SEMIGROUP DOMINATION ON A RIEMANNIAN MANIFOLD WITH BOUNDARY

ICHIRO SHIGEKAWA

Dedicated to Professor Takeyuki Hida on his 70th birthday

ABSTRACT. We discuss the semigroup domination on a Riemannian manifold with boundary. Our main interest is the Hodge-Kodaira Laplacian for differential forms. We consider two kinds of boundary conditions; the absolutely boundary condition and the relative boundary condition. Our main tool is the square field operator. We also develop a general theory of semigroup commutation.

1. INTRODUCTION

In this paper, we discuss a theory of semigroup domination. To be precise, let $(X, \mathcal{B}, m)$ be a σ-finite measure space. Suppose we are given a semigroup $\{T_t\}$ on $L^2 = L^2(m)$. We assume that $\{T_t\}$ is positivity preserving. Besides, we are given a semigroup $\{\vec{T}_t\}$ acting on Hilbert space valued square integrable functions. We denote the norm of the Hilbert space by $|\cdot|$. If we have

$$|\vec{T}_t u| \le T_t |u|, \quad \forall u \in L^2, \tag{1.1}$$

we say that the semigroup $\{\vec{T}_t\}$ is dominated by $\{T_t\}$. We are interested in when this inequality holds.

A necessary and sufficient condition for (1.1) is given by the abstract Kato theorem due to B. Simon [18, 19]. Later E. Ouhabaz [13] gave a necessary and sufficient condition in terms of bilinear form under the sector condition. In my previous paper [16], we discuss this problem in the framework of square field operator. Typical example to which our theorem is applicable is the Hodge-Kodaira Laplacian for differential forms on a Riemannian manifold (see also [10] in this direction).

In this paper, we consider the Hodge-Kodaira Laplacian on a Riemannian manifold with boundary. We consider two kinds of boundary condition: the relative boundary condition and the absolute boundary condition. In this case, we cannot apply the result in [16] and so we generalize the notion of $\vec{\Gamma}$ that corresponds to the Bakry-Emery Γ_2. So far, $\vec{\Gamma}$ is an L^1 function. But in our formulation, it is no more a function; it is a smooth measure in the sense of Dirichlet form. The positivity of the smooth measure is essential. Using this notion, we give a sufficient condition to (1.1).

We remark that this kind of problem was also discussed by Donnelly-Li [6]. They proved the heat kernel domination. We take a different approach. Méritet [12] also proved the cohomology vanishing theorem.

1991 *Mathematics Subject Classification.* 60J60, 58G32.

Key words and phrases. semigroup domination, Hodge-Kodaira Laplacian, absolute boundary condition, relative boundary condition.

The organization of this paper is as follows. In Section 2, we give a generalization of $\vec{\Gamma}$ and prepare the general theory for the semigroup domination. To do this, we use the Ouhabaz criterion for the semigroup domination. We also discuss the theory of semigroup commutation in Section 3. Combining the semigroup domination with the semigroup commutation, we can reformulate the Bakry-Emery criterion for the logarithmic Sobolev inequality. In Section 4, we consider the Hodge-Kodaira Laplacian and apply our theory to it. To give a sufficient condition for the domination, the second fundamental form on the boundary is crucial.

2. Square field operator and the contraction semigroup

In this section, we discuss the contraction semigroup in the framework of the square field operator, which is called "opérateur carré du champ" in the French literature. Our main interest is the semigroup domination on a Riemannian manifold with boundary in Section 4, but we prepare a general theory in this section.

Let $(X, \mathcal{B})$ be a measurable space and m be a σ-finite measure. Suppose we are given a strongly continuous contraction symmetric semigroup $\{T_t\}$ on L^2. We further assume that T_t is Markovian, i.e., if $f \in L^2$ satisfies $0 \le f \le 1$, then $0 \le T_t f \le 1$. Here these inequalities hold a.e. But we do not specify 'a.e.' either in the sequel. We denote the generator by A and the associated Dirichlet form by $\mathcal{E}$.

We note that we can regard $\{T_t\}$ as a semigroup on $L^p(m)$, $p \ge 1$ by the Riesz-Thorin interpolation theorem. We denote the generator of $\{T_t\}$ on $L^p(m)$ by A_p.

To introduce the square field operator, we assume the following condition (see Bouleau-Hirsch [3, Chapter 1, §4] for details):

(Γ) For f, $g \in \mathrm{Dom}(A_2)$, we have $f \cdot g \in \mathrm{Dom}(A_1)$.

Under the above assumption, we set

$$\Gamma(f, g) = \frac{1}{2}\{A_1(f \cdot g) - A_2 f \cdot g - f \cdot A_2 g\}. \tag{2.1}$$

Here we remark that our definition of Γ is different from that of [3] up to a constant.

Furthermore we suppose that $\mathcal{E}$ has the local property in the following sense (see [3, Definition I.5.1.2]):

(L) For any real valued function $f \in \mathrm{Dom}(\mathcal{E})$, $F, G \in C_0^\infty(\mathbb{R})$,

$$\operatorname{supp} F \cap \operatorname{supp} G = \emptyset \implies \mathcal{E}(F_0(f), G_0(f) = 0, \tag{2.2}$$

where $F_0(x) = F(x) - F(0)$, $G_0(x) = G(x) - G(0)$.

The above condition is satisfied as soon as it is satisfied for each element of a dense subset of $\mathrm{Dom}(\mathcal{E})$. In particular, the following identity is most commonly used: For f, $g \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$,

$$\Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h) \quad \text{for } {}^\forall h \in \mathrm{Dom}(\mathcal{E}). \tag{2.3}$$

In addition, we are given a strongly continuous semigroup $\{\vec{T}_t\}$ on $L^2(m; K)$ where $L^2(m; K)$ is the set of all square integrable K-valued functions, K being a separable Hilbert space. We denote the generator by $\vec{A}$ (or $\vec{A}_2$ to specify the space $L^2(m; K)$). We also assume that $\{\vec{T}_t\}$ is symmetric and is associated with a bilinear form $\vec{\mathcal{E}}$. Of course, $\vec{\mathcal{E}}$ is bounded from below.

Our interest is the following semigroup domination:

$$|\vec{T}_t u| \leq T_t|u|, \quad {}^{\forall}u \in L^2, \tag{2.4}$$

To give a sufficient condition for (2.4), we now define the square field operator for $\{\vec{T}_t\}$. To do this, we imposed the following condition in [16]:

$(\vec{\Gamma}_\lambda)$ For u, $v \in \mathrm{Dom}(\vec{A}_2)$, we have $(u|v)_K \in \mathrm{Dom}(A_1)$ and there exists $\lambda \in \mathbb{R}$ such that

$$A_1|u|^2 - 2(\vec{A}_2 u|u)_K + 2\lambda|u|^2 \geq 0. \tag{2.5}$$

Under the above condition we define $\vec{\Gamma}$ by

$$\vec{\Gamma}(u,v) = \frac{1}{2}\{A_1(u|v)_K - (\vec{A}_2 u|v)_K - (u|\vec{A}_2 v)_K\}. \tag{2.6}$$

The condition $(\vec{\Gamma}_\lambda)$ is rather restrictive. In particular we have to assume that $(u|v)_K \in \mathrm{Dom}(A_1)$ for u, $v \in \mathrm{Dom}(\vec{A}_2)$. We can define $\vec{\Gamma}$ without this condition. In fact, under the above definition, a formal calculation leads to

$$-\mathcal{E}((u,v),f) + \vec{\mathcal{E}}(fu,v) + \vec{\mathcal{E}}(u,fv) = 2\int_X \vec{\Gamma}(u,v) f dm. \tag{2.7}$$

In the above expression it is only necessary to assume that $(u,v) \in \mathrm{Dom}(\mathcal{E})$, fu, $fv \in \mathrm{Dom}(\vec{\mathcal{E}})$.

Keeping in mind this observation, we modify the condition $(\vec{\Gamma}_\lambda)$ as follows:

$(\vec{\Gamma}'_\lambda\text{-}1)$ For u, $v \in \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$ and $f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$, we have $(u,v) \in \mathrm{Dom}(\mathcal{E})$, fu, $fv \in \mathrm{Dom}(\vec{\mathcal{E}})$ and moreover $\mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$ is dense in $\mathrm{Dom}(\vec{\mathcal{E}})$.

$(\vec{\Gamma}'_\lambda\text{-}2)$ There exists $\vec{\Gamma}$: $\mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty \times \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty \to L^1$, a smooth measure σ, and a non-negative symmetric tensor $\vec{\gamma}$ (i.e, $\vec{\gamma}$ is an $H^* \otimes H^*$-valued function on X) such that

$$\begin{gathered} -\mathcal{E}((u,v),f) + \vec{\mathcal{E}}(fu,v) + \vec{\mathcal{E}}(u,fv) = 2\int_X \vec{\Gamma}(u,v) f dm + 2\int_X \vec{\gamma}(\tilde{u},\tilde{v})\tilde{f} d\sigma \\ \forall f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty, \quad \forall u,v \in \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty. \end{gathered} \tag{2.8}$$

Here $\tilde{u}$, $\tilde{v}$, $\tilde{f}$ are quasi-continuous modification of u, v, f, respectively. Of course, we have assumed that each element of $\mathrm{Dom}(\vec{\mathcal{E}})$ admits a quasi-continuous modification.

$(\vec{\Gamma}'_\lambda\text{-}3)$ For the real constant λ, it holds that

$$\vec{\Gamma}(u,u) + \lambda|u|^2 \geq 0 \quad \text{for } u \in \mathrm{Dom}(\vec{A}_2). \tag{2.9}$$

$(\vec{\Gamma}'_\lambda\text{-}4)$ It holds that

$$\vec{\mathcal{E}}(u,v) = \int_X \vec{\Gamma}(u,v) dm + \int_X \vec{\gamma}(\tilde{u},\tilde{v}) d\sigma \quad \forall u,v \in \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty. \tag{2.10}$$

In the sequel, $(\vec{\Gamma}'_\lambda)$ refers to these four conditions. We emphasize that we assumed the *positivity* of $\vec{\gamma}$ and σ. Since $\mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$ is dense in $\mathrm{Dom}(\vec{\mathcal{E}})$, it is easy to see that $\vec{\Gamma}$ is well-defined on $\mathrm{Dom}(\vec{\mathcal{E}}) \times \mathrm{Dom}(\vec{\mathcal{E}})$ as a continuous bilinear mapping into L^1 and (2.10) holds for u, $v \in \mathrm{Dom}(\vec{\mathcal{E}})$

Next we introduce the following condition on $\vec{\Gamma}$ and $\vec{\mathcal{E}}$. It was already appeared in [16]. It is related to the derivation property of $\vec{\Gamma}$.

$(\vec{D})$ For $u, v \in \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$, $f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$, it holds that

$$2f\vec{\Gamma}(u,v) = -\Gamma(f,(v|u)_K) + \vec{\Gamma}(u,fv) + \vec{\Gamma}(fu,v), \tag{2.11}$$

From this, we have

$$2f\vec{\Gamma}(u,u) = -\Gamma(f,|u|^2) + 2\vec{\Gamma}(u,fu). \tag{2.12}$$

Further, by (2.8), we have

$$-\mathcal{E}((u,u),f) + 2\vec{\mathcal{E}}(fu,u) = 2\int_X \vec{\Gamma}(u,u) f dm + 2\int_X \vec{\gamma}(\tilde{u},\tilde{u})\tilde{f} d\sigma. \tag{2.13}$$

Substituting (2.12) into this equation, we have

$$\begin{aligned}-\mathcal{E}((u,u),f) + 2\vec{\mathcal{E}}(fu,u) &= -\int_X \Gamma(f,|u|^2)dm + 2\int_X \vec{\Gamma}(fu,u)dm + 2\int_X \vec{\gamma}(\tilde{u},\tilde{u})\tilde{f}d\sigma \\ &= -\mathcal{E}((u,u),f) + 2\int_X \vec{\Gamma}(fu,u)dm + 2\int_X \vec{\gamma}(\tilde{u},\tilde{u})\tilde{f}d\sigma\end{aligned}$$

Thus we have

$$\vec{\mathcal{E}}(fu,u) = \int_X \vec{\Gamma}(fu,u)dm + \int_X \vec{\gamma}(\tilde{u},\tilde{u})\tilde{f}d\sigma. \tag{2.14}$$

In particular, for $f \geq 0$, it holds that

$$\vec{\mathcal{E}}(fu,u) \geq \int_X \vec{\Gamma}(fu,u)dm. \tag{2.15}$$

We remark that if $\mathbf{1} \in \mathrm{Dom}(\mathcal{E})$, (2.10) follows from (2.14). Here $\mathbf{1}$ denotes the function identically equal to 1.

We can give a sufficient condition for (2.4) in terms of square field operator. To do this, we need the following Ouhabaz' criterion. For the semigroup domination (2.4), the following condition is necessary and sufficient:

1. If $u \in \mathrm{Dom}(\mathcal{E})$, then $|u| \in \mathrm{Dom}(\mathcal{E})$ and
$$\mathcal{E}(|u|,|u|) \leq \vec{\mathcal{E}}(u,u) + \lambda\|u\|_2^2. \tag{2.16}$$
2. If $u \in \mathrm{Dom}(\vec{\mathcal{E}})$ and $f \in \mathrm{Dom}(\mathcal{E})$ with $0 \leq f \leq |u|$, then $f \operatorname{sgn} u \in \mathrm{Dom}(\vec{\mathcal{E}})$ and
$$\mathcal{E}(f,|u|) \leq \vec{\mathcal{E}}(u, f\operatorname{sgn} u) + \lambda(f,|u|) \tag{2.17}$$
where $\operatorname{sgn} u = u/|u|$.

Now we have the following theorem.

Theorem 2.1. *Assume conditions* (Γ), (L), $(\vec{\Gamma}_\lambda)$ *and* $(\vec{D})$. *Then, for* $u \in \mathrm{Dom}(\vec{\mathcal{E}})$ *we have* $|u| \in \mathrm{Dom}(\mathcal{E})$ *and*

$$\Gamma(|u|,|u|) \leq \vec{\Gamma}(u,u) + \lambda|u|^2. \tag{2.18}$$

In addition, for $f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$ *and* $u \in \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$, *we have*

$$\{\vec{\Gamma}(fu,fu) + \lambda|fu|^2\}^{1/2} \leq |f|\{\vec{\Gamma}(u,u) + \lambda|u|^2\}^{1/2} + |u|\Gamma(f,f)^{1/2}. \tag{2.19}$$

Furthermore we have the following semigroup domination:

$$|\vec{T}_t u| \le e^{\lambda t} T_t |u| \quad \textit{for } u \in L^2(X;K) \tag{2.20}$$

Proof. Many parts are the same as in [16]. We give a proof for completeness. For simplicity, we give a proof in the case $\lambda = 0$. Take $u \in \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$ and $f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$. From the assumption, $v = fu \in \mathrm{Dom}(\vec{\mathcal{E}})$ and we substitute v in (2.11):

$$\vec{\Gamma}(fu, fu) + \vec{\Gamma}(u, |f|^2 u) = 2f\vec{\Gamma}(u, fu) + \Gamma(f, f|u|^2).$$

Hence

$$\begin{aligned}
\vec{\Gamma}(fu, fu) &= -\vec{\Gamma}(u, |f|^2 u) + 2f\vec{\Gamma}(u, fu) + \Gamma(f, f|u|^2) \\
&= -|f|^2\vec{\Gamma}(u,u) - \frac{1}{2}\Gamma(|f|^2, |u|^2) + f\{2f\vec{\Gamma}(u,u) + \Gamma(f, |u|^2)\} + \Gamma(f, f|u|^2) \\
&= -|f|^2\vec{\Gamma}(u,u) - \frac{1}{2}f\Gamma(f, |u|^2) - \frac{1}{2}f\Gamma(f, |u|^2) + 2|f|^2\vec{\Gamma}(u,u) \\
&\quad + f\Gamma(f, |u|^2) + f\Gamma(f, |u|^2) + |u|^2\Gamma(f,f) \\
&= |f|^2\vec{\Gamma}(u,u) + \frac{1}{2}f\Gamma(f, |u|^2) + \frac{1}{2}f\Gamma(f, |u|^2) + |u|^2\Gamma(f,f).
\end{aligned}$$

In particular, if we take $f = |u|^2$, we get

$$\vec{\Gamma}(|u|^2 u, |u|^2 u) = |u|^4\vec{\Gamma}(u,u) + 2|u|^2\Gamma(|u|^2, |u|^2). \tag{2.21}$$

On the other hand, substituting $v = u$ and $f = |u|^2$ in (2.11), we have

$$2|u|^2\vec{\Gamma}(u,u) + \Gamma(|u|^2, |u|^2) = 2\vec{\Gamma}(|u|^2 u, u).$$

Taking square,

$$\begin{aligned}
4|u|^4\vec{\Gamma}(u,u)^2 &+ 4|u|^2\vec{\Gamma}(u,u)\Gamma(|u|^2, |u|^2) + \Gamma(|u|^2, |u|^2)^2 \\
&= 4\vec{\Gamma}(|u|^2 u, u)^2 \\
&\le 4\vec{\Gamma}(|u|^2 u, |u|^2 u)\vec{\Gamma}(u,u) \quad \text{(by the Schwarz inequality)} \\
&= 4\{|u|^4\vec{\Gamma}(u,u) + 2|u|^2\Gamma(|u|^2, |u|^2)\}\vec{\Gamma}(u,u) \quad (\because\ (2.21)) \\
&= 4|u|^4\vec{\Gamma}(u,u)^2 + 8|u|^2\Gamma(|u|^2, |u|^2)\vec{\Gamma}(u,u).
\end{aligned}$$

Thus we have

$$\Gamma(|u|^2, |u|^2) \le 4|u|^2\vec{\Gamma}(u,u). \tag{2.22}$$

Now for $\varepsilon > 0$, set $\varphi_\varepsilon(t) = \sqrt{t + \varepsilon^2} - \varepsilon$. Then by the derivation property and (2.22),

$$\Gamma(\varphi_\varepsilon(|u|^2), \varphi_\varepsilon(|u|^2)) \le \frac{1}{4(|u|^2 + \varepsilon^2)}\Gamma(|u|^2, |u|^2) \le \frac{4|u|^2}{4(|u|^2 + \varepsilon^2)}\vec{\Gamma}(u,u) \le \vec{\Gamma}(u,u).$$

From this we can show that $\{\varphi_\varepsilon(|u|^2)\}_{\varepsilon > 0}$ is a bounded set in $\mathrm{Dom}(\mathcal{E})$. In fact

$$\begin{aligned}
\mathcal{E}(\varphi_\varepsilon(|u|^2), \varphi_\varepsilon(|u|^2)) &= \int_X \Gamma(\varphi_\varepsilon(|u|^2), \varphi_\varepsilon(|u|^2))dm \\
&\le \int_X \vec{\Gamma}(u,u)dm \\
&\le \vec{\mathcal{E}}(u,u).
\end{aligned}$$

Hence we can take a sequence $\{\varepsilon_j\}$ tending to 0 such that $\{\varphi_{\varepsilon_j}(|u|^2)\}_j$ converges weakly in $\mathrm{Dom}(\mathcal{E})$. The limit is $|u|$ since it converges to $|u|$ strongly in L^2. This means that $|u| \in \mathrm{Dom}(\mathcal{E})$. Further, taking a subsequence if necessary, we may assume that the Cesaro mean converges strongly in $\mathrm{Dom}(\mathcal{E})$, i.e.,

$$\frac{1}{n}\sum_{j=1}^{n}\varphi_{\varepsilon_j}(|u|^2) \to |u| \quad \text{strongly in } \mathrm{Dom}(\mathcal{E}).$$

By the continuity of Γ, we have

$$\Gamma(\frac{1}{n}\sum_{j=1}^{n}\varphi_{\varepsilon_j}(|u|^2), \frac{1}{n}\sum_{j=1}^{n}\varphi_{\varepsilon_j}(|u|^2)) \to \Gamma(|u|,|u|) \quad \text{strongly in } L^1.$$

On the other hand, by the Minkowski inequality, it follows that

$$\Gamma(\frac{1}{n}\sum_{j=1}^{n}\varphi_{\varepsilon_j}(|u|^2), \frac{1}{n}\sum_{j=1}^{n}\varphi_{\varepsilon_j}(|u|^2))^{1/2} \le \frac{1}{n}\sum_{j=1}^{n}\Gamma(\varphi_{\varepsilon_j}(|u|^2), \varphi_{\varepsilon_j}(|u|^2))^{1/2} \le \vec{\Gamma}(u,u)^{1/2}.$$

Therefore we have

$$\Gamma(|u|,|u|) \le \vec{\Gamma}(u,u). \tag{2.23}$$

In fact, it is enough to take a sequence $\{\varepsilon_n\}$ such that the Cesaro mean of $\{\varphi_{\varepsilon_n}(|u|^2)\}$ converges to $|u|$ in $\mathrm{Dom}(\mathcal{E})$.

Now we return to $\vec{\Gamma}(fu,fu)$:

$$\begin{aligned}
\vec{\Gamma}(fu,fu) &= |f|^2\vec{\Gamma}(u,u) + \frac{1}{2}f\Gamma(f,|u|^2) + \frac{1}{2}f\Gamma(f,|u|^2) + |u|^2\Gamma(f,f) \\
&\le |f|^2\vec{\Gamma}(u,u) + |f|\Gamma(|u|^2,|u|^2)^{1/2}\Gamma(f,f)^{1/2} + |u|^2\Gamma(f,f) \\
&\le |f|^2\vec{\Gamma}(u,u) + 2|f||u|\vec{\Gamma}(u,u)^{1/2}\Gamma(f,f)^{1/2} + |u|^2\Gamma(f,f) \\
&= \{|f|\vec{\Gamma}(u,u)^{1/2} + |u|\Gamma(f,f)^{1/2}\}^2
\end{aligned}$$

which shows (2.19).

We show that for $g \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty_+$ and $u \in \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$ we have

$$\vec{\mathcal{E}}(u,gu) \ge \mathcal{E}(g|u|,|u|). \tag{2.24}$$

To see this,

$$\begin{aligned}
\vec{\mathcal{E}}(u,gu) &\ge \int_X \vec{\Gamma}(u,gu)dm \quad (\because\ (2.15)) \\
&= \int_X \{g\vec{\Gamma}(u,u) + \frac{1}{2}\Gamma(g,|u|^2)\}dm \\
&\ge \int_X \{g\Gamma(|u|,|u|) + |u|\Gamma(g,|u|)\}dm \\
&= \int_X \Gamma(g|u|,|u|)dm \\
&= \mathcal{E}(g|u|,|u|).
\end{aligned}$$

Now we have to check Ouhabaz' condition. Since we have assumed that $\mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$ is dense in $\mathrm{Dom}(\vec{\mathcal{E}})$, for any $u \in \mathrm{Dom}(\vec{\mathcal{E}})$, we take a sequence $\{u_n\} \subseteq \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$ that converges to u in $\mathrm{Dom}(\vec{\mathcal{E}})$. Set

$$\psi_\varepsilon(t) = \frac{1}{1+\varepsilon t}, \quad t \geq 0.$$

Then $\psi_\varepsilon(|u_n|)|u_n| \in \mathrm{Dom}(\mathcal{E})$ and by (2.19) we have

$$\begin{aligned}
\vec{\Gamma}(\psi_\varepsilon(|u_n|)u_n, \psi_\varepsilon(|u_n|)u_n)^{1/2} &\leq \psi_\varepsilon(|u_n|)\vec{\Gamma}(u_n, u_n)^{1/2} + |u_n|\Gamma(\psi_\varepsilon(|u_n|), \psi_\varepsilon(|u_n|))^{1/2} \\
&\leq \psi_\varepsilon(|u_n|)\vec{\Gamma}(u_n, u_n)^{1/2} + |u_n||\psi_\varepsilon'|\Gamma(|u_n|, |u_n|)^{1/2} \\
&\leq \Big\{\psi_\varepsilon(|u_n|) + \frac{\varepsilon|u_n|}{(1+\varepsilon|u_n|)^2}\Big\}\vec{\Gamma}(u_n, u_n)^{1/2} \\
&\leq 2\psi_\varepsilon(|u_n|)\vec{\Gamma}(u_n, u_n)^{1/2} \\
&\leq 2\vec{\Gamma}(u_n, u_n)^{1/2}.
\end{aligned}$$

Now

$$\begin{aligned}
&\vec{\mathcal{E}}(\psi_\varepsilon(|u_n|)u_n, \psi_\varepsilon(|u_n|)u_n) \\
&\quad = \int_X \vec{\Gamma}(\psi_\varepsilon(|u_n|)u_n, \psi_\varepsilon(|u_n|)u_n)dm + \int_X \vec{\gamma}(\psi_\varepsilon(|u_n|)u_n, \psi_\varepsilon(|u_n|)u_n)d\sigma \\
&\quad \leq \int_X \vec{\Gamma}(u_n, u_n)dm + \int_X \vec{\gamma}(u_n, u_n)d\sigma \\
&\quad = \vec{\mathcal{E}}(u_n, u_n).
\end{aligned}$$

Thus we have $\{\psi_\varepsilon(|u_n|)u_n\}$ is $\vec{\mathcal{E}}_1$ bounded and converges to $\psi_\varepsilon(|u|)u$ strongly in L^2 by taking a subsequence if necessary. Therefore we have $\psi_\varepsilon(|u|)u \in \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$.

For any $f \in \mathrm{Dom}(\mathcal{E})$ and $u \in \mathrm{Dom}(\vec{\mathcal{E}})$ with $0 \leq f \leq |u|$. We further assume that u is bounded. Set

$$\varphi_\varepsilon(t) = \frac{1}{t+\varepsilon}$$

and

$$g = f\varphi_\varepsilon(|u|) = \frac{f}{|u|+\varepsilon}.$$

Clearly $g \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$. Now substituting g into (2.24), we have

$$\mathcal{E}(|u|, \frac{f}{|u|+\varepsilon}|u|) \leq \vec{\mathcal{E}}(u, \frac{f}{|u|+\varepsilon}u).$$

We note that the boundedness of u is necessary to obtain this inequality. Letting $\varepsilon \to 0$, we get the desired result. To do this, we note

$$\vec{\Gamma}(f\varphi_\varepsilon(|u|)u, f\varphi_\varepsilon(|u|)u)^{1/2} \leq f\varphi_\varepsilon(|u|)\vec{\Gamma}(u, u) + |u|\Gamma(f\varphi_\varepsilon(|u|), f\varphi_\varepsilon(|u|))^{1/2}$$

and further

$$\begin{aligned}
\vec{\Gamma}(f\varphi_\varepsilon(|u|), f\varphi_\varepsilon(|u|))^{1/2} &\leq f\Gamma(\varphi_\varepsilon(|u|), \varphi_\varepsilon(|u|))^{1/2} + \varphi_\varepsilon(|u|)\Gamma(f, f) \\
&\leq f|\varphi_\varepsilon'(|u|)|\Gamma(|u|, |u|)^{1/2} + \varphi_\varepsilon(|u|)\Gamma(f, f) \\
&\leq \varphi_\varepsilon(|u|)\{\Gamma(|u|, |u|)^{1/2} + \Gamma(f, f)\}.
\end{aligned}$$

Combining both of them, we have

$$\begin{aligned}\vec{\Gamma}(f\varphi_\varepsilon(|u|)u, f\varphi_\varepsilon(|u|)u)^{1/2} &\le f\varphi_\varepsilon(|u|)\vec{\Gamma}(u,u) + |u|\varphi_\varepsilon(|u|)\{\Gamma(|u|,|u|)^{1/2} + \Gamma(f,f)\} \\ &\le 2\vec{\Gamma}(u,u)^{1/2} + \Gamma(f,f)^{1/2}.\end{aligned}$$

Now we have $\{f\varphi_\varepsilon(|u|)u; \varepsilon > 0\}$ is bounded in $\mathrm{Dom}(\vec{\mathcal{E}})$. Clearly $f\varphi_\varepsilon(|u|)u \to f \operatorname{sgn} u$ in L^2 and hence we have $f \operatorname{sgn} u \in \mathrm{Dom}(\vec{\mathcal{E}})$. Moreover, we can take a sequence whose Cesaro mean converges strongly in $\mathrm{Dom}(\vec{\mathcal{E}})$, and hence we have

$$\vec{\Gamma}(f \operatorname{sgn} u, f \operatorname{sgn} u)^{1/2} \le 2\vec{\Gamma}(u,u)^{1/2} + \Gamma(f,f)^{1/2} \tag{2.25}$$

and

$$\mathcal{E}(|u|, f) \le \vec{\mathcal{E}}(u, f \operatorname{sgn} u). \tag{2.26}$$

Lastly we remove the restriction of the boundedness of u. So now we do not assume that u is bounded. So first we notice that

$$\psi_\varepsilon(|u|)|u| \ge \psi_\varepsilon(f)f$$

and both are bounded. We take a sequence ε_j converging to 0 such that

$$u_N = \frac{1}{N}\sum_{j=1}^N \psi_{\varepsilon_j}(|u|)u \to u \quad \text{strongly in } \mathrm{Dom}(\vec{\mathcal{E}}),$$

$$f_N = \frac{1}{N}\sum_{j=1}^N \psi_{\varepsilon_j}(f)f \to f \quad \text{strongly in } \mathrm{Dom}(\mathcal{E}).$$

Now u_N is bounded and $f_N \le |u_N|$ and hence, by using (2.26) we have

$$\mathcal{E}(|u_N|, f_N) \le \vec{\mathcal{E}}(u_N, f_N \operatorname{sgn} u_N). \tag{2.27}$$

By taking a subsequence if necessary, we may assume that $\{|u_N|\}$ converges to $|u|$ weakly in $\mathrm{Dom}(\mathcal{E})$ and $\{f_N \operatorname{sgn} u_N\}$ converges to $f \operatorname{sgn} u$ weakly in $\mathrm{Dom}(\vec{\mathcal{E}})$. Hence, by taking limit, we have

$$\mathcal{E}(|u|, f) \le \vec{\mathcal{E}}(u, f \operatorname{sgn} u) \tag{2.28}$$

which is the desired result. This completes the proof. □

As before, it is sufficient to assume $(\vec{D})$ for elements of a core. We state it as a proposition. We also include an analogue of (Γ').

Proposition 2.2. *Assume that the assumptions* (Γ) *and* (L) *hold. Furthermore assume that there exist an algebra* $\mathcal{C} \subseteq \mathrm{Dom}(\mathcal{E}) \cap L^\infty$ *and a subspace* $\mathcal{D} \subseteq \mathrm{Dom}(\vec{A}_2) \cap L^\infty$ *such that* $fu \in \mathcal{D}$ *(i.e,* $\mathcal{D}$ *is a* $\mathcal{C}$*-module),* $(u,v) \in \mathcal{C}$ *for* $f \in \mathcal{C}$*,* $u, v \in \mathcal{D}$ *and further* $(\vec{\Gamma}'_\lambda)$ *and* $(\vec{D})$ *hold for* $f \in \mathcal{C}$*,* $u, v \in \mathcal{D}$*, e.g.,* $(\vec{\Gamma}'_\lambda\text{-}2)$ *means that any element in* $\mathcal{D}$ *admits a quasi-continuous modification and* (2.8) *holds. We also suppose that* $\mathcal{C}$ *is a core for* $\mathcal{E}$ *and* $\mathcal{D}$ *is a core for* $\vec{\mathcal{E}}$*.*

Then $(\vec{\Gamma}_\lambda)$ *and* $(\vec{D})$ *are satisfied for* $f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$*,* $u, v \in \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$*.*

Proof. We may suppose $\lambda = 0$. Take $u \in \mathcal{D}$ and $f \in \mathcal{C}$. From the assumption, $v = fu \in \mathcal{D}$. Now by the same argument as in the proof of Theorem 2.1, we have

$$\begin{aligned}\vec{\Gamma}(fu, fu) &= f^2\vec{\Gamma}(u,u) + \frac{1}{2}f\Gamma(f, |u|^2) + \frac{1}{2}f\Gamma(f, |u|^2) + |u|^2\Gamma(f,f) \\ &\le f^2\vec{\Gamma}(u,u) + |f|\Gamma(f,f)^{1/2}\Gamma(|u|^2, |u|^2)^{1/2} + |u|^2\Gamma(f,f).\end{aligned}$$

Clearly this shows that

$$\begin{aligned}\vec{\mathcal{E}}(fu, fu) &= \int_X \vec{\Gamma}(fu, fu)dm + \int_X \vec{\gamma}(fu, fu)d\sigma \\ &\le \|f\|_\infty^2 \int_X \vec{\Gamma}(u,u)dm + \|f\|_\infty \int_X \Gamma(f,f)^{1/2}\Gamma(|u|^2, |u|^2)^{1/2}dm \\ &\quad + \|u\|_\infty^2 \int_X \Gamma(f,f)dm + \|f\|_\infty^2 \int_X \vec{\gamma}(u,u)d\sigma \\ &\le \|f\|_\infty^2 \int_X \vec{\Gamma}(u,u)dm + \|f\|_\infty \Big\{\int_X \Gamma(f,f)dm\Big\}^{1/2}\Big\{\int_X \Gamma(|u|^2, |u|^2)dm\Big\}^{1/2} \\ &\quad + \|u\|_\infty^2 \int_X \Gamma(f,f)dm + \|f\|_\infty^2 \int_X \vec{\gamma}(u,u)d\sigma \\ &= \|f\|_\infty^2 \vec{\mathcal{E}}(u,u) + \|f\|_\infty \mathcal{E}(f,f)^{1/2}\mathcal{E}(|u|^2, |u|^2)^{1/2} + \|u\|_\infty^2 \mathcal{E}(f,f).\end{aligned}$$

We claim that for $f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$, $u \in \mathcal{D}$, we have $fu \in \mathrm{Dom}(\vec{\mathcal{E}})$ and (2.11), (2.8) hold. To show this, take any real valued function $f \in \mathcal{C}$. Then, for any C^1-function $F : \mathbb{R} \to \mathbb{R}$, we take a sequence of polynomials $\{P_n\}$ such that $P_n \to F$ uniformly on any compact sets up to the first derivative. Since $\vec{\mathcal{E}}(P_n(f)u, P_n(f)u)$ is bounded and $P_n(f)u \to F(f)u$ in L^2. Thus we have $F(f)u \in \mathrm{Dom}(\vec{\mathcal{E}})$ and

$$2F(f)\vec{\Gamma}(u,v) = \Gamma(F(f), (u|v)_K) - \vec{\Gamma}(u, F(f)v) - \vec{\Gamma}(F(f)u, v) \tag{2.29}$$

$$\begin{aligned}\vec{\mathcal{E}}(F(f)u, F(f)u) &\le 2\|F(f)\|_\infty^2\vec{\mathcal{E}}(u,u) + 2\|F(f)\|_\infty\mathcal{E}(F(f), F(f))^{1/2}\mathcal{E}(|u|^2, |u|^2)^{1/2} \\ &\quad + 2\|u\|_\infty^2\mathcal{E}(F(f)f, F(f)).\end{aligned} \tag{2.30}$$

Now for any $f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$, we take a bounded C^1 function φ such that $\varphi(t) = t$ for $|t| \le \|f\|_\infty$. Let $\{f_n\} \subseteq \mathcal{C}$ be a sequence converging to f in $\mathrm{Dom}(\mathcal{E})$. Clearly $g_n = \varphi(f_n) \to f$ weakly in $\mathrm{Dom}(\mathcal{E})$ and $\{g_n\}$ is uniformly bounded. Moreover, by (2.30) we have $\sup_n \vec{\mathcal{E}}(g_n u, g_n u) < \infty$. Hence we can extract a subsequence $\{g_{n_j}u\}$ whose Cesaro mean converges strongly in $\mathrm{Dom}(\vec{\mathcal{E}})$. Together with the fact that $g_n u \to fu$ in L^2, we can see that for $f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$ and $u \in \mathcal{D}$, we have $fu \in \mathrm{Dom}(\vec{\mathcal{E}})$ and (2.11) hold.

Since (2.11) holds for $u, v \in \mathcal{D}$ and $f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$, we can take $f = |u|^2$. By repeating the argument in the proof of Theorem 2.1, for $u \in \mathcal{D}$ and $f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$, we have $fu \in \mathrm{Dom}(\vec{\mathcal{E}})$ and

$$\vec{\Gamma}(fu, fu)^{1/2} \le |f|\vec{\Gamma}(u,u)^{1/2} + |u|\Gamma(f,f)^{1/2} \tag{2.31}$$

$$\Gamma(|u|, |u|) \le \vec{\Gamma}(u,u) \tag{2.32}$$

Since $\mathcal{D}$ is dense in $\mathrm{Dom}(\vec{\mathcal{E}})$, it is easy to see that (2.32) holds for $u \in \mathrm{Dom}(\vec{\mathcal{E}})$.

Now we can prove (2.11) for $f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$ and $u \in \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$. Take any $u \in \mathcal{D}$. For $\varepsilon > 0$, set

$$\psi_\varepsilon(t) = \frac{1}{1+\varepsilon t}, \quad t \in [0, \infty).$$

Note that $\psi'_\varepsilon = -\frac{\varepsilon}{(1+\varepsilon t)^2}$. Since $\psi_\varepsilon(|u|) \in \mathrm{Dom}(\mathcal{E})$, we have $\psi_\varepsilon(|u|)u \in \mathrm{Dom}(\vec{\mathcal{E}})$, and

$$\begin{aligned}
&\vec{\Gamma}(f\psi_\varepsilon(|u|)u, f\psi_\varepsilon(|u|)u)^{1/2} \\
&\quad \le |f|\psi_\varepsilon(|u|)\vec{\Gamma}(u,u)^{1/2} + |u|\Gamma(f\psi_\varepsilon(|u|), f\psi_\varepsilon(|u|))^{1/2} \\
&\quad \le |f|\psi_\varepsilon(|u|)\vec{\Gamma}(u,u)^{1/2} + |u|\{\psi_\varepsilon(|u|)^2\Gamma(f,f) + 2\psi_\varepsilon(|u|)\psi'_\varepsilon(|u|)f\Gamma(|u|,f) \\
&\qquad\qquad + f^2\psi'_\varepsilon(|u|)^2\Gamma(|u|,|u|)\}^{1/2} \\
&\quad \le |f|\psi_\varepsilon(|u|)\vec{\Gamma}(u,u)^{1/2} + |u|\{\psi_\varepsilon(|u|)^2\Gamma(f) + 2\psi_\varepsilon(|u|)\psi'_\varepsilon(|u|)|f|\Gamma(|u|,|u|)^{1/2}\Gamma(f,f)^{1/2} \\
&\qquad\qquad + f^2\psi'_\varepsilon(|u|)^2\Gamma(|u|,|u|)\}^{1/2} \\
&\quad \le |f|\psi_\varepsilon(|u|)\vec{\Gamma}(u,u)^{1/2} + |u|\psi_\varepsilon(|u|)\Gamma(f,f)^{1/2} \\
&\qquad + 2\psi_\varepsilon(|u|)|\psi'_\varepsilon(|u|)||f|\Gamma(|u|,|u|)^{1/2}\Gamma(f,f)^{1/2} + f^2|\psi'_\varepsilon(|u|)|^2\Gamma(|u|,|u|)\}^{1/2} \\
&\quad \le |f|\psi_\varepsilon(|u|)\vec{\Gamma}(u,u)^{1/2} + |u|\psi_\varepsilon(|u|)\Gamma(f,f)^{1/2} + |u||f||\psi'_\varepsilon(|u|)|\vec{\Gamma}(u,u)^{1/2}.
\end{aligned}$$

Now for any $v \in \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$, we take a sequence $\{u_n\} \subseteq \mathcal{D}$ converging to v in $\mathrm{Dom}(\vec{\mathcal{E}})$. Then by the above estimate, we can see that

$$\sup_n \vec{\mathcal{E}}(f\psi_\varepsilon(|u_n|)u_n, f\psi_\varepsilon(|u_n|)u_n) < \infty.$$

Hence, by virtue of Banach-Saks theorem, we have, by taking a subsequence if necessary, the Cesaro mean of $\{f\psi_\varepsilon(|u_n|)u_n\}$ converges to $f\psi_\varepsilon(|u|)u$ in $\mathrm{Dom}(\vec{\mathcal{E}})$. Moreover, by noting that

$$\begin{aligned}
&\vec{\Gamma}\Big(\frac{1}{n}\sum_{k=1}^n f\psi_\varepsilon(|u_k|)u_k, \frac{1}{n}\sum_{k=1}^n f\psi_\varepsilon(|u_k|)u_k\Big)^{1/2} \\
&\quad \le \frac{1}{n}\sum_{k=1}^n \vec{\Gamma}(f\psi_\varepsilon(|u_k|)u_k, f\psi_\varepsilon(|u_k|)u_k)^{1/2} \\
&\quad \le \frac{1}{n}\sum_{k=1}^n \{|f|\psi_\varepsilon(|u_k|)\vec{\Gamma}(u_k,u_k)^{1/2} + |u_k|\psi_\varepsilon(|u_k|)\Gamma(f,f)^{1/2} + |u_k||f||\psi'_\varepsilon(|u_k|)|\vec{\Gamma}(u_k,u_k)^{1/2}\}.
\end{aligned}$$

Letting $n \to \infty$, we have for $v \in \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$,

$$\begin{aligned}
\vec{\Gamma}(f\psi_\varepsilon(|u|)v, f\psi_\varepsilon(|v|)v)^{1/2} &\le |f|\psi_\varepsilon(|v|)\vec{\Gamma}(v,v)^{1/2} + |v|\psi_\varepsilon(|v|)\Gamma(f,f)^{1/2} \\
&\quad + |v||f||\psi'_\varepsilon(|v|)|\vec{\Gamma}(v,v)^{1/2}.
\end{aligned}$$

Next, letting $\varepsilon \to 0$, we eventually obtain that $fv \in \mathrm{Dom}(\vec{\mathcal{E}})$ and

$$\vec{\Gamma}(fv, fv)^{1/2} \le |f|\vec{\Gamma}(v,v)^{1/2} + |v|\Gamma(f,f)^{1/2}.$$

Next, we see that (2.11) holds for $u, v \in \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$ and $f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$. To see this, we first note the following identity: for $u, v \in \mathcal{D}$ and $f, \psi \in \mathrm{Dom}(\mathcal{E})$,

$$2f\vec{\Gamma}(\psi u, v) = \Gamma(f, (v|\psi u)_K) - \vec{\Gamma}(\psi u, fv) - \vec{\Gamma}(f\psi u, v).$$

This identity is clear from the assumption if $\psi \in \mathcal{C}$. By an approximating argument, we can see it for $\psi \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$. Using this, we can repeat the above argument and get the desired result.

So far, we have obtained that $\mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$ is a $\mathrm{Dom}(\mathcal{E}) \cap L^\infty$-module. We can also prove (2.8). Here we remark that every element of $\mathrm{Dom}(\vec{\mathcal{E}})$ admits a quasi-continuous modification. To see this, we note that for $u \in \mathcal{D}$,

$$\begin{aligned}\mathrm{Cap}_1(|u| \geq \varepsilon) &\leq \frac{1}{\varepsilon^2}\mathcal{E}_1(|u|, |u|) \\ &\leq \frac{1}{\varepsilon^2}\int_X \Gamma(|u|, |u|)dm + \int_X |u|^2 dm \\ &\leq \frac{1}{\varepsilon^2}\int_X \vec{\Gamma}(u, u)dm + \int_X |u|^2 dm \\ &\leq \frac{1}{\varepsilon^2}\vec{\mathcal{E}}_1(u, u).\end{aligned}$$

The rest is easy by a standard Borel-Cantelli argument.

This completes the proof. □

3. Commutation relation

In this section, we discuss the commutation relation between two semigroups. Combining this with semigroup domination, we can discuss the logarithmic Sobolev inequality along the Bakry-Emery argument.

Let B and $\hat{B}$ be Banach spaces. We assume that $\hat{B}$ is reflexive. Suppose we are given strongly continuous semigroups $\{T_t\}$ on B and $\{\hat{T}_t\}$ on $\hat{B}$. We denote their generators by A and $\hat{A}$, respectively. We also denote the resolvents of A and $\hat{A}$ by G_α and $\hat{G}_\alpha$, respectively. Further, we are given a closed linear operator D from B to $\hat{B}$ with the dense domain $\mathrm{Dom}(D)$. The dual operator of D is denoted by D^*. We assume that $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(D)$. Under this condition, we have the following theorem.

Theorem 3.1. *Then the following conditions are equivalent to each other.*
(i) $\mathrm{Dom}(\hat{A}^*) \subseteq \mathrm{Dom}(D^*)$ *and*

$$\langle Au, D^*\theta\rangle = \langle Du, \hat{A}^*\theta\rangle, \quad \forall u \in \mathrm{Dom}(A),\ \forall \theta \in \mathrm{Dom}(\hat{A}^*). \tag{3.1}$$

(ii) *For sufficiently large* α,

$$DG_\alpha u = \hat{G}_\alpha Du, \quad \forall u \in \mathrm{Dom}(D). \tag{3.2}$$

(iii) *For any* $t \geq 0$, $T_t \mathrm{Dom}(D) \subseteq \mathrm{Dom}(D)$ *and*

$$DT_t u = \hat{T}_t Du, \quad \forall u \in \mathrm{Dom}(D). \tag{3.3}$$

Proof. First we show the implication (i) $\Rightarrow$ (ii). We note that $\{\hat{T}_t^*\}$ is also a strongly continuous semigroup with the generator $\hat{A}^*$ from the reflexivity of $\hat{B}$. From (i), we have, for $u \in \mathrm{Dom}(A)$ and $\theta \in \mathrm{Dom}(\hat{A}^*)$,

$$\langle(\alpha - A)u, D^*\theta\rangle = \langle Du, (\alpha - \hat{A}^*)\theta\rangle.$$

Substituting $u = G_\alpha v$, $\theta = \hat{G}_\alpha^*\xi$, we have

$$\langle v, D^*\hat{G}_\alpha^*\xi\rangle = \langle DG_\alpha v, \xi\rangle.$$

If $v \in \mathrm{Dom}(D)$, it follows

$$\langle \hat{G}_\alpha Dv, \xi \rangle = \langle DG_\alpha v, \xi \rangle$$

which implies (ii).

To show (ii) $\Rightarrow$ (iii), we note that for any integer n,

$$DG_\alpha^n u = \hat{G}_\alpha^n Du$$

which can be easily shown by the induction on n. Since the semigroup can be expresses in terms of the resolvents as

$$T_t u = \lim_{n\to\infty} \Big(\frac{n}{t} G_{nt}\Big)^n u,$$

we have for $u \in \mathrm{Dom}(D)$,

$$\begin{aligned} \hat{T}_t Du &= \lim_{n\to\infty} \Big(\frac{n}{t} \hat{G}_{nt}\Big)^n Du \\ &= \lim_{n\to\infty} D\Big(\frac{n}{t} \hat{G}_{nt}\Big)^n u. \end{aligned}$$

This, combined with the closeness of D, implies $T_t u \in \mathrm{Dom}(D)$ and $\hat{T}_t Du = DT_t u$.

The implication (iii) $\Rightarrow$ (ii) can be similarly shown by noting

$$G_\alpha u = \int_0^\infty e^{-\alpha t} T_t u dt.$$

Lastly we show (ii) $\Rightarrow$ (i). From assumption $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(D)$, the closed operator $S = DG_\alpha$ is defined whole space B and hence bounded. Hence, for $u \in \mathrm{Dom}(D)$, $\theta \in \hat{B}^*$, we have

$$\langle u, S^*\theta \rangle = \langle Su, \theta \rangle = \langle DG_\alpha u, \theta \rangle = \langle \hat{G}_\alpha Du, \theta \rangle = \langle Du, \hat{G}_\alpha^* \theta \rangle.$$

Now, setting $\theta = (\alpha - \hat{A}^*)\xi$, $\xi \in \mathrm{Dom}(\hat{A}^*)$, we have

$$\langle u, S^*(\alpha - \hat{A}^*)\xi \rangle = \langle Du, \xi \rangle.$$

From this, it follows $\xi \in \mathrm{Dom}(D^*)$. Thus, for $\xi \in \mathrm{Dom}(\hat{A}^*)$, it holds that

$$\langle Su, (\alpha - \hat{A}^*)\xi \rangle = \langle u, D^*\xi \rangle, \quad \forall u \in \mathrm{Dom}(D). \tag{3.4}$$

Since $\mathrm{Dom}(D)$ is dense in B, the above identity holds for all $u \in B$. In particular, if we set $u = (\alpha - A)v$, $v \in \mathrm{Dom}(A)$, we have

$$\langle S(\alpha - A)v, (\alpha - \hat{A}^*)\xi \rangle = \langle (\alpha - A)v, D^*\xi \rangle.$$

By noting $S(\alpha - A)v = DG_\alpha(\alpha - A)v = Dv$, we get (i). This completes the proof. □

Next we give an sufficient condition for the above theorem. We consider the same situation as in Theorem 3.1. In the sequel, any domain of an operator is equipped with a topology given by the graph norm.

Proposition 3.2. *Suppose that there exist a subspace $\mathcal{D} \subset \mathrm{Dom}(A)$ satisfying one of the following conditions:*

(i) *$A\mathcal{D} \subseteq \mathrm{Dom}(D)$, $D\mathcal{D} \subseteq \mathrm{Dom}(\hat{A})$, $\mathcal{D}$ is dense in $\mathrm{Dom}(A)$ and $\mathrm{Dom}(\hat{A}^*) \subseteq \mathrm{Dom}(D^*)$, Further for any $u \in \mathcal{D}$, it holds that*

$$DAu = \hat{A}Du. \tag{3.5}$$

(ii) *$A\mathcal{D} \subseteq \mathrm{Dom}(D)$, $D\mathcal{D} \subseteq \mathrm{Dom}(\hat{A})$ and $(\alpha - A)\mathcal{D}$ is dense in $\mathrm{Dom}(D)$ for sufficiently large α. Further (3.5) holds for any $u \in \mathcal{D}$.*

Then, one of (and hence all) three conditions in Theorem 3.1 holds.

Proof. First we assume (i). Then, for any $u \in \mathcal{D}$ and $\theta \in \mathrm{Dom}(\hat{A}^*)$, we have

$$\langle Au, D^*\theta\rangle = \langle DAu, \theta\rangle = \langle \hat{A}Du, \theta\rangle = \langle Du, \hat{A}^*\theta\rangle.$$

Since $\mathcal{D}$ is dense in $\mathrm{Dom}(A)$, we can see that $\langle Au, D^*\theta\rangle = \langle Du, \hat{A}^*\theta\rangle$ holds for $u \in \mathrm{Dom}(A)$ and $\theta \in \mathrm{Dom}(\hat{A}^*)$. Here, we use that D is a bounded operator from $\mathrm{Dom}(A)$ to $\hat{B}$. Hence (i) in Theorem 3.1 holds.

Next assume (ii). From the assumption, we have for $u \in \mathcal{D}$ and $\theta \in \mathrm{Dom}(\hat{A}^*)$,

$$\langle D(\alpha - A)u, \theta\rangle = \langle (\alpha - \hat{A})Du, \theta\rangle = \langle Du, (\alpha - \hat{A}^*)\theta\rangle.$$

If we see $v = (\alpha - A)u \in (\alpha - A)\mathcal{D}$, it follows that

$$\langle Dv, \theta\rangle = \langle DG_\alpha v, (\alpha - \hat{A}^*)\theta\rangle.$$

Now, using the denseness of $(\alpha - A)\mathcal{D}$ in $\mathrm{Dom}(D)$, the above identity holds for all $v \in \mathrm{Dom}(D)$. Further, as was seen in the proof of Theorem 3.1, $S = DG_\alpha$ is a bounded operator on B, we have

$$\langle Dv, \theta\rangle = \langle v, S^*(\alpha - \hat{A}^*)\theta\rangle \quad \forall v \in \mathrm{Dom}(D).$$

Then it follows $\theta \in \mathrm{Dom}(D^*)$ which implies $\mathrm{Dom}(\hat{A}^*) \subseteq \mathrm{Dom}(D^*)$. Hence, this case is reduce to the case (i). □

Now, we take Hilbert spaces H and $\hat{H}$ in place of Banach spaces B and $\hat{B}$. Moreover generators A and $\hat{A}$ are self-adjoint and associated with the closed quadratic forms $\mathcal{E}$ and $\hat{\mathcal{E}}$, respectively. Of course, $\mathcal{E}$ and $\hat{\mathcal{E}}$ are bounded from below. Then we have the following;

Proposition 3.3. *Suppose that there exist a subspace $\mathcal{D} \subset \mathrm{Dom}(A^{3/2})$ satisfying the following conditions.* $\mathrm{Dom}(\mathcal{E}) \subseteq \mathrm{Dom}(D)$, $D\mathcal{D} \subseteq \mathrm{Dom}(\hat{A})$, *$\mathcal{D}$ is dense in* $\mathrm{Dom}(A^{3/2})$, *and it holds that*

$$DAu = \hat{A}Du, \quad \forall u \in \mathcal{D}. \tag{3.6}$$

Then, one of (and hence all) three conditions in Theorem 3.1 holds.

Proof. From the assumption $\mathcal{D} \subseteq \mathrm{Dom}(A^{3/2})$, we have $(\alpha - A)\mathcal{D} \subseteq \mathrm{Dom}(A^{1/2}) = \mathrm{Dom}(\mathcal{E})$. Moreover $(\alpha - A)\mathcal{D}$ is dense in $\mathrm{Dom}(\mathcal{E})$ by virtue of the denseness of $\mathcal{D}$ in $\mathrm{Dom}(A^{3/2})$. By an argument similar to the proof of Proposition 3.2 (ii), we can get the conclusion. □

We can also give another sufficient condition.

Proposition 3.4. *We assume the following conditions:*

$$\begin{aligned}
&\mathcal{D} \subseteq \mathrm{Dom}(A) \subseteq \mathrm{Dom}(\mathcal{E}) \subseteq \mathrm{Dom}(D),\\
&\hat{\mathcal{D}} \subseteq \mathrm{Dom}(\hat{A}) \subseteq \mathrm{Dom}(\hat{\mathcal{E}}) \subseteq \mathrm{Dom}(D^*),\\
&D\,\mathrm{Dom}(A) \subseteq \mathrm{Dom}(\hat{\mathcal{E}}),\\
&D^*\hat{\mathcal{D}} \subseteq \mathrm{Dom}(\mathcal{E}).
\end{aligned}$$

$\mathcal{D}$ is a dense subspace of $\mathrm{Dom}(\mathcal{E})$ *and $\hat{\mathcal{D}}$ is a dense subspace of* $\mathrm{Dom}(\hat{\mathcal{E}})$. *Further, we assume that it holds*

$$(Au, D^*\theta) = (Du, \hat{A}\theta), \quad \text{for } u \in \mathcal{D}, \theta \in \hat{\mathcal{D}}. \tag{3.7}$$

Then, one of (and hence all) three conditions in Theorem 3.1 holds.

Proof. Take any $u \in \mathcal{D}$, $\theta \in \hat{\mathcal{D}}$. From the assumption, $D^*\theta \in \mathrm{Dom}(\mathcal{E})$ and hence

$$\mathcal{E}(u, D^*\theta) = (Au, D^*\theta) = (Du, \hat{A}\theta).$$

Now using the denseness of $\mathcal{D}$ in $\mathrm{Dom}(\mathcal{E})$, we have

$$\mathcal{E}(u, D^*\theta) = (Du, \hat{A}\theta), \quad \text{for } u \in \mathrm{Dom}(\mathcal{E}), \theta \in \hat{\mathcal{D}}.$$

In particular, $Du \in \mathrm{Dom}(\hat{\mathcal{E}})$ for $u \in \mathrm{Dom}(\hat{\mathcal{E}})$, we have, for $\theta \in \hat{\mathcal{D}}$

$$(Au, D^*\theta) = \mathcal{E}(u, D^*\theta) = (Du, \hat{A}\theta), = \hat{\mathcal{E}}(Du, \theta).$$

Since $\hat{\mathcal{D}}$ is dense in $\mathrm{Dom}(\hat{\mathcal{E}})$, we obtain

$$(Au, D^*\theta) = \hat{\mathcal{E}}(Du, \theta) \quad \text{for } u \in \mathrm{Dom}(A), \theta \in \mathrm{Dom}(\hat{\mathcal{E}}).$$

Taking θ from $\mathrm{Dom}(\hat{A}) \subseteq \mathrm{Dom}(\mathcal{E})$, it holds that

$$(Au, D^*\theta) = \hat{\mathcal{E}}(Du, \theta) = (Du, \hat{A}\theta), \quad \text{for } u \in \mathrm{Dom}(A), \theta \in \mathrm{Dom}(\hat{A}),$$

which is the desired result. □

Remark 3.1. The condition (3.7) can be replaced by $DAu = \hat{A}Du$ for $u \in \mathcal{D}$. Of course in this case we assume that they are well-defined.

In many applications, $\mathcal{E}$ is given by

$$\mathcal{E}(u, v) = (Du, Dv)_{\hat{H}}. \tag{3.8}$$

Hence the assumption $\mathrm{Dom}(D) \subseteq \mathrm{Dom}(\mathcal{E})$ is automatic, i.e., the identity holds.

Now we formulate the Bakry-Emery criterion for the logarithmic Sobolev inequality in our setting. Let the notations be as before and we impose the following conditions. The Hilbert space H is $L^2(X, m)$ on a measure space (X, m). We assume that m is a *probability* measure. The quadratic form $\mathcal{E}$ is a local Dirichlet form with $\mathbf{1} \in \mathrm{Dom}(\mathcal{E})$ and $\mathcal{E}(\mathbf{1}, \mathbf{1}) = 0$. D is a closed operator (usually first order differential operator) from $L^2(X, m)$ to the space of L^2 sections of a vector bundle E over X. We assume, for the notational simplicity, that E is a trivial bundle with a fiber K that is a Hilbert space. We denote ∇ instead of D. We assume that the square field operator is given by

$$\Gamma(f, g)(x) = (\nabla f(x), \nabla g(x))_K \tag{3.9}$$

The Dirichlet form $\mathcal{E}$ is written as

$$\mathcal{E}(f, g) = \int_X (\nabla f(x), \nabla g(x))_K dm \tag{3.10}$$

Suppose further that we are given another symmetric semigroup $\{\vec{T_t}\}$ on $L^2(m; K)$. We assume the following semigroup domination for which we have given a sufficient condition in the previous section.

$$|\vec{T_t}\theta| \leq e^{-\lambda t} T_t |\theta| \tag{3.11}$$

Lastly we assume that the following commutativity condition:

$$\vec{T_t} \nabla f = \nabla T_t f. \tag{3.12}$$

As we have seen, we already have sufficient conditions for this.

Under the above conditions, we can show the logarithmic Sobolev inequality for the Dirichlet form $\mathcal{E}$. Before that, let us see the relation to the Bakry-Emery Γ_2-criterion. Suppose the following commutativity condition on a space $\mathcal{D}$,

$$\nabla A f = \vec{A} \nabla f, \quad \forall f \in \mathcal{D}. \tag{3.13}$$

Of course, we have to choose a suitable space $\mathcal{D}$. We ignore the technical detail for a while. We also assume the condition $(\vec{\Gamma}_\lambda)$. Hence $\vec{\Gamma}$ is realized as an L^1 function. Under these conditions,

$$\begin{aligned} 2\vec{\Gamma}(\nabla f, \nabla g) &= A_1(\nabla f, \nabla g) - (\vec{A}\nabla f, \nabla g) - (\nabla f, \vec{A}\nabla g) \\ &= A_1(\nabla f, \nabla g) - (\nabla A f, \nabla g) - (\nabla f, \nabla A g) \\ &= A_1 \Gamma(f, g) - \Gamma(Af, g) - \Gamma(f, Ag). \end{aligned}$$

The right hand side is usually called the Bakry-Emery Γ_2. So our condition (2.9) corresponds to

$$\Gamma_2(f, f) \geq \lambda \Gamma(f, f)$$

which is the famous Bakry-Emery criterion.

Now we can state the Bakry-Emery theorem in our formulation as follows:

Theorem 3.5. *Under assumptions* (3.11) *and* (3.12) *for* $\lambda > 0$, *the following logarithmic Sobolev inequality holds:*

$$\int_X f^2 \log(f^2/\|f\|_2^2) dm \leq \frac{2}{\lambda} \mathcal{E}(f, f), \qquad \forall f \in \mathrm{Dom}(\mathcal{E}). \tag{3.14}$$

Proof. From our assumption, it holds that

$$|\vec{T}_t \omega| \leq e^{-\lambda t} T_t |\omega|.$$

Therefore, taking $\omega = \nabla f$, we have

$$|\nabla T_t f| \leq e^{-\lambda t} T_t |\nabla f|.$$

Now the rest is the standard argument (see e.g., Deuschel-Stroock [4, Proof of Theorem 6.2.42]). □

4. Riemannian manifold with boundary

In this section, we consider a Riemannian manifold with boundary.

Let M be a d-dimensional compact Riemannian manifold with a boundary ∂M. We assume that everything is smooth. As usual, the Hodge-Kodaira Laplacian is defined as follows:

$$\Box = -(d\delta + \delta d). \tag{4.1}$$

Here d is the exterior differentiation and δ is its (formal) dual. To specify that it acts p-forms, we denote it by $\Box_{(p)}$. Moreover we suppose that $\Box$ is defined on a set of all C^∞ differential forms. We always assume that differential forms are always smooth, i.e., C^∞. Later we consider symmetric operators but they are all essentially self-adjoint on smooth differential forms (to be precise, under a suitable boundary condition) and hence it is enough to achieve formal calculation on smooth forms. We denote the p-th exterior bundle of T^*M by $\bigwedge^p T^*M$. The set of all smooth differential p-forms, i.e., smooth sections of $\bigwedge^p T^*M$, will be denoted by $A^p(M)$.

Since the manifold M has a boundary, we have to impose boundary condition. Several boundary conditions are known but we consider the two conditions: the absolute boundary condition and the relative boundary condition. To describe boundary condition, we introduce the following local coordinate $(x^1, x^2, \ldots, x^{d-1}, r)$ near the boundary: The local coordinate gives an diffeomorphism between a neighborhood in M and $U \times [0, \varepsilon) \subset \mathbb{R}^{d-1} \times R_+$. We identify the neighborhood in M with $U \times [0, \varepsilon)$ through the local coordinate. We assume that, on the boundary ∂M, $r = 0$ and $\frac{\partial}{\partial x^j} \perp \frac{\partial}{\partial r}$, $j = 1, \ldots, d-1$. We denote the inner normal vector by N, i.e., $N = \frac{\partial}{\partial r}$. Further we assume that for fixed $(x_0^1, \ldots x_0^{d-1}) \in U$, $\gamma_t = (x_0^1, \ldots x_0^{d-1}, t)$ is a geodesic with the velocity 1. For any 1-form θ on M, we can decompose it as

$$\theta = \theta_t + \theta_n \wedge dr \tag{4.2}$$

where neither θ_n nor θ_t contains dr. This decomposition is well-defined on the boundary. Using this decomposition, we define

$$B_r(\theta) = \theta_t \tag{4.3}$$

$$B_a(\theta) = \theta_n. \tag{4.4}$$

Recall that θ_t and θ_n are defined only on ∂M. The suffix r stands for the relative boundary condition and a for the absolute boundary condition. We use these operators to define the boundary condition.

In the sequel, B stands for either B_r or B_a. Now we restrict the domain of $\Box$ according to boundary condition. Set

$$\mathrm{Dom}(\Box_{(p)}^B) = \{\phi \in A^p(M);\ B\phi = 0,\ B(d+\delta)\phi = 0\}. \tag{4.5}$$

By an integration by parts formula for differential forms, we can see that $\Box_{(p)}^B$ is symmetric and moreover it is essentially self-adjoint on $\mathrm{Dom}(\Box_{(p)}^B)$. We can think of the associated bilinear form $\mathcal{E}_{(p)}^B$, which is given as follows:

$$\mathrm{Dom}(\mathcal{E}_{(p)}^B) = \{\phi \in A^p(M);\ B\phi = 0\} \tag{4.6}$$

and

$$\mathcal{E}_{(p)}^B(\omega, \eta) = (d\omega, d\eta) + (\delta\omega, \delta\eta). \tag{4.7}$$

Let see why this is so. We recall the integration by parts formula for differential forms. Take any $\omega \in A^p(M)$ and $\eta \in A^{p+1}(M)$, then it holds that

$$\int_M (d\omega, \eta) dm = \int_M (\omega, \delta\eta) dm + \int_{\partial M} \omega \wedge *\eta. \tag{4.8}$$

Here $*$ is the Hodge star operator which sends p-form to $d-p$-form. Now we have

$$\begin{aligned}
-(\Box_{(p)}^{B_a}\omega, \eta) &= -\int_M (\Box_{(p)}^{B_a}\omega, \eta) dm \\
&= \int_M (d\delta\omega + \delta d\omega, \eta) dm \\
&= \int_M (\delta\omega, \delta\eta) dm + \int_{\partial M} \delta\omega \wedge *\eta + \int_M (d\omega, d\eta) dm - \int_{\partial M} \eta \wedge *d\omega \\
&= \int_M (\delta\omega, \delta\eta) dm + \int_M (d\omega, d\eta) dm
\end{aligned}$$

$$= \mathcal{E}^{B_a}_{(p)}(\omega, \eta).$$

Here $d\sigma$ is the surface element of ∂M. We used the boundary condition B_a to get $\delta\omega \wedge *\eta = 0$ on ∂M. To see this, $*\eta$ contains dr since $B_a\eta = 0$. On the other hand, $\delta\omega$ does not contain dr since $B_a\delta\omega = 0$. Hence $\delta\omega \wedge *\eta$ contains dr. Therefore $\delta\omega \wedge *\eta = 0$ on ∂M since $dr = 0$ on ∂M. $\eta \wedge *d\omega$ also vanishes on ∂M.

We can show similar identity for $\Box^{B_r}_{(p)}$

Next we see the commutation relation. Formally the commutation relation $d\Box = \Box d$ holds i.e., at least in the interior of M. But the boundary condition is involved and we need some arguments.

Take $\omega \in \mathrm{Dom}(\Box^{B_a}_{(p)})$, $\eta \in \mathrm{Dom}(\Box^{B_a}_{(p+1)})$

$$\begin{aligned}
-\int_M (\Box\omega, \delta\eta)dm &= \int_M (d\delta\omega + \delta d\omega, \delta\eta)dm \\
&= \int_M (\delta\omega + \delta d\omega, \delta\delta\eta)dm + \int_{\partial M} \delta\omega \wedge *\delta\eta \\
&\quad + \int_M (d\omega, d\delta\eta)dm - \int_{\partial M} \delta\eta \wedge *dw \\
&= \int_M (d\omega, d\delta\eta)dm + \int_M (dd\omega, d\eta)dm \\
&= \int_M (d\omega, d\delta\eta)dm + \int_M (d\omega, \delta d\eta)dm + \int_{\partial M} d\omega \wedge *d\eta \\
&= -\int_M (d\omega, \Box\eta)dm.
\end{aligned}$$

We used the boundary condition B_a. Absolutely same argument works for B_r.

Now invoking the essential self-adjointness of $\Box^{B_a}_{(p)}$ and $\Box^{B_r}_{(p)}$, we can get the semigroup commutation. Of course, we must choose the same boundary condition.

We proceed to the issue of semigroup domination. For 0-forms, i.e., scalar functions, the boundary condition B_a corresponds to the Neumann condition: $\frac{\partial f}{\partial r} = 0$ on ∂M. On the other hand, B_r corresponds to the Dirichlet boundary condition: $f = 0$ on ∂M. From now on, for scalar functions, we always consider the Neumann Laplacian $\Box^{B_a}_{(0)}$ and we denote it by Δ for simplicity. We also denote the associated Dirichlet form by $\mathcal{E}$, i.e., $\mathcal{E} = \mathcal{E}^{B_a}_{(0)}$. We do not use the Dirichlet form (for scalar functions) with Dirichlet boundary condition. The reason is that the semigroup generated by Hodge-Kodaira Laplacian can not be dominated by the Dirichlet semigroup since the diffusion dies after hitting the boundary.

In order to apply our theorem, we have to check condition $(\vec{\Gamma}_\lambda)$. In particular, to show (2.8), we have to calculate $-\mathcal{E}((\theta, \eta), f) + \vec{\mathcal{E}}(f\theta, \eta) + \vec{\mathcal{E}}(\theta, f\eta)$.

First we note that

$$\begin{aligned}
-\int_M \Delta(\theta, \eta) f dm &= \int_M \nabla^*\nabla(\theta, \eta) f dm \\
&= \int_M \delta d(\theta, \eta) f dm \\
&= \int_M (d(\theta, \eta), df)dm + \int_{\partial M} f\langle d(\theta, \eta), N\rangle d\sigma
\end{aligned}$$

$$(\because \text{ Green-Stokes formula})$$
$$= \mathcal{E}((\theta,\eta),f) + \int_{\partial M} f\nabla_N(\theta,\eta)d\sigma.$$

Here $d\sigma$ denotes the surface element of ∂M and we recall that N is the inner normal vector. Hence

$$\mathcal{E}((\theta,\eta),f) = -\int_M \Delta(\theta,\eta)f dm - \int_{\partial M} f\nabla_N(\theta,\eta)d\sigma.$$

On the other hand, it holds that

$$\Delta(\theta,\eta) + (\nabla^*\nabla\theta,\eta) + (\theta,\nabla^*\nabla\eta) = 2(\nabla\theta,\nabla\eta).$$

By the Weitzenböck formula $-\Box^B_{(p)} = \nabla^*\nabla + R_{(p)}$, we have

$$\Delta(\theta,\eta) - (\Box^B_{(p)}\theta,\eta) - (\theta,\Box^B_{(p)}\eta) = 2(\nabla\theta,\nabla\eta) + 2R_{(p)}(\theta,\eta).$$

We do not give the explicit form of $R_{(p)}$, but we just say that $R_{(p)}$ can be written in terms of the curvature. Now by combining these identities, we can obtain the following identity:

$$\begin{aligned} &-\mathcal{E}((\theta,\eta),f) + \mathcal{E}^B_{(p)}(f\theta,\eta) + \mathcal{E}^B_{(p)}(\theta,f\eta) \\ &= 2\int_M (\nabla\theta,\nabla\eta)f dm + 2\int_M R_{(p)}(\theta,\eta)f dm + 2\int_{\partial M} \nabla_N(\theta,\eta)f d\sigma \end{aligned} \tag{4.9}$$

In fact

$$\begin{aligned} &-\mathcal{E}((\theta,\eta),f) + \mathcal{E}^B_{(p)}(f\theta,\eta) + \mathcal{E}^B_{(p)}(\theta,f\eta) \\ &= \int_M \Delta(\theta,\eta)\cdot f dm + 2\int_{\partial M} \nabla_N(\theta,\eta)f d\sigma - \int_M \{(\Box^B_{(p)}\theta,\eta)f - (\theta,\Box^B_{(p)}\eta)f\}dm \\ &= 2\int_M (\nabla\theta,\nabla\eta)f dm + 2\int_M R_{(p)}(\theta,\eta)f dm + 2\int_{\partial M} \nabla_N(\theta,\eta)f d\sigma \end{aligned}$$

Therefore $\vec{\Gamma}$ in ($\vec{\Gamma}'_\lambda$-2) is given as

$$\vec{\Gamma}(\theta,\eta) = (\nabla\theta,\nabla\eta) + R_{(p)}(\theta,\eta). \tag{4.10}$$

We can easily check the condition ($\vec{D}$) in this case. The rest is to show that the third part of L.H.S. in (4.9) corresponds to $\vec{\gamma}d\sigma$ in (2.8).

We take a local frame $\{\omega^1,\ldots,\omega^{d-1},dr\}$ in T^*M so that it forms a orthonormal basis and ω^j is parallel along the geodesic $t \to (x^1,\ldots,x^{d-1},t)$. We also denote the dual basis of $\{\omega^1,\ldots,\omega^{d-1},dr\}$ by $\{X_1,\ldots,X_{d-1},N\}$. Note that On the other hand,

$$\begin{aligned} d\omega^k[X_j,N] &= \langle\nabla_N\omega^k, X_j\rangle - \langle\nabla_{X_j}\omega^k, N\rangle \\ &= -\langle\nabla_{X_j}\omega^k, N\rangle \end{aligned}$$

Here we used that $\nabla_N\omega^k = 0$ since ω^k is parallel along the path $t \mapsto (x^1,\ldots,x^{d-1},t)$. Moreover we note that $\langle\nabla_{X_j}\omega^k, N\rangle = (\alpha(X_j,X_k),N)$ where α is the second fundamental form on ∂M. Thus we have

$$d\omega^k[X_j,N] = -(\alpha(X_j,X_k),N). \tag{4.11}$$

Using the second fundamental form, we introduce an operator A as follows:

$$A\omega^i = \sum_{j=1}^{d-1}(\alpha(X_i,X_j),N)\omega^j. \tag{4.12}$$

A can be extended to a linear operator from $A^1(\partial M)$ to $A^1(\partial M)$ and it is independent of the choice of $\{\omega^j\}$. Moreover we define $d\Gamma(A)\colon A^p(\partial M) \to A^p(\partial M)$ as

$$d\Gamma(A)(\theta_1 \wedge \cdots \wedge \theta_p) = \sum_{j=1}^{p} \theta_1 \wedge \cdots \wedge A\theta_j \wedge \cdots \wedge \theta_p. \tag{4.13}$$

Then we can have the follolwing.

Lemma 4.1. *On the boundary ∂M, it holds that for $I = \{i_1 < \cdots < i_p\} \subseteq \{1, \ldots, d-1\}$,*

$$(d\omega^I)_n = -(-1)^{|I|} d\Gamma(A)\omega^I. \tag{4.14}$$

Proof. From (4.11) and (4.12), it holds that

$$\sum_{k=1}^{d-1} d\omega^j[X_k, N]\omega^k = -A\omega^j.$$

Hence

$$\begin{aligned}
d\omega^I &= d(\omega^{i_1} \wedge \cdots \wedge \omega^{i_p}) \\
&= \sum_{\mu=1}^{p} (-1)^\mu \omega^{i_1} \wedge \cdots \wedge d\omega^{i_\mu} \wedge \cdots \wedge \omega^{i_p} \\
&= \sum_{\mu=1}^{p} \sum_{1 \le k \le l \le d-1} (-1)^\mu \omega^{i_1} \wedge \cdots \wedge d\omega^{i_\mu}[X_k, X_l](\omega^k \wedge \omega^l) \wedge \cdots \wedge \omega^{i_p} \\
&\quad + \sum_{\mu=1}^{p} \sum_{k=1}^{d-1} (-1)^\mu \omega^{i_1} \wedge \cdots \wedge d\omega^{i_\mu}[X_k, N](\omega^k \wedge dr) \wedge \cdots \wedge \omega^{i_p} \\
&= \sum_{\mu=1}^{p} \sum_{1 \le k \le l \le d-1} (-1)^\mu \omega^{i_1} \wedge \cdots \wedge d\omega^{i_\mu}[X_k, X_l](\omega^k \wedge \omega^l) \wedge \cdots \wedge \omega^{i_p} \\
&\quad - (-1)^p \sum_{\mu=1}^{p} \sum_{k=1}^{d-1} \omega^{i_1} \wedge \cdots \wedge A\omega^{i_\mu} \wedge \cdots \wedge \omega^{i_p} \wedge dr.
\end{aligned}$$

Now, recalling the definition of $d\Gamma(A)$, we have

$$(d\omega^I)_n = -(-1)^p d\Gamma(A)\omega^I$$

which is the desired result. $\square$

Now we can deal with the absolute boundary condition.

Theorem 4.2. *For p-forms θ, $\eta \in \mathrm{Dom}(\mathcal{E}^{B_a}_{(p)})$ and $f \in C^\infty(M)$, it holds that*

$$\begin{aligned}
&-\mathcal{E}((\theta, \eta), f) + \mathcal{E}^{B_a}_{(p)}(f\theta, \eta) + \mathcal{E}^{B_a}_{(p)}(\theta, f\eta) \\
&\quad = 2\int_M (\nabla\theta, \nabla\eta) f\, dm + 2\int_M R_{(p)}(\theta, \eta) f\, dm + 2\int_{\partial M} (d\Gamma(A)\theta_t, \eta_t) f\, d\sigma
\end{aligned} \tag{4.15}$$

Proof. First we write the boundary condition in terms of local coordinate. We decompose θ as follows:

$$\theta = \sum_{I:|I|=p} f_I \omega^I + \sum_{J:|J|=p-1} g_J \omega^J \wedge dr.$$

Noting that $g_J = 0$ on ∂M, we have,

$$\begin{aligned} d\theta &= \sum_{I:|I|=p} df_I \wedge \omega^I + \sum_{I:|I|=p} f_I d\omega^I \\ &\quad + \sum_{J:|J|=p-1} dg_J \wedge \omega^J \wedge dr + \sum_{J:|J|=p-1} g_J d\omega^J \wedge dr \\ &= \sum_{I:|I|=p} df_I \wedge \omega^I + \sum_{I:|I|=p} f_I d\omega^I. \end{aligned}$$

Since $(d\theta)_n = 0$, we have

$$\begin{aligned} 0 &= \sum_{I:|I|=p} \langle df_I, N\rangle dr \wedge \omega^I + \sum_{I:|I|=p} f_I (d\omega^I)_n \\ &= \sum_{I:|I|=p} (-1)^{|I|} \langle df_I, N\rangle \omega^I \wedge dr - \sum_{I:|I|=p} (-1)^{|I|} f_I d\Gamma(A)\omega^I \wedge dr \\ &= \sum_{I:|I|=p} (-1)^{|I|} \langle df_I, N\rangle \omega^I \wedge dr - (-1)^{|I|} d\Gamma(A)\theta_t \wedge dr. \end{aligned}$$

Now we have

$$\sum_{I:|I|=p} \langle df_I, N\rangle \omega^I = d\Gamma(A)\theta_t.$$

Now we calculate $(\nabla_N \theta, \eta)$.

$$\begin{aligned} (\nabla_N \theta, \eta) &= \Big(\nabla_N \Big\{ \sum_{I:|I|=p} f_I \omega^I + \sum_{J:|J|=p-1} g_J \omega^J \wedge dr \Big\}, \eta_t \Big) \\ &= \big(\sum_{I:|I|=p} (\nabla_N f_I) \omega^I, \eta_t \big) \qquad (\because \nabla_N \omega^I = 0) \\ &= (d\Gamma(A)\theta_t, \eta_t). \end{aligned}$$

This completes the proof $\square$

Nest we consider the relative boundary condition. This can be easily done by noting that the Hodge operation gives an isometry which interchanges the absolute boundary condition and the relative boundary condition. We denote the Hodge operation on ∂M by $*$. Then we have

Theorem 4.3. *For p-forms θ, $\eta \in \mathrm{Dom}(\mathcal{E}_{(p)}^{B_r})$ and $f \in C^\infty(M)$, it holds that*

$$\begin{aligned} &-\mathcal{E}((\theta, \eta), f) + \mathcal{E}_{(p)}^{B_r}(f\theta, \eta) + \mathcal{E}_{(p)}^{B_r}(\theta, f\eta) \\ &\quad = 2\int_M (\nabla\theta, \nabla\eta) f dm + 2\int_M R_{(p)}(\theta, \eta) f dm + 2\int_{\partial M} (*^{-1} d\Gamma(A) * \theta_n, \eta_n) f d\sigma. \end{aligned} \tag{4.16}$$

Now we can apply Theorem 2.1. For example, if $\mathrm{Ric} \geq \lambda I$ and A is non-negative definite, then we have

$$|\vec{T}_t\theta| \leq e^{-\lambda t}|\theta|, \qquad \text{for } \theta \in A^1(M)$$

where $\vec{T}_t$ is a semigroup generated by $\Box_{(1)}^{B_a}$. Moreover we can give an estimate of the constant for the logarithmic Sobolev inequality.

Taking $f = 1$ in (4.15), we have

$$\mathcal{E}_{(p)}^{B_a}(\theta, \eta) = \int_M (\nabla\theta, \nabla\eta)dm + \int_M R_{(p)}(\theta, \eta)dm + \int_{\partial M} (d\Gamma(A)\theta_t, \eta_t)d\sigma. \tag{4.17}$$

This identity is already known (see e.g., Schwarz [17, Thm 2.1.5]).

In particular, if $p = 1$, it holds that

$$\mathcal{E}_{(1)}^{B_a}(\theta, \eta) = \int_M (\nabla\theta, \nabla\eta)dm + \int_M \mathrm{Ric}(\theta, \eta)dm + \int_{\partial M} (A\theta_t, \eta_t)d\sigma. \tag{4.18}$$

Here Ric denotes the Ricci curvature. Using this identity, we can show the Lichnèrowicz theorem. Let Δ be the Laplacian with Neumann boundary condition on M. We assume that $\mathrm{Ric} \geq (d-1)\rho I$ with $\rho > 0$ and A is non-negative definite. Then we have

$$\lambda_1 \geq \rho d \tag{4.19}$$

where λ_1 is the first non-zero eigenvalue of Δ.

To see this, note that

$$\begin{aligned}
(\Delta f, \Delta f) &= -(ddf, ddf) - (\delta df, \delta df) \\
&= \mathcal{E}_{(1)}^{B_a}(\nabla f, \nabla f) \\
&= \int_M (\nabla^2 f, \nabla^2 f)dm + \int_M \mathrm{Ric}(\nabla f, \nabla f)dm + \int_{\partial M} (A(\nabla f)_t, (\nabla f)_t)d\sigma.
\end{aligned}$$

Now, by the standard argument, we easily have

$$\frac{d-1}{d}(\Delta f, \Delta f) \geq \int_M \mathrm{Ric}(\nabla f, \nabla f)dm \geq (d-1)\rho\mathcal{E}(f, f).$$

Now we can get the desired result.

Moreover it is known that the identity holds if and only if M is isomorphic to the hemisphere (Xia [20]). Similar result holds for the Laplacian with Dirichlet boundary condition, see Reilly [14].

References

[1] S. Aida, T. Masuda and I. Shigekawa, Logarithmic Sobolev inequalities and exponential integrability, *J. Func. Anal.*, **126** (1994), 83–101.

[2] D. Bakry and M. Emery, Diffusions hypercontractives, *Séminaire de Prob.* XIX, Lecture Notes in Math., vol. 1123, pp. 177–206, Springer-Verlag, Berlin-Heidelberg-New York, 1985.

[3] N. Bouleau and F. Hirsch, "*Dirichlet forms and analysis on Wiener space*," Walter de Gruyter, Berlin-New York, 1991.

[4] J-D. Deuschel and D. W. Stroock, "*Large deviations*," Academic Press, San Diego, 1989.

[5] J-D. Deuschel and D. W. Stroock, Hypercontractivity and spectral gap of symmetric diffusions with applications to stochastic Ising models, *J. Funct. Anal.*, **92** (1990), 30–48.

[6] H. Donnelly and P. Li, Lower bounds for the eigenvalues of Rimannian manifolds, *Michigan Math. J.*, **29** (1982), 149–161.

[7] M. Fukushima, "*Dirichlet forms and Markov Processes*," North Holland/Kodansha, Amsterdam/ Tokyo, 1980.
[8] L. Gross, Logarithmic Sobolev inequalities, *Amer. J. Math.*, **97** (1975), 1061–1083.
[9] H. Hess, R. Schrader and D.A. Uhlenbrock, Domination of semigroups and generalization of Kato's inequality, *Duke Math. J.*, **44** (1977), 893–904.
[10] H. Hess, R. Schrader and D.A. Uhlenbrock, Kato's inequality and the spectral distribution of Laplacians on compact Riemannian manifolds, *J. Diff. Geom.*, **15** (1980), 27–37.
[11] Z.-M. Ma and M. Röckner, "*Introduction to the theory of (non-symmetric) Dirichlet forms*," Springer-Verlag, Berlin-Heidelberg-New York, 1992.
[12] A. Méritet, Théorème d'anulation pour la cohomologie absolue d'une variété riemannienne a bord, *Bull. Sc. math.*, 2^e série, **103** (1979), 379-400.
[13] E. Ouhabaz, Invariance of closed convex sets and domination criteria for semigroups *Potential Analysis*, **5** (1996), 611–625.
[14] R. C. Reilly, Applications of the Hessian operator in a Riemannina manifold, *Indiana Univ. Math. J.*, **26** (1977), 459–472.
[15] D. B. Ray and I. M. Singer, R-torsion and the Laplacian on Riemannian manifolds, *Adv. in Math.*, **7** (1971), 145–210.
[16] I. Shigekawa, L^p contraction semigroups for vector valued functions, *J. Funct. Anal.*, **147** (1997), 69–108.
[17] G. Schwarz, "*Hodge-decomposition—a method for solving boundary value problems*," Lecture Notes in Math., vol. 1607, Springer-Verlag, Berlin-Heidelberg-New York, 1995.
[18] B. Simon, An abstract Kato's inequality for generators of positivity preserving semigroups, *Indiana Univ. Math. J.*, **26** (1997), 1069–1073.
[19] B. Simon, Kato's inequality and the comparison of semigroups, *J. Funct. Anal.*, **32** (1979), 97–101.
[20] C.-Y. Xia, The first nonzero eigenvalue for manifolds with Ricci curvature having positive lower bound, in "it Chinese mathematics into the 21st century," ed. W.-T. Wu and M.-D. Cheng, pp. 243–249, Peking Univ. Press, Beijing, 1991.

Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan
E-mail address: `ichiro@kusm.kyoto-u.ac.jp`
URL: `http://www.kusm.kyoto-u.ac.jp/~ichiro/`