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Orlicz norm equivalence for the Ornstein-Uhlenbeck operator

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Dedicated to Professor Kiyosi Ito on the occasion of his 88th birthday

Abstract.

The Meyer equivalence on an abstract Wiener space states that the L^p-norm of square root of the Ornstein-Uhlenbeck operator is equivalent to L^p-norm of the Malliavin derivative. We prove the equivalence in the framework of Orlicz space. We also discuss the logarithmic Sobolev inequality in L^p setting and higher order logarithmic Sobolev inequality.

§1. Introduction

Let (B, H, μ) be an abstract Wiener space: B is a separable real Banach space, H is a separable real Hilbert space which is embedded densely and continuously in B and μ is a Gaussian measure with

$$\int_B \exp\{\sqrt{-1}\,{}_{B^*}\langle l, x\rangle_B\}\mu(dx) = \exp\Big\{-\frac{1}{2}|l|_{H^*}^2\Big\}, \quad l \in B^* \hookrightarrow H^*.$$

On an abstract Wiener space, the Ornstein-Uhlenbeck semigroup is defined as

$$T_t f(x) = \int_B f(e^{-t}x + \sqrt{1-e^{-2t}}y)\mu(dy). \tag{1}$$

The generator of the semigroup $\{T_t\}$ is called the Ornstein-Uhlenbeck operator and we denote it by L. Then the following Meyer equivalence is well-known: for any $1 < p < \infty$, there exists positive constants C_1 and C_2 such that

$$C_1\{\|Df\|_p + \|f\|_p\} \le \|\sqrt{1-L}f\|_p \le C_2\{\|Df\|_p + \|f\|_p\}. \tag{2}$$

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Here D is the Malliavin H-derivation and $\| \ \|_p$ is the L^p-norm. The constants C_1 and C_2 depend only on p.

In this paper we show that similar inequalities hold in the framework of Orlicz space, i.e., the above inequalities hold for the Orlicz norm in place of L^p-norm. Typical example we are in mind is $L^p \log^\beta L$. As an application, we discuss the logarithmic Sobolev inequality in L^p setting and higher order logarithmic Sobolev inequality.

§2. Orlicz space

In this section, we review the Orlicz space (see, e.g., [1] or [8] for details). First we need the notion of Young function. Young function is a function Φ defined as

$$\Phi(x) = \int_0^x \phi(t)dt, \quad x \geq 0 \tag{3}$$

where ϕ is a non-negative, right continuous, non-decreasing function. If, in addition, ϕ satisfies $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$, $\phi(\infty) = \infty$, then Φ is called a nice Young function or N-function. Define ψ by

$$\psi(u) = \inf\{t\,;\,\phi(t) > u\}.$$

ψ is right continuous and non-decreasing. The function Ψ defined by

$$\Psi(y) = \int_0^y \psi(u)du, \quad y \geq 0$$

is called a complementary function. The following properties are fundamental.

$$xy \leq \Phi(x) + \Psi(y), \tag{4}$$

$$x\phi(x) = \Phi(x) + \Psi(\phi(x)). \tag{5}$$

(4) is called the Young inequality.

The Orlicz space associated with Φ is defined as follows. Let (M, m) be a measure space and Φ be a nice Young function. Define a norm $\| \ \|_\Phi$ by

$$\|f\|_\Phi := \inf\{\lambda > 0\,;\, \int_M \Phi(|f|/\lambda)dm \leq 1\}. \tag{6}$$

$L^\Phi(m)$ is the set of all measurable functions f which satisfy $\|f\|_\Phi < \infty$. We call $L^\Phi(m)$ an Orlicz space. It is a Banach space with the norm $\| \ \|_\Phi$. If Φ satisfies the Δ_2 condition, i.e., there exists a constant C such that $\Phi(2x) \leq C\Phi(x)$, then the dual space is identified with $L^{\Psi}(m)$, Ψ being the complementary function of Φ.

We introduce some classes of functions.

Definition 2.1. For non-negative constant α, we define a set of functions $L(\alpha)$, $U(\alpha)$ as follows:

(i) $\phi \in L(\alpha) \overset{\text{def}}{\iff} \alpha\phi(t) \leq t\phi'(t), \quad \forall t > 0.$

(ii) $\phi \in U(\alpha) \overset{\text{def}}{\iff} t\phi'(t) \leq \alpha\phi(t), \quad \forall t > 0.$

The following inequality for semimartingales is important in our later argument.

Let (Z_t) $(t \in [0, \infty])$ be a non-negative submartingale. We assume that (Z_t) is right continuous and has left hand limits. By the Doob-Meyer decomposition theorem, (Z_t) can be decomposed as

$$Z_t = M_t + A_t$$

where (M_t) is a martingale and (A_t) is an increasing process. We assume that (A_t) is continuous and $A_0 = 0$. If $\Phi \in U(\alpha)$, then the following inequality holds (see [4, Theorem VI.99]):

$$E[\Phi(A_\infty)] \leq E[\Phi(\alpha Z_\infty)]. \tag{7}$$

Further, a generalization of the Doob's inequality also holds. It is stated as follows (see [4, Chapter VI, Section 3]). We assume that $\Phi \in L(\alpha)$ for an $\alpha > 1$. Then, setting $Z_t^* := \sup_{s \leq t} Z_s$, it holds that

$$E[\Phi(Z_\infty^*)] \leq E[\Phi(\alpha Z_\infty)]. \tag{8}$$

From this inequality, we can have the following maximal ergodic inequality.

$$\int_B \Phi(\sup_{t \geq 0} |T_t f(x)|)\mu(dx) \leq \int_B \Phi(\alpha |f(x)|)\mu(dx). \tag{9}$$

Here $\{T_t\}$ is the Ornstein-Uhlenbeck semigroup on an abstract Wiener space (B, H, μ).

§3. Littlewood-Paley inequality

Let (B, H, μ) be an abstract Wiener space and K be a separable Hilbert space. $\{T_t\}$ is the Ornstein-Uhlenbeck semigroup on $L^p(E, \mu; K)$ defined by (1). For $\alpha > 0$, set

$$T_t^{(\alpha)} = e^{-\alpha t} T_t.$$

Then the generator of $\{T_t^{(\alpha)}\}$ is $L-\alpha$. We further define a semigroup $\{Q_t^{(\alpha)}\}$ by subordination as follows:

$$Q_t^{(\alpha)} = \int_0^\infty T_s^{(\alpha)} \lambda_t(ds) = \int_0^\infty e^{-\alpha s} T_s \lambda_t(ds)$$

Here λ_t is a probability measure on $[0,\infty)$ whose Laplace transform is given by

$$\int_0^\infty e^{-\gamma s} \lambda_t(ds) = e^{-\sqrt{\gamma} t}.$$

When $\alpha = 0$, $Q_t^{(0)}$ is simply denoted by Q_t and called the Cauchy semigroup. For $F \in L^\Phi(B,\mu;K)$, it holds that

$$\|Q_t^{(\alpha)} F\|_\Phi \le e^{-\sqrt{\alpha} t} \|F\|_\Phi$$

and $\{Q_t^{(\alpha)}\}$ is a strongly continuous semigroup on L^Φ. The generator will be denoted by $-\sqrt{\alpha - L}$.

We denote by $\mathcal{P}(K)$ a set of all functions $f\colon B \to K$ which can be expressed as

$$f(x) = \sum_i p_i(\langle l_1, x\rangle), \dots, \langle l_n, x\rangle) k_i$$

where p_i is a polynomial on $\mathbb{R}^n$ and $k_1, \dots, k_n \in K$, $l_1, \dots, l_n \in B^*$.

For $f \in \mathcal{P}(K)$, define

$$\begin{aligned} g^{\to} f(x,t) &= |\partial_t Q_t^{(\alpha)}(x,f)|_K, \\ g^{\uparrow} f(x,t) &= |D Q_t^{(\alpha)}(x,f)|_{\mathrm{HS}}, \\ g f(x,t) &= \sqrt{g^{\to} f(x,t)^2 + g^{\uparrow} f(x,t)^2}. \end{aligned}$$

Here $Q_a^{(\alpha)}(x,f) = Q_a^{(\alpha)} f(x)$ and the norm $|\ |_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm. $g^{\to} f$, $g^{\uparrow} f$, gf all depend on α but we fix it throughout the argument and suppress it for simplicity. We further define

$$\begin{aligned} G^{\to} f(x) &= \left\{ \int_0^\infty t g^{\to} f(x,t)^2 dt \right\}^{1/2}, \\ G^{\uparrow} f(x) &= \left\{ \int_0^\infty t g^{\uparrow} f(x,t)^2 dt \right\}^{1/2}, \\ G f(x) &= \left\{ \int_0^\infty t g f(x,t)^2 dt \right\}^{1/2}. \end{aligned}$$

We call them Littlewood-Paley G-functions.

Our aim in this section is to prove the following theorem.

Theorem 3.1. *Assume that $\Phi \in L(\alpha) \cap U(\beta)$ for constants $1 < \alpha < \beta$. Further assume that ϕ is either convex or concave. Then we have*

$$\|\Phi(Gf)\|_1 \lesssim \|\Phi(|f|)\|_1, \tag{10}$$

$$\|\Phi(|f|)\|_1 \lesssim \|\Phi(G^{\rightharpoonup} f)\|_1. \tag{11}$$

In the above theorem, $A \lesssim B$ stands for $A \le CB$ for a positive constant C that is independent of f. We use this convention in the sequel without mentioning.

We give a probabilistic proof. To do this, we take the Ornstein-Uhlenbeck process (X_t) on B, i.e., the diffusion process generated by L. We also take a process (B_t) on $\mathbb{R}$ generated by $\frac{d^2}{da^2}$. We assume that the initial distribution of (X_t) is the stationary measure μ so that the process becomes stationary. We denote the starting point of the Brownian motion (B_t) by N. E_N stands for the integration with respect to this measure. Later we let $N \to \infty$.

Now, for $f \in \mathcal{P}(K)$, set $u(x,a) = Q_a^{(\alpha)}(x,f)$. Then $u(x,a)$ satisfies

$$\begin{cases} u(x,0) = f(x) \\ L_x u(x,a) + \partial_a^2 u(x,a) - \alpha u(x,a) = 0. \end{cases} \tag{12}$$

Define a stopping time τ by

$$\tau = \inf\{t > 0 \mid B_t = 0\}.$$

Then we can think of $u(X_t, B_t)$ for $t \le \tau$. Set

$$\begin{aligned} M_t &= Q_{B_{t\wedge\tau}}(X_{t\wedge\tau}, f) - \alpha \int_0^{t\wedge\tau} Q_{B_s}(X_s, f) ds \\ &= u(X_{t\wedge\tau}, B_{t\wedge\tau}) - \alpha \int_0^{t\wedge\tau} Q_{B_s}(X_s, f) ds. \end{aligned}$$

Then (M_t) is a martingale with $M_0 = Q_{B_0} f(X_0)$. The quadratic variation is given as

$$\langle M \rangle_t = 2 \int_0^{t\wedge\tau} g f^2(X_s, B_s) ds. \tag{13}$$

Therefore we have

$$\begin{aligned} d|u|^2 &= 2(u, dM) + 2\alpha |u|^2 dt + \langle dM, dM \rangle \\ &= 2(u, dM) + (2\alpha |u|^2 + 2 g f^2) dt \end{aligned} \tag{14}$$

Now set

$$Z_t = |u(X_{t\wedge\tau}, B_{t\wedge\tau})|^2.$$

(Z_t) is a non-negative submartingale. To compute $\Phi(\sqrt{Z_t})$, we approximate it as follows. Take any $\varepsilon > 0$ and set $F(x) = \Phi(\sqrt{x+\varepsilon})$. Recall that ϕ is either convex or concave. We divide into tow cases.

(i) ϕ is concave.

We need the following proposition.

Proposition 3.2. *Assume $\Phi \in L(\alpha)$ $(\alpha > 1)$. Then it holds that, for u, $v \geq 0$,*

$$\Phi(v) \leq \frac{1}{\alpha - 1}\Big(\frac{1}{2}\phi'(u)v^2 + \Phi(u)\Big). \tag{15}$$

Proof. From the assumption, $\alpha\Phi(x) \leq x\phi(x)$ holds. Since ϕ is concave, $\Phi(x) \geq \frac{1}{2}x\phi(x)$ which leads to $\alpha \leq 2$. Hence (15) clearly holds when $u \geq v$.

If $v \geq u$, we have

$$\begin{aligned}
&|\{(x,y)\,;\, 0 \leq x \leq u,\, \phi(x) \leq y \leq \phi(u)\}| \leq \frac{1}{2}u\phi(u) \\
&|\{(x,y)\,;\, 0 \leq x \leq u,\, \phi(u) \leq y \leq \phi(v)\}| \leq u(\phi(v) - \phi(u)) \leq u\phi'(u)(v-u) \\
&|\{(x,y)\,;\, u \leq x \leq v,\, \phi(x) \leq y \leq \phi(v)\}| \leq \frac{1}{2}(v-u)^2\phi'(u).
\end{aligned}$$

These are easily obtained by observing the graph.

Summing up three terms of the left-hand side and $\Phi(v)$, we have $v\phi(v)$. Therefore

$$\frac{1}{2}u\phi(u) + u\phi'(u)(v-u) + \frac{1}{2}(v-u)^2\phi'(u) + \Phi(v) \geq v\phi(v) \geq \alpha\Phi(v).$$

Hence we have

$$\begin{aligned}
(\alpha - 1)\Phi(v) &\leq \frac{1}{2}u\phi(u) + \phi'(u)(v-u)\Big(u + \frac{1}{2}v - \frac{1}{2}u\Big) \\
&= \frac{1}{2}u\phi(u) + \frac{1}{2}\phi'(u)(v^2 - u^2) \\
&\leq \Phi(u) + \frac{1}{2}\phi'(u)v^2
\end{aligned}$$

which is the desired result. Q.E.D.

The derivatives of $F(x) = \Phi(\sqrt{x+\varepsilon})$ are

$$F'(x) = \Phi'(\sqrt{x+\varepsilon})\frac{1}{2\sqrt{x+\varepsilon}},$$
$$F''(x) = \Phi''(\sqrt{x+\varepsilon})\frac{1}{4(x+\varepsilon)} + \Phi'(\sqrt{x+\varepsilon})\frac{1}{2}\Big(-\frac{1}{2}\Big)\frac{1}{\sqrt{x+\varepsilon}^3}.$$

By the Itô formula,

$$\begin{aligned}
d\Phi(\sqrt{Z_t+\varepsilon}) &= \frac{\Phi'(\sqrt{Z_t+\varepsilon})}{2\sqrt{Z_t+\varepsilon}}dZ_t \\
&\quad + \frac{1}{2}\Big\{\frac{\Phi''(\sqrt{Z_t+\varepsilon})}{4(Z_t+\varepsilon)} - \frac{1}{4}\frac{\Phi'(\sqrt{Z_t+\varepsilon})}{\sqrt{Z_t+\varepsilon}^3}\Big\}\langle dZ, dZ\rangle \\
&= \frac{\phi(\sqrt{|u|^2+\varepsilon})}{2\sqrt{|u|^2+\varepsilon}}\{2(u,dM) + 2(\alpha|u|^2+gf^2)dt\} \\
&\quad + \frac{1}{2}\Big\{\frac{\phi'(\sqrt{|u|^2+\varepsilon})}{4(|u|^2+\varepsilon)} - \frac{1}{4}\frac{\phi(\sqrt{|u|^2+\varepsilon})}{\sqrt{|u|^2+\varepsilon}^3}\Big\}\langle dZ, dZ\rangle \\
&= \frac{\phi(\sqrt{|u|^2+\varepsilon})}{\sqrt{|u|^2+\varepsilon}}(u,dM) + \frac{\phi(\sqrt{|u|^2+\varepsilon})}{\sqrt{|u|^2+\varepsilon}}(\alpha|u|^2+gf^2)dt \\
&\quad + \frac{1}{8}\frac{1}{|u|^2+\varepsilon}\Big\{\phi'(\sqrt{|u|^2+\varepsilon}) - \frac{\phi(\sqrt{|u|^2+\varepsilon})}{\sqrt{|u|^2+\varepsilon}}\Big\}\langle dZ, dZ\rangle.
\end{aligned}$$

Now we note $\langle dZ, dZ\rangle \le 4|u|^2\langle dM, dM\rangle = 8|u|^2 gf^2 dt$. Further $\phi'(t) \le \phi(t)/t$ since ϕ is concave. We therefore have

$$\begin{aligned}
&\frac{\phi(\sqrt{|u|^2+\varepsilon})}{\sqrt{|u|^2+\varepsilon}}gf^2dt + \frac{1}{8}\frac{1}{|u|^2+\varepsilon}\Big\{\phi'(\sqrt{|u|^2+\varepsilon}) - \frac{\phi(\sqrt{|u|^2+\varepsilon})}{\sqrt{|u|^2+\varepsilon}}\Big\}\langle dZ, dZ\rangle \\
&\quad \ge \frac{\phi(\sqrt{|u|^2+\varepsilon})}{\sqrt{|u|^2+\varepsilon}}gf^2dt + \Big\{\phi'(\sqrt{|u|^2+\varepsilon}) - \frac{\phi(\sqrt{|u|^2+\varepsilon})}{\sqrt{|u|^2+\varepsilon}}\Big\}gf^2dt \\
&\quad = \phi'(\sqrt{|u|^2+\varepsilon})gf^2dt.
\end{aligned}$$

Integrating from 0 to τ and taking expectation, we have

$$\|\Phi(\sqrt{|f|^2+\varepsilon})\|_1 \ge E_N\Big[\int_0^\tau \phi'(\sqrt{|u|^2+\varepsilon})gf^2dt\Big]. \tag{16}$$

We will give an estimate from below of the right-hand side. We note that $f^*(x) := \sup_{t\ge 0}|T_tf(x)| \ge \sup_{a\ge 0}|u(x,a)|$.

$$E_N\Big[\int_0^\tau \phi'(\sqrt{|u|^2+\varepsilon})gf^2dt\Big] = \Big\|\int_0^\infty \phi'(\sqrt{|u|^2+\varepsilon})gf^2(\,\cdot\,,a)(a\wedge N)da\Big\|_1$$

$$\geq \left\| \int_0^\infty \phi'(\sqrt{f^{*2}+\varepsilon}) g f^2(\,\cdot\,, a)(a \wedge N) da \right\|_1 .$$
$$(\because \phi' \text{ is non-increasing})$$

Combining this with (16) and letting $N \to \infty$

$$\begin{aligned}\|\Phi(\sqrt{|f|^2+\varepsilon})\|_1 &\geq \left\| \int_0^\infty \phi'(\sqrt{f^{*2}+\varepsilon}) g f^2(\,\cdot\,, a) a da \right\|_1 \\ &= \|\phi'(\sqrt{f^{*2}+\varepsilon}) G f^2\|_1\end{aligned}$$

Now we use the inequality $\Phi(v) \leq \frac{1}{\alpha-1}(\frac{1}{2}\phi'(u)v^2 + \Phi(u))$ in Proposition 3.2 and get

$$\begin{aligned}\|\Phi(Gf)\|_1 &\lesssim \|\phi'(\sqrt{f^{*2}+\varepsilon}) G f^2\|_1 + \|\Phi(\sqrt{f^{*2}+\varepsilon})\|_1 \\ &\leq \|\Phi(\sqrt{|f|^2+\varepsilon})\|_1 + \|\Phi(\sqrt{f^{*2}+\varepsilon})\|_1 .\end{aligned}$$

Letting $\varepsilon \to 0$ and using the maximal ergodic inequality (9), we have

$$\|\Phi(Gf)\|_1 \lesssim \|\Phi(|f|)\|_1 + \|\Phi(f^*)\|_1 \lesssim \|\Phi(|f|)\|_1 .$$

This completes the proof in the case that ϕ is concave.

(ii) ϕ is convex.

Set $\tilde{\Phi}(x) = \Phi(\sqrt{x})$. Then $\tilde{\Phi}$ is convex. In fact, by differentiating, we have

$$\frac{d}{dx}\Phi(\sqrt{x}) = \Phi'(\sqrt{x})\frac{1}{2\sqrt{x}} = \frac{\phi(\sqrt{x})}{2\sqrt{x}} .$$

The function is increasing since ϕ is convex and so the convexity of $\tilde{\Phi}$ follows. Further $\tilde{\Phi} \in U(\alpha/2)$ since

$$\frac{x\tilde{\Phi}'(x)}{\tilde{\Phi}(x)} = \frac{x\Phi'(\sqrt{x})}{2\sqrt{x}\Phi(\sqrt{x})} = \frac{\sqrt{x}\Phi'(\sqrt{x})}{2\Phi(\sqrt{x})} .$$

The submartingale $Z_t = |u(X_{t\wedge\tau}, B_{t\wedge\tau})|^2$ is decomposed as a sum of a martingale and an increasing process as in (14). By using (7), we get

$$\begin{aligned}E_N[\tilde{\Phi}(\int_0^\tau gf(X_s, B_s)^2 ds)] &\lesssim E_N[\tilde{\Phi}(Z_\infty)] = E_N[\tilde{\Phi}(|f(X_\tau)|^2)] \\ &= E_N[\Phi(|f(X_\tau)|)] = \|\Phi(|f|)\|_1\end{aligned} \tag{17}$$

Now we introduce H-functions as follows.

$$H^{\rightarrow} f(x) = \left\{ \int_0^\infty tQ_t g^{\rightarrow} f(x,t)^2 \, dt \right\}^{1/2},$$
$$H^{\uparrow} f(x) = \left\{ \int_0^\infty tQ_t g^{\uparrow} f(x,t)^2 \, dt \right\}^{1/2},$$
$$Hf(x) = \left\{ \int_0^\infty tQ_t gf(x,t)^2 \, dt \right\}^{1/2}.$$

Then we have

$$\begin{aligned}
\|\Phi(Hf)\|_1 &= \|\tilde{\Phi}(Hf^2)\|_1 \\
&= \lim_{N\to\infty} \left\| \tilde{\Phi}\left(\int_0^\infty Q_a gf(\,\cdot\,,a)^2 (a \wedge N) da \right) \right\|_1 \\
&= \lim_{N\to\infty} \int_B \tilde{\Phi}\left(E_N \left[\int_0^\tau gf^2(X_t,B_t)dt \middle| X_\tau = x \right] \right) \mu(dx) \\
&\le \lim_{N\to\infty} \int_B E_N \left[\tilde{\Phi}\left(\int_0^\tau gf^2(X_t,B_t)dt \right) \middle| X_\tau = x \right] \mu(dx) \\
&\le \lim_{N\to\infty} E_N \left[\tilde{\Phi}\left(\int_0^\tau gf^2(X_t,B_t)dt \right) \right] \\
&\lesssim \|\Phi(|f|)\|_1. \quad (\because\ (17))
\end{aligned}$$

It is well-known that Gf is dominated by Hf (see [7]) and so (10) follows. (11) can be shown by a standard duality argument. This completes the proof of Theorem 3.1.

Using this theorem, the Meyer equivalence in Orlicz space, which is of our main interest, follows easily. In fact, the same proof as in L^p setting works (see e.g., [9]).

Theorem 3.3. *Assume that $\Phi \in L(\alpha) \cap U(\beta)$ for $1 < \alpha < \beta$ and that ϕ is either convex or concave. Then there exist positive constants C_1 and C_2 such that*

$$C_1\{\|Df\|_\Phi + \|f\|_\Phi\} \le \|\sqrt{1-L}f\|_\Phi \le C_2\{\|Df\|_\Phi + \|f\|_\Phi\}. \tag{18}$$

§4. Examples

We give some example of nice Young functions that satisfy the condition of Theorem 3.3. For indicies $p > 1$, $\beta \in \mathbb{R}$, $k \ge 1$, we set

$$\phi_{p,\beta,k}(x) = x^{p-1} \log^{p\beta}(k+x), \quad x \ge 0 \tag{19}$$

and define

$$\Phi_{p,\beta,k}(x) = \int_0^x \phi_{p,\beta,k}(y)dy. \tag{20}$$

We regards this as a Young function. The function does not satisfy the condition of Young function in general since β might be negative. We see when it is a Young function. To avoid complexity, we simply denote ϕ and Φ in place of $\phi_{p,\beta,k}$ and $\Phi_{p,\beta,k}$, respectively. Differentiating ϕ, we have

$$\begin{aligned}\phi'(x) &= (p-1)x^{p-2}\log^{p\beta}(k+x) + p\beta x^{p-1}\{\log^{p\beta-1}(k+x)\}\frac{1}{k+x} \\ &= x^{p-2}\log^{p\beta}(k+x)\Big\{p-1+p\beta\frac{x}{(k+x)\log(k+x)}\Big\}.\end{aligned}$$

We look for the condition so that ϕ' is positive. To do this, set

$$f(x) = \frac{x}{(k+x)\log(k+x)}. \tag{21}$$

If $k=1$, f takes its maximum 1 at $x=0$. If $k>1$, f takes its maximum at $x=\alpha$ where α is the solution of $k\log(k+x)-x=0$. We can see that $f(\alpha) \le \frac{1}{1+\log k}$. Therefore, in all cases of k, it holds that

$$0 \le \frac{x}{(k+x)\log(k+x)} \le \frac{1}{1+\log k}. \tag{22}$$

Now it is easy to see that Φ is a nice Young function if $p\Big(1+\dfrac{\beta}{1+\log k}\Big) \ge 1$. Further we easily have the following proposition.

Proposition 4.1. *ϕ satisfies following inequalities:*

$$(p-1)\phi(x) \le x\phi'(x) \le \Big(p-1+\frac{p\beta}{1+\log k}\Big)\phi(x), \quad \textit{for } \beta \ge 0, \tag{23}$$

$$\Big(p-1+\frac{p\beta}{1+\log k}\Big)\phi(x) \le x\phi'(x) \le (p-1)\phi(x), \quad \textit{for } \beta < 0. \tag{24}$$

Similar inequalities hold for Φ. To see this, we need the following proposition.

Proposition 4.2. *For positive constant α, it holds that*

(i) *if $\phi \in L(\alpha)$, then $\Phi \in L(\alpha+1)$,*
(ii) *if $\phi \in U(\alpha)$, then $\Phi \in U(\alpha+1)$.*

Proof. Suppose $\phi \in L(\alpha)$, i.e., $\alpha\phi(t) \le t\phi'(t)$. By integrating both hands, we have

$$\alpha\Phi(x) \le \int_0^x t\phi'(t)dt. \tag{25}$$

On the other hand, since $(t\phi(t))' = \phi(t) + t\phi'(t)$, we have

$$x\phi(x) = \int_0^x \{\phi(t) + t\phi'(t)\}dt$$

and hence

$$x\phi(x) - \Phi(x) = \int_0^x t\phi'(t)dt.$$

This combined with (25) leads

$$x\phi(x) \ge (\alpha+1)\Phi(x).$$

(ii) can be shown similarly. Q.E.D.

Now the following proposition easily follows.

Proposition 4.3. *The following inequalities hold:*

$$p\Phi(x) \le x\Phi'(x) \le p\Big(1 + \frac{\beta}{1+\log k}\Big)\Phi(x), \quad (\beta \ge 0), \tag{26}$$

$$p\Big(1 + \frac{\beta}{1+\log k}\Big)\Phi(x) \le x\Phi'(x) \le p\Phi(x), \quad (\beta < 0). \tag{27}$$

Lastly, we will see the asymptotic behavior of the complementary function Ψ. We use the notation $f \sim g$ when $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ holds.

Proposition 4.4. *Assume $p > 1$ and let q be the conjugate exponent of p: $\frac{1}{p} + \frac{1}{q} = 1$. Then it holds that*

$$p\Phi(x) \sim x^p \log^{p\beta} x, \tag{28}$$

$$(q-1)^{q\beta} q\Psi(x) \sim x^q \log^{-q\beta} x. \tag{29}$$

Proof. By the l'Hôpital theorem, we have

$$\begin{aligned}\lim_{x\to\infty} \frac{p\Phi(x)}{x^p \log^{p\beta} x} &= \lim_{x\to\infty} \frac{p\phi(x)}{px^{p-1}\log^{p\beta} x + x^p p\beta(\log^{p\beta} x)/x} \\ &= \lim_{x\to\infty} \frac{px^{p-1}\log^{p\beta}(k+x)}{px^{p-1}\log^{p\beta} x + p\beta x^{p-1}\log^{p\beta-1} x}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to\infty} \frac{\log^{p\beta}(k+x)}{\log^{p\beta} x + \beta \log^{p\beta-1} x} \\
&= 1
\end{aligned}
$$

which shows (28).

As for Ψ, we have, by the l'Hôpital theorem,

$$
\begin{aligned}
&\lim_{x\to\infty} \frac{q\Psi(x)}{x^q \log^{-q\beta} x} \\
&= \lim_{x\to\infty} \frac{q\Psi(\phi(x))}{\phi(x)^q \log^{-q\beta} \phi(x)} \\
&= \lim_{x\to\infty} \frac{q\Psi'(\phi(x))\phi'(x)}{q\phi(x)^{q-1}\phi'(x)\log^{-q\beta}\phi(x) + \phi(x)^q(-q\beta)\{\log^{-q\beta-1}\phi(x)\}\phi'(x)/\phi(x)} \\
&= \lim_{x\to\infty} \frac{x}{\phi(x)^{q-1}\log^{-q\beta}\phi(x) - \beta\phi(x)^{q-1}\log^{-q\beta-1}\phi(x)} \\
&= \lim_{x\to\infty} \frac{x}{\phi(x)^{q-1}\log^{-q\beta}\phi(x)(1-\beta/\log\phi(x))} \\
&= \lim_{x\to\infty} \frac{x}{\phi(x)^{q-1}\log^{-q\beta}\phi(x)} \\
&= \lim_{x\to\infty} \frac{x}{\{x^{p-1}\log^{p\beta}(k+x)\}^{q-1}\log^{-q\beta}(x^{p-1}\log^{p\beta}(k+x))} \\
&= \lim_{x\to\infty} \frac{x}{x^{(p-1)(q-1)}\log^{p\beta(q-1)}(k+x)\{(p-1)\log x + p\beta\log\log(k+x)\}^{-q\beta}} \\
&= \lim_{x\to\infty} \frac{\{(p-1)\log x + p\beta\log\log(k+x)\}^{q\beta}}{\log^{p\beta(q-1)}(k+x)} \qquad (\because (p-1)(q-1)=1) \\
&= \lim_{x\to\infty} \left\{\frac{\{(p-1)\log x + p\beta\log\log(k+x)}{\log(k+x)}\right\}^{q\beta} \qquad (\because q = p(q-1)) \\
&= (p-1)^{q\beta}
\end{aligned}
$$

which shows (29). Q.E.D.

We denotes the Orlicz space L^Φ associated with Φ by $L^p\log^{p\beta} L$. We do not specify k since it does not affect the asymptotic behavior at infinity. Since the Wiener measure is finite, $L^p\log^{p\beta} L$ is independent of k. The above theorem means that the dual space of $L^p\log^{p\beta} L$ is $L^q\log^{-q\beta} L$.

§5. Logarithmic Sobolev inequality

The logarithmic Sobolev inequality in L^p setting was discussed by D. Bakry-P. A. Meyer [3] and higher order Logarithmic Sobolev inequality was discussed by G. F. Feissner [5] and R. A. Adams [2]. They all used the interpolation theorem. Here we take a different approach.

The following logarithmic Sobolev inequality holds for the Ornstein-Uhlenbeck process.

$$E\left[f^2 \log(f^2/\|f\|_2^2)\right] \leq 2E\left[|Df|^2\right].$$

Here $E[\ ]$ stands for the integration with respect to μ. Hereafter we use this notation. Recall that $\int_B (Df, Dg)_{H^*} d\mu$ is the Dirichlet form associated with the Ornstein-Uhlenbeck process. We remark that the following argument works for the diffusion Dirichlet form satisfying the logarithmic Sobolev inequality if we assume the Dirichlet form is of the gradient type.

We introduce a new Young function. Set

$$\theta(x) = \{x^2 \log(e + x^2)\}^{(p-2)/4} \log^{p\beta/4}(k + x^2 \log(e + x)) \tag{30}$$

and define

$$\Theta(x) = \int_0^x \theta(y) dy. \tag{31}$$

Then we have the following proposition.

Proposition 5.1. *For sufficient large k if necessary, there exists a positive constant K such that*

$$\begin{aligned} x^p \log^{p/2}(e + x^2) \log^{p\beta/2}(k + x^2 \log(e + x^2)) \\ \leq K\Theta(x)^2 \log(e + \Theta(x)^2) \end{aligned} \tag{32}$$

Proof. We divide the proof into two cases.
(a) $\beta \geq 0$, $k = 1$.

Let us see the asymptotic behavior as $x \to 0$.

$$\text{LHS} \sim x^p \cdot x^{(p\beta/2)2} = x^{p(1+\beta)}.$$

On the other hand,

$$\theta(x) \sim x^{(p-2)/2 + (p\beta)/2} = x^{p(\beta+1)/2 - 1}.$$

and hence

$$\Theta(x) \sim \frac{2}{p(\beta + 1)} x^{p(\beta+1)/2}$$

$$\Theta(x)^2 \sim \frac{4}{p^2(\beta+1)^2} x^{p(\beta+1)}.$$

Thus both hands have the same asymptotic behavior.

As $x \to \infty$,

$$\text{LHS} \sim x^p 2^{p/2} (\log^{p/2} x) 2^{p\beta/2} \log^{p\beta/2} x = 2^{p(\beta+1)/2} x^p \log^{p(\beta+1)/2} x.$$

On the other hand,

$$\begin{aligned}
\theta(x) &\sim x^{(p-2)/2} 2^{(p-2)/4} (\log^{(p-2)/4} x) 2^{p\beta/4} \log^{p\beta/4} x \\
\Theta(x) &\sim (2/p) 2^{(p+p\beta-2)/4} x^{p/2} \log^{(p+p\beta-2)/4} x \\
\Theta(x)^2 \log(e+\Theta(x)^2) &\sim p^{-2} 2^{(p+p\beta+2)/2} x^p (\log^{(p+p\beta-2)/2} x) p \log x \\
&= p^{-1} 2^{(p+p\beta+2)/2} x^p \log^{(p+p\beta)/2} x.
\end{aligned}$$

Hence they have the same asymptotic behavior.

(b) $\beta < 0$ and large k.

The asymptotic behavior at $x = \infty$ can be obtained similarly as in the case $\beta \geq 0$.

As $x \to 0$, LHS $\sim x^p$ is clear. Further we have

$$\begin{aligned}
\theta(x) &\sim x^{(p-2)/2} \\
\Theta(x) &\sim \frac{2}{p} x^{p/2} \\
\Theta(x)^2 \log(e+\Theta(x)^2) &\sim \frac{4}{p^2} x^p
\end{aligned}$$

Thus we have the desired result. Q.E.D.

We recall the following fact. Let U and V be a non-negative functions on a measure space (M, m). Assume that

$$\begin{aligned}
&\int_M U\phi(U) dm < \infty, \\
&\int_M U\phi(U) dm \leq \int_M V\phi(U) dm + C.
\end{aligned}$$

Then it follows that

$$\int_M \Phi(U) dm \leq \int_M \Phi(V) dm + C. \tag{33}$$

For the proof, see [4, Lemma VI.98]. Now we have the following theorem. In the sequel, we denote by $\Phi_{p,\beta}$ in place of $\Phi_{p,\beta,k}$ because the index k is not essential.

Proposition 5.2. *For $p > 2$, $\beta \in \mathbb{R}$, there exists a positive constant C such that*

(34)
$$E[\Phi_{p,(\beta+1)/2}(|f|)] \leq CE[\Phi_{p,(1+\beta)/2-(1/p)}(|f|)] + CE[\Phi_{p,\beta/2}(|Df|)].$$

Proof. Set $g = \sqrt{\Theta(|f|)^2 + e}$. Then

$$Dg = \frac{2\Theta(|f|)\Theta'(|f|)D|f|}{2\sqrt{\Theta(|f|)^2 + e}}$$

and hence $|Dg| \leq \theta(|f|)|Df|$. Now, by using the logarithmic Sobolev inequality

$$E\left[g^2 \log(g^2/\|g\|_2^2)\right] \leq 2E\left[|Dg|^2\right],$$

we have

$$\begin{aligned}
&E\left[\{\Theta(|f|)^2 + e\} \log(e + \Theta(|f|)^2)\right] \\
&\leq E\left[\Theta(|f|)^2 + e\right] \log E\left[e + \Theta(|f|)^2\right] \\
&\quad + E\left[|Df|^2\{|f|^2 \log(e + |f|^2)\}^{(p-2)/2} \log^{p\beta/2}(k + |f|^2 \log(e + |f|^2))\right],
\end{aligned}$$

and

$$\begin{aligned}
&E\left[\Theta(|f|)^2 \log(e + \Theta(|f|)^2)\right] \\
&\leq E\left[\Theta(|f|)^2\right] \log E\left[e + \Theta(|f|)^2\right] \\
&\quad + E\left[|Df|^2\{|f|^2 \log(e + |f|^2)\}^{(p-2)/2} \log^{p\beta/2}(k + |f|^2 \log(e + |f|^2))\right].
\end{aligned}$$

We set

$$\begin{aligned}
\phi(x) &= \phi_{p/2,\beta,k}(x) = x^{(p/2)-1} \log^{p\beta/2}(k + x), \\
U &= |f|^2 \log(e + |f|^2).
\end{aligned}$$

Then

$$\begin{aligned}
U\phi(U) &= |f|^2 \log(e + |f|^2) \\
&\qquad \times \{|f|^2 \log(e + |f|^2)\}^{(p/2)-1} \log^{p\beta/2}(k + |f|^2 \log(e + |f|^2)) \\
&= |f|^p \log^{p/2}(e + |f|^2) \log^{p\beta/2}(k + |f|^2 \log(e + |f|^2)) \\
&\leq K\Theta(|f|)^2 \log(e + \Theta(|f|)^2). \quad (\because\ (32))
\end{aligned}$$

Combining this with the previous result, we have

$$K^{-1}E[U\phi(U)] \leq E[e + \Theta(|f|)^2] \log E[e + \Theta(|f|)^2] + E[|Df|^2\phi(U)].$$

Now, by (33), it follows that

$$E[\Phi(U)] \leq KE[e+\Theta(|f|)^2]\log E[e+\Theta(|f|)^2] + KE[\Phi(|Df|^2)].$$

Here Φ is the integral of ϕ. Since $\Phi = \Phi_{p/2,\beta}$,

$$\begin{aligned}\Phi(x^2) &\leq c_1 x^2\phi(x^2)\\ &\leq c_1 x^2 (x^2)^{(p/2)-1}\log^{p\beta/2}(k+x^2)\\ &\leq c_1 x^p \log^{p\beta/2}(k+x^2)\\ &\leq c_2\Phi_{p,\beta/2}(x).\end{aligned}$$

Further

$$\begin{aligned}\Phi(x^2\log(e+x^2)) &\geq c_3 x^2\log(e+x^2)\phi(x^2\log(e+x^2))\\ &= c_3 x^2\log(e+x^2)\{x^2\log(e+x^2)\}^{(p/2)-1}\\ &\quad\times\log^{p\beta/2}(k+x^2\log(e+x^2))\\ &= c_3 x^p\log^{p/2}(e+x^2)\log^{p\beta/2}(k+x^2\log(e+x^2))\\ &\geq c_4 x^p\log^{p/2}(e+x)\log^{p\beta/2}(k+x)\\ &\geq c_5 x^p \log^{p(1+\beta)/2}(k+x)\\ &\geq c_6\Phi_{p,(\beta+1)/2}(x)\end{aligned}$$

and

$$\begin{aligned}\Theta(x)^2 &\leq x^2\theta(x)^2\\ &\leq x^2\{x^2\log(e+x^2)\}^{(p-2)/2}\log^{p\beta/2}(k+x^2\log(e+x))\\ &\leq c_7 x^p\log^{(p-2)/2}(e+x)\log^{p\beta/2}(k+x)\\ &\leq c_7 x^p\log^{(p+p\beta-2)/2}(k+x)\\ &\leq c_8\Phi_{p,(1+\beta)/2-(1/p)}(x).\end{aligned}$$

Thus we have eventually obtained

$$E[\Phi_{p,(\beta+1)/2}(|f|)] \leq CE[\Phi_{p,(1+\beta)/2-(1/p)}(|f|)] + CE[\Phi_{p,\beta/2}(|Df|)].$$

This completes the proof. Q.E.D.

If $p=2$ and $\beta \geq 0$, the above proof works as well in this case. We only state the result.

Proposition 5.3. *For $p=2$, $\beta\geq 0$, there exists a positive constant C such that*

$$E[\Phi_{2,(\beta+1)/2}(|f|)] \leq CE[\Phi_{2,\beta/2}(|f|)] + CE[\Phi_{p,\beta/2}(|Df|)]. \tag{35}$$

In Section 3, we showed that the right hand side of (34) is equivalent to $E\left[\Phi_{p,\beta/2}(\sqrt{1-L}f)\right]$. Therefore we easily get the following theorem.

Theorem 5.4. *For $p>1$, $\beta\geq 0$, the following map is continuous:*

$$\sqrt{1-L}^{\,-1}\colon L^p\log^{p\beta}L\to L^p\log^{p(\beta+1/2)}L. \tag{36}$$

Recall that the dual space of $L^p\log^{p\beta}$ is $L^q\log^{-q\beta}L$ (see Proposition 4.4). Hence, when $1<p<2$, the above equation (36) is shown by the duality. By iterating the map $\sqrt{1-L}^{\,-1}$, we can have the continuity of $(1-L)^{-1}$ from $L^p\log^{p\beta}L$ to $L^p\log^{p(\beta+1)}L$.

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