in $(-\infty, -p]$.*

On the other hand, assuming $d^*\omega_N \geq 0$ on ∂M, we have that there is no harmonic 1-form satisfying the relative boundary condition.

For the relative boundary condition case, we will give an example satisfying the condition $d^*\omega_N \geq 0$ on ∂M. Before that, let us compute $d^*\omega_N$ explicitly. First

$$D|DF| = D\sqrt{(DF, DF)} = \frac{2D^2F(DF^\sharp, \cdot)}{2\sqrt{(DF, DF)}} = D^2F(N, \cdot).$$

Hence

$$\begin{aligned} d^*\omega_N &= -\frac{D^*DF}{|DF|} + \Big(DF, D\frac{1}{|DF|}\Big) \\ &= \frac{LF}{|DF|} - \Big(DF, \frac{D^2F(N, \cdot)}{|DF|^2}\Big) \\ &= \frac{LF - D^2F(N, N)}{|DF|}. \end{aligned}$$

Thus $d^*\omega_N \geq 0$ is equivalent to $LF - D^2F(N, N) \geq 0$.

Now let us take

$$F(x) = \int_0^T x(t)^2 dt.$$

Here $\{x(t)\}$ is a one-dimensional Brownian motion. In this case, an abstract Wiener space is $W_0 = \{x \in C([0, T] \to \mathbb{R}); x(0) = 0\}$ with the Wiener measure μ. The Cameron-Martin space H is the set of all absolutely continuous function $h \in W_0$ with square integrable derivative. The inner product of H is given by

$$(h, k) = \int_0^T h'(t)k'(t)\, dt.$$

Set $M = \{F \leq C\}$. Then, we easily see

$$\begin{aligned} LF(x) &= 2\int_0^T (x(t)^2 - t)dt = 2F(x) - T^2, \\ DF(x) &= 2\int_0^T x(t)h^t dt. \end{aligned}$$

Here $h^t \in H$ is defined as follows:

$$h^t(s) = \min\{s, t\}.$$

$DF(x)$ is an element of H^* but we identify H^* with H. Further we have

$$D^2F(x) = 2\int_0^T h^t \otimes h^t \, dt.$$

Hence we have

$$|D^2F(x)|^2 = 4\int_0^T \int_0^T (h^t, h^s)_H^2 \, dtds = 4\int_0^T \int_0^T \min\{t, s\}^2 \, dtds = \frac{2}{3}T^4.$$

Therefore, on the set $\partial M = \{F = C\}$,

$$LF(x) - D^2F(x)(N, N) \geq 2C - T^2 - |D^2F(x)| = 2C - \Big(1 + \frac{2}{\sqrt{6}}\Big)T^2.$$

Thus, assuming $2C \geq (1 + \frac{2}{\sqrt{6}})T^2$, we have the vanishing theorem for 1-forms with the relative boundary condition.

References

[1] T. Kazumi and I. Shigekawa, Differential calculus on a submanifold of an abstract Wiener space, I. Covariant derivative, in "*Stochastic analysis on infinite dimensional spaces*," ed. by H. Kunita and H.-H. Kuo, pp. 117–140, Longman, Harlow, 1994.

[2] T. Kazumi and I. Shigekawa, Differential calculus on a submanifold of an abstract Wiener space, II. Weitzenböck formula, in "*Dirichlet forms and stochastic processes,*" Proceedings of the International Conference held in Beijing, China, October 25-31, 1993, ed. by Z. M. Ma, M. Röckner and J. A. Yan, pp. 235–251, Walter de Gruyter, Berlin-New York, 1995.

[3] D. B. Ray and I. M. Singer, R-torsion and the Laplacian on Riemannian manifolds, *Adv. in Math.*, **7** (1971), 145–210.

[4] R. C. Reilly, Applications of the Hessian operator in a Riemannian manifold, *Indiana Univ. Math. J.*, **26** (1977), 459–472.

[5] I. Shigekawa, De Rham-Hodge-Kodaira's decomposition on an abstract Wiener space, *J. Math. Kyoto Univ.*, **26** (1986), 191–202.

[6] I. Shigekawa, Semigroup domination on a Riemannian manifold with boundary, *Acta Applicandae Math.*, **63** (2000), 385–410.

[7] G. Schwarz, "*Hodge-decomposition—a method for solving boundary value problems*," Lecture Notes in Math., vol. 1607, Springer-Verlag, Berlin-Heidelberg-New York, 1995.

[8] C.-Y. Xia, The first nonzero eigenvalue for manifolds with Ricci curvature having positive lower bound, in "*Chinese mathematics into the 21st century*," ed. W.-T. Wu and M.-D. Cheng, pp. 243–249, Peking Univ. Press, Beijing, 1991.

Department of Mathematics
Graduate School of Science
Kyoto University,
Kyoto 606-8502, Japan
e-mail: ichiro@kusm.kyoto-u.ac.jp
URL: http://www.kusm.kyoto-u.ac.jp/~ichiro/