L^p multiplier theorem for the Hodge–Kodaira operator

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Summary. We discuss the L^p multiplier theorem for a semigroup acting on vector valued functions. A typical example is the Hodge–Kodaira operator on a Riemannian manifold. We give a probabilistic proof. Our main tools are the semigroup domination and the Littlewood–Paley inequality.

1 Introduction

We discuss the L^p multiplier theorem. In L^2 setting, it is well known that $\varphi(-L)$ is bounded if and only if φ is bounded where L is a non-positive self-adjoint operator. In L^p setting, the criterion above is no more true in general.

E. M. Stein [9] gave a sufficient condition when L is a generator of a symmetric Markov process. It reads as follows: define a function φ on $[0, \infty)$ by

$$\varphi(\lambda) = \lambda \int_0^\infty \mathrm{e}^{-2t\lambda} m(t)\,\mathrm{d}t. \tag{1.1}$$

Here we assume that m is a bounded function. A typical example is $\varphi(\lambda) = \lambda^{i\alpha}$ ($\alpha \in \mathbb{R}$). Then Stein proved that $\varphi(-L)$ is a bounded operator in L^p for $1 < p < \infty$. He also proved that the operator norm of $\varphi(-L)$ depends only on p and the bound of m.

In the meanwhile we consider the Hodge–Kodaira operator on a compact Riemannian manifold M. It is of the form $\mathbf{L} = -(dd^* + d^*d)$ where d is the exterior differentiation. A typical feature is that $\mathbf{L}$ acts on vector valued functions, to be precise, differential forms on M. In this case, we can get the following theorem:

* This research was partially supported by the Japanese Ministry of Education, Culture, Sports, Science and Technology, Grant-in-Aid for Scientific Research (B), No. 11440045, 2001

Theorem 1.1. *For sufficiently large κ, $\varphi(\kappa-\mathbf{L})$ is a bounded operator in L^p. Further the operator norm is estimated in terms of m and p only.*

To show this theorem, we use the following facts.

1. The semigroup domination.
2. The Littlewood–Paley inequality.

As for the first, we can show that

$$\left|\mathrm{e}^{t(\mathbf{L}-\kappa)}\theta\right| \leqslant \mathrm{e}^{tL}|\theta|. \tag{1.2}$$

Here L is the Laplace–Beltrami operator on M and the inequality holds pointwisely. This inequality can be shown by means of Ouhabaz criterion ([3]). To use the criterion, the following inequality is essential.

$$L|\theta|^2 - 2(\mathbf{L}\theta,\theta) + \kappa|\theta|^2 \geqslant 0.$$

As for the second, we need the Littlewood–Paley function. This is somehow different from usual one. We may call it the Littlewood–Paley function of parabolic type. It is defined as follows:

$$\mathcal{P}\theta(x) = \left\{\int_0^\infty |\nabla \mathbf{T}_t\theta(x)|^2 \,\mathrm{d}t\right\}^{1/2}.$$

Here $\mathbf{T}_t$ denotes the semigroup $\mathrm{e}^{t(\mathbf{L}-\kappa)}$. We can show the following inequality: there exists positive constant C independent of θ such that

$$\|\mathcal{P}\theta\|_p \leqslant C\|\theta\|_p$$

where $\|\,.\,\|_p$ stands for the L^p-norm. This inequality is called the Littlewood–Paley inequality.

Combining these two inequality we can show that

$$\left|\big(\varphi(\kappa-\mathbf{L})\theta,\eta\big)\right| \leqslant C_1\|\mathcal{P}\theta\|_p\,\|\mathcal{P}\eta\|_q \leqslant C_2\|\theta\|_p\,\|\eta\|_q.$$

Here q is the conjugate exponent of p. Now the desired result follows easily.

The organization of the paper is as follows. We discuss this problem in the general framework of symmetric diffusion process. We give this formulation in §2. We introduce the square field operator not only in the scalar valued case but also in the vector valued case. We give conditions to assure the semigroup domination which plays an important role in the paper. In §3, we discuss the Littlewood–Paley inequality. We use the Littlewood–Paley function of parabolic type. After preparing those, we give a proof of the multiplier theorem. In §4, we give an example of the Hodge–Kodaira operator. The crucial issue is the intertwining property of these operators.

2 Symmetric Markov processes and the semigroup domination

In the introduction, we stated the theorem for the Hodge–Kodaira operator but it can be discussed under more general setting. We give it in the framework of symmetric Markov diffusion process.

Let (M,μ) be a measure space and suppose that we are given a conservative diffusion process (X_t, P_x) on M. Here P_x denotes a measure on $C([0,\infty)\to M)$ that stands for the law of the diffusion process starting at $x \in M$. We assume that (X_t) is symmetric with respect to μ and hence the semigroup $\{T_t\}$ defined by

$$T_t f(x) = E_x[f(X_t)], \tag{2.1}$$

is a strongly continuous symmetric semigroup in $L^2(m)$. Here E_x stands for the expectation with respect to P_x. We denote the associated Dirichlet form by $\mathcal{E}$ and the generator by L. We assume further that there exists a continuous bilinear map $\Gamma\colon \operatorname{Dom}(\mathcal{E})\times\operatorname{Dom}(\mathcal{E})\to L^1(m)$ such that

$$2\int_M \Gamma(f,g)h\,\mathrm{d}\mu = \mathcal{E}(fg,h)-\mathcal{E}(f,gh)-\mathcal{E}(g,fh), \qquad \text{for } f,\ g,\ h\in\operatorname{Dom}(\mathcal{E})\cap L^\infty. \tag{2.2}$$

Γ is called the square field operator ("opérateur carré du champ" in French literature) and we impose on Γ the following derivation property:

$$\Gamma(fg,h) = f\Gamma(g,h)+g\Gamma(f,h), \qquad \text{for } f,\ g,\ h\in\operatorname{Dom}(\mathcal{E})\cap L^\infty. \tag{2.3}$$

We are dealing with a semigroup acting on vector valued functions (to be precise, sections of a vector bundle) and so we are given another semigroup $\{\mathbf{T}_t\}$. The semigroup acts on L^2-sections of a vector bundle E. Here E is equipped with a metric $(\,.\,,\,.\,)_E$ and L^2-sections are measurable sections θ with

$$\|\theta\|_2^2 = \int_M |\theta(x)|_E^2\,\mu(\mathrm{d}x) < \infty.$$

The norm $|\,.\,|_E$ is defined by $|\theta|_E = \sqrt{(\theta,\theta)_E}$. We denote the set of all L^2-sections by $L^2\Gamma(E)$. The typical example of E is a exterior bundle of T^*M over a Riemannian manifold M and in this case $L^2\Gamma(E)$ is the set of all square integrable differential forms. $\mathbf{L}$ denotes the generator of $\{\mathbf{T}_t\}$ and $\boldsymbol{\mathcal{E}}$ denotes the associated bilinear form. We assume that $\mathbf{L}$ is decomposed as

$$\mathbf{L} = \hat{L} - \kappa - R. \tag{2.4}$$

Here R is a symmetric section of $\operatorname{Hom}(E;E)$ and κ is a positive constant. Later we take κ to be large enough. $\hat{L}$ is self-adjoint and non-negative definite. It generates a contraction semigroup which we denote by $\{\hat{T}_t\}$. $\hat{L}$

and L satisfy the following relation: there exists a square field operator $\hat{\Gamma}\colon \mathrm{Dom}(\hat{\mathcal{E}}) \times \mathrm{Dom}(\hat{\mathcal{E}}) \to L^1(\mu)$ such that

$$2\int_M \hat{\Gamma}(\theta,\eta)h\,\mathrm{d}\mu = \hat{\mathcal{E}}\big((\theta,\eta)_E, h\big) - \hat{\mathcal{E}}(\theta, h\eta) - \hat{\mathcal{E}}(h\theta,\eta),$$
$$\text{for } \theta,\ \eta \in \mathrm{Dom}(\hat{\mathcal{E}}) \cap L^\infty,\ h \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty. \quad (2.5)$$

We assume that $\hat{\Gamma}$ enjoys the positivity $\hat{\Gamma}(\theta,\theta) \geqslant 0$ and

$$2h\boldsymbol{\Gamma}(\theta,\eta) = -\Gamma\big(h,(\theta,\eta)_E\big) + \boldsymbol{\Gamma}(\theta,h\eta) + \boldsymbol{\Gamma}(h\theta,\eta) \quad (2.6)$$

for $\theta,\ \eta \in \mathrm{Dom}(\hat{\mathcal{E}}) \cap L^\infty,\ h \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$. These properties lead to the semigroup domination (see e.g., [5]):

$$\big|\hat{T}_t\theta\big|_E \leqslant T_t|\theta|_E. \quad (2.7)$$

Since R is bounded, we may assume that $\kappa + R$ is non-negative definite at any point of M by taking κ large enough. We assume further that there exists a positive constant $\delta > 0$ such that

$$\kappa(\theta,\theta)_E + (R\theta,\theta)_E \geqslant \delta(\theta,\theta)_E. \quad (2.8)$$

Then the semigroup domination for $\{\mathbf{T}_t\}$ also holds as follows:

$$|\mathbf{T}_t\theta|_E \leqslant \mathrm{e}^{-\delta t} T_t|\theta|_E. \quad (2.9)$$

We give a correspondence to the Hodge–Kodaira operator when M is a Riemannian manifold. $L = \Delta$ (i.e., the Laplace–Beltrami operator), $E = \bigwedge^q T^*M$ (the exterior product of the cotangent bundle) and $L^2\Gamma(E)$ is the set of all square integrable q-forms. $\hat{L} = -\nabla^*\nabla$ is the covariant Laplacian (Bochner Laplacian), $\mathbf{L} = -(dd^* + d^*d) - \kappa = \hat{L} - \kappa - R_{(q)}$. The explicit form of $R_{(q)}$ is given by the Weitzenböck formula and can be written in terms of the curvature tensor. We do not give the explicit form because we do not need it. We only need the boundedness of $R_{(q)}$. $\hat{\Gamma}$ is given by

$$\hat{\Gamma}(\theta,\eta) = \frac{1}{2}\left\{\Delta(\theta,\eta)_E + (\nabla^*\nabla\theta,\eta)_E + (\theta,\nabla^*\nabla\eta)_E\right\} = (\nabla\theta,\nabla\eta)_{E\otimes T^*M}.$$

The positivity of $\hat{\Gamma}$ clearly holds and (2.6) follows from the derivation property of ∇.

We now return to the general framework. We assume that $\hat{\Gamma}$ is expressed as

$$\hat{\Gamma}(\theta,\eta) = (D\theta, D\eta) \quad (2.10)$$

for some operator D. For instance, the covariant Laplacian satisfies this condition. In this case, D is the covariant derivation ∇. Later we need this condition when the exponent p is greater than 2.

D is an operator from $L^2\Gamma(E)$ to $L^2\Gamma(\widetilde{E})$, $\widetilde{E}$ being another vector bundle over M. The domain of D is not necessarily the whole space $L^2\Gamma(E)$ but we do assume that D is a closed operator. Last assumption is the following intertwining property: there exists a self-adjoint operator $\mathbf{\Lambda}$ satisfying

$$D\mathbf{L} = \mathbf{\Lambda} D + K \tag{2.11}$$

where K is a bounded section of $\mathrm{Hom}(E; E')$. For $\mathbf{\Lambda}$, we assume the same conditions as $\mathbf{L}$. In particular, we need the semigroup domination for $\mathbf{S}_t = \mathrm{e}^{t\mathbf{\Lambda}}$:

$$|\mathbf{S}_t \xi|_{\widetilde{E}} \leqslant \mathrm{e}^{-\delta t} T_t |\xi|_{\widetilde{E}}, \qquad \xi \in L^2\Gamma(\widetilde{E}). \tag{2.12}$$

Due to the boundedness of K, this is possible by taking κ large enough. Moreover the intertwining property (2.11) implies

$$D\mathbf{T}_t\theta = \mathbf{S}_t D\theta + \int_0^t \mathbf{S}_{t-s} K \mathbf{T}_s \theta \,\mathrm{d}s, \qquad \forall \theta \in \mathrm{Dom}(D), \tag{2.13}$$

(see [8]).

3 Littlewood–Paley inequality

We introduce the Littlewood–Paley function of parabolic type. They are given as follows:

$$\mathcal{P}\theta(x) = \left\{ \int_0^\infty \hat{\Gamma}(\mathbf{T}_t\theta, \mathbf{T}_t\theta)(x)\,\mathrm{d}t \right\}^{1/2}, \tag{3.1}$$

$$\mathcal{H}\theta(x) = \left\{ \int_0^\infty T_t\hat{\Gamma}(\mathbf{T}_t\theta, \mathbf{T}_t\theta)(x)\,\mathrm{d}t \right\}^{1/2}. \tag{3.2}$$

We fix a time N and set

$$u(x,t) = T_{N-t}\theta(x), \qquad 0 \leqslant t \leqslant N.$$

Then we have

$$\begin{aligned}
&(\partial_t + L)|u(x,t)|_E^2 \\
&\quad = (\partial_t + L)(\mathbf{T}_{N-t}\theta, \mathbf{T}_{N-t}\theta) \\
&\quad = -2(\mathbf{L}\mathbf{T}_{N-t}\theta, \mathbf{T}_{N-t}\theta) + 2(\hat{L}\mathbf{T}_{N-t}\theta, \mathbf{T}_{N-t}\theta) + 2\hat{\Gamma}(\mathbf{T}_{N-t}\theta, \mathbf{T}_{N-t}\theta) \\
&\quad = -2\big((\hat{L} - \kappa - R)\mathbf{T}_{N-t}\theta, \mathbf{T}_{N-t}\theta\big) \\
&\qquad + 2(\hat{L}\mathbf{T}_{N-t}\theta, \mathbf{T}_{N-t}\theta) + 2\hat{\Gamma}(\mathbf{T}_{N-t}\theta, \mathbf{T}_{N-t}\theta) \\
&\quad = 2\big((\kappa + R)\mathbf{T}_{N-t}\theta, \mathbf{T}_{N-t}\theta\big) + 2\hat{\Gamma}(\mathbf{T}_{N-t}\theta, \mathbf{T}_{N-t}\theta)
\end{aligned}$$

For notational simplicity, we use the following convention. We write $\|A\theta\|_p \lesssim \|\theta\|_p$ if there exists a constant C such that $\|A\theta\|_p \leqslant C\|\theta\|_p$. C is independent of θ but may depend on p and A. We use this convention without mention. Now we have the following.

Proposition 3.1. *For $1 < p \leqslant 2$, it holds that*

$$\|\mathcal{P}\theta\|_p \lesssim \|\theta\|_p. \tag{3.3}$$

Proof. Define a martingale (M_t) by

$$\begin{aligned} M_t &= |u(X_t,t)|_E^2 - |u(X_0,0)|_E^2 - \int_0^t (\partial_s + L)|u(X_s,s)|_E^2 \,\mathrm{d}s \\ &= |u(X_t,t)|_E^2 - |u(X_0,0)|_E^2 \\ &\quad - 2\int_0^t \Big\{ \big((\kappa+R)\mathbf{T}_{N-s}\theta(X_s), \mathbf{T}_{N-s}\theta(X_s)\big) \\ &\qquad\qquad + \hat{\Gamma}\big(\mathbf{T}_{N-s}\theta(X_s), \mathbf{T}_{N-s}\theta(X_s)\big)\Big\} \,\mathrm{d}s. \end{aligned}$$

Then the quadratic variation of (M_t) is written as

$$\begin{aligned} \langle M, M\rangle_t &= 2\int_0^t \Gamma\big(|u(\,.\,,s)|_E^2, |u(\,.\,,s)|_E^2\big)(X_s) \,\mathrm{d}s \\ &= 8\int_0^t |u(X_s,s)|_E^2 \Gamma\big(|u(\,.\,,s)|_E, |u(\,.\,,s)|_E\big)(X_s) \,\mathrm{d}s. \end{aligned}$$

In particular, $Z_t = |u(X_t,t)|_E^2$ is a non-negative submartingale:

$$Z_t = |u(X_0,0)|_E^2 + M_t + B_t \tag{3.4}$$

where an increasing process B_t is given by

$$\begin{aligned} B_t = 2\int_0^t \Big\{ \big((\kappa+R)\mathbf{T}_{N-s}\theta(X_s), \mathbf{T}_{N-s}\theta(X_s)\big) \\ + \hat{\Gamma}\big(\mathbf{T}_{N-s}\theta(X_s), \mathbf{T}_{N-s}\theta(X_s)\big)\Big\} \,\mathrm{d}s. \end{aligned} \tag{3.5}$$

Take any $\varepsilon > 0$ and apply the Itô formula to $(|u|_E^2 + \varepsilon)^{p/2}$, we have

$$\begin{aligned} \mathrm{d}\big(|u|_E^2 + \varepsilon\big)^{p/2} &= \frac{p}{2}\big(|u|_E^2 + \varepsilon\big)^{p/2-1} \mathrm{d}\big(|u|_E^2 + \varepsilon\big) \\ &\quad + \frac{1}{2}\frac{p}{2}\Big(\frac{p}{2} - 1\Big)\big(|u|_E^2 + \varepsilon\big)^{p/2-2} \mathrm{d}\langle M, M\rangle_t \\ &= \frac{p}{2}\big(|u|_E^2 + \varepsilon\big)^{p/2-1} \mathrm{d}M_t \\ &\quad + \Big[\frac{p}{2}\big(|u|_E^2 + \varepsilon\big)^{p/2-1} 2\big\{\big((\kappa+R)u, u\big) + \hat{\Gamma}(u,u)\big\} \\ &\quad + p(p-2)\big(|u|_E^2 + \varepsilon\big)^{p/2-2} |u|_E^2 \Gamma(|u|_E, |u|_E)\Big] \,\mathrm{d}t. \end{aligned}$$

Here, in the above identity, $u(X_t,t)$ is simply denoted by u. Therefore

$$\big(|u(X_t,t)|_E^2 + \varepsilon\big)^{p/2} = \big(|u(X_0,0)|_E^2 + \varepsilon\big)^{p/2} + \int_0^t \frac{p}{2}\big(|u|_E^2 + \varepsilon\big)^{p/2-1} \mathrm{d}M_s + A_t.$$

Here A_t is defined by

$$A_t = \int_0^t \Big[p\big(|u|_E^2 + \varepsilon\big)^{p/2-1} \big\{ \big((\kappa + R)u, u\big) + \hat{\Gamma}(u,u) \big\} \\ + p(p-2)\big(|u|_E^2 + \varepsilon\big)^{p/2-2} |u|_E^2 \Gamma(|u|_E, |u|_E) \Big] \,\mathrm{d}t.$$

(A_t) is an increasing process. To see this, recalling the inequality

$$\Gamma(|u|_E, |u|_E) \leqslant \hat{\Gamma}(u,u),$$

we have

$$\begin{aligned} \mathrm{d}A_t &\geqslant p\big(|u|_E^2 + \varepsilon\big)^{p/2-1} \big\{ \big((\kappa + R)u, u\big) + \hat{\Gamma}(u,u) \big\} \\ &\quad + p(p-2)\big(|u|_E^2 + \varepsilon\big)^{p/2-2} |u|^2 \hat{\Gamma}(u,u) \\ &\geqslant \big(p + p(p-2)\big)\big(|u|_E^2 + \varepsilon\big)^{p/2-1} \hat{\Gamma}(u,u) \\ &\quad + p\big(|u|_E^2 + \varepsilon\big)^{p/2-1} \big((\kappa + R)u, u\big) \\ &\geqslant p(p-1)\big(|u|_E^2 + \varepsilon\big)^{p/2-1} \hat{\Gamma}(u,u) \end{aligned}$$

which implies that A_t is increasing. By taking expectation of $(|u(X_N, N)|_E^2 + \varepsilon)^{p/2}$, we obtain

$$\begin{aligned} p(p-1) E\Big[\int_0^N \big(|u|^2 + \varepsilon\big)^{p/2-1} \hat{\Gamma}(u,u) \,\mathrm{d}t \Big] &\leqslant E\Big[\big(|u(X_N, N)|_E^2 + \varepsilon\big)^{p/2} \Big] \\ &\leqslant E\Big[\big(|\theta(X_N)|_E^2 + \varepsilon\big)^{p/2} \Big] \\ &\leqslant \Big\| \big(|\theta|_E^2 + \varepsilon\big)^{1/2} \Big\|_p^p. \end{aligned}$$

We proceed to the estimation of the left hand side. By the semigroup domination

$$|\mathbf{T}_{N-t}\theta(x)| \leqslant T_{N-t}|\theta|(x) \leqslant \sup_{s \geqslant 0} T_s|\theta|(x) =: \theta^*(x)$$

The maximal ergodic theorem implies $\|\theta^*\|_p \lesssim \|\theta\|_p$. Now, noting $p/2 - 1 \leqslant 0$,

$$\begin{aligned} E\Big[\int_0^N \big(|u|^2 + \varepsilon\big)^{p/2-1} \hat{\Gamma}(u,u) \,\mathrm{d}t \Big] &= \Big\| \int_0^N \big(|u|^2 + \varepsilon\big)^{p/2-1} \hat{\Gamma}(\mathbf{T}_{N-t}\theta, \mathbf{T}_{N-t}\theta) \,\mathrm{d}t \Big\|_1 \\ &\gtrsim \Big\| \big((\theta^*)^2 + \varepsilon\big)^{p/2-1} \int_0^N \hat{\Gamma}(\mathbf{T}_t\theta, \mathbf{T}_t\theta) \,\mathrm{d}t \Big\|_1. \end{aligned}$$

Letting $N \to \infty$,

$$\Big\| \big(|\theta|^2 + \varepsilon\big)^{1/2} \Big\|_p^p \gtrsim \Big\| \big((\theta^*)^2 + \varepsilon\big)^{p/2-1} \int_0^\infty \hat{\Gamma}(\mathbf{T}_t\theta, \mathbf{T}_t\theta) \,\mathrm{d}t \Big\|_1$$

$$\gtrsim \left\| \left((\theta^*)^2 + \varepsilon\right)^{p/2-1} \mathcal{P}\theta^2 \right\|_1 = \left\| \left((\theta^*)^2 + \varepsilon\right)^{(p-2)/4} \mathcal{P}\theta \right\|_2^2 .$$

Therefore

$$\begin{aligned}
\|\mathcal{P}\theta\|_p &= \left\| \left((\theta^*)^2 + \varepsilon\right)^{(2-p)/4} \left((\theta^*)^2 + \varepsilon\right)^{(p-2)/4} \mathcal{P}\theta \right\|_p \\
&\leqslant \left\| \left((\theta^*)^2 + \varepsilon\right)^{(2-p)/4} \right\|_{2p/(2-p)} \left\| \left((\theta^*)^2 + \varepsilon\right)^{(p-2)/4} \mathcal{P}\theta \right\|_2 \\
&\qquad \left(\text{since } \frac{1}{p} = \frac{2-p}{2p} + \frac{1}{2}\right) \\
&\lesssim \left\| \left((\theta^*)^2 + \varepsilon\right)^{1/2} \right\|_p^{(2-p)/2} \left\| \left(|\theta|^2 + \varepsilon\right)^{1/2} \right\|_p^{p/2} .
\end{aligned}$$

Finally, letting $\varepsilon \to 0$, we obtain

$$\|\mathcal{P}\theta\|_p \lesssim \|\theta^*\|_p^{(2-p)/2} \, \||\theta|\|_p^{p/2} \lesssim \|\theta\|_p^{(2-p)/2} \, \|\theta\|_p^{p/2} = \|\theta\|_p .$$

The proof is complete. □

Next we show the case $p \geqslant 2$. First we need the following easy lemma.

Lemma 3.1. *Let j be a non-negative fonction on $M \times [0, N]$. Then it holds that*

$$E_\mu\left[\int_0^N j(X_t, t)\,\mathrm{d}t \;\middle|\; X_N = x\right] = \int_0^N T_t\big(j(\,.\,, N-t)\big)(x)\,\mathrm{d}t. \tag{3.6}$$

Here E_μ stands for the integration with respect to $P_\mu = \int_M P_x\,\mu(\mathrm{d}x)$.

Proof. It is enough to show that

$$E_\mu\left[\left\{\int_0^N j(X_t, t)\,\mathrm{d}t\right\} f(X_N)\right] = \int_E \left\{\int_0^N T_{N-t} j(x, t)\,\mathrm{d}t\right\} f(x)\,\mu(\mathrm{d}x) \tag{3.7}$$

for any non-negative function f. To see this,

$$\begin{aligned}
E_\mu\left[\left\{\int_0^N j(X_t, t)\,\mathrm{d}t\right\} f(X_N)\right] &= \int_0^N E_\mu[j(X_t, t) f(X_N)]\,\mathrm{d}t \\
&= \int_0^N E_\mu[j(X_t, t) E_\mu[f(X_N) | \mathcal{F}_t]]\,\mathrm{d}t \\
&= \int_0^N E_\mu[j(X_t, t) T_{N-t} f(X_t)]\,\mathrm{d}t \\
&\qquad \text{(by the Markov property)} \\
&= \int_0^N \mathrm{d}t \int_M j(x, t) T_{N-t} f(x)\,\mu(\mathrm{d}x)
\end{aligned}$$

$$= \int_0^N \mathrm{d}t \int_M T_{N-t}\big(j(\,.\,,t)\big)(x) f(x)\,\mu(\mathrm{d}x)$$
(by symmetry)
$$= \int_M \left\{ \int_0^N T_{N-t}\big(j(\,.\,,t)\big)(x)\,\mathrm{d}t \right\} f(x)\,\mu(\mathrm{d}x)$$

which shows (3.7). □

Proposition 3.2. *For $p \geqslant 2$, we have*

$$\|\mathcal{H}\theta\|_p \lesssim \|\theta\|_p. \tag{3.8}$$

Proof. We consider a submartingale $Z_t = |u(X_t,t)|^2$. As was seen in (3.4), Z_t is decomposed as

$$Z_t = |u(X_0,0)|_E^2 + M_t + B_t.$$

Then the following inequality is well-known (see, [2]): for $q \geqslant 1$,

$$E[B_N^q] \lesssim E[Z_N^q]. \tag{3.9}$$

Using Lemma 3.1, we have

$$\begin{aligned}
&\int_M \mu(\mathrm{d}x) \left\{ \int_0^N T_t \hat{\Gamma}(\mathbf{T}_t\theta, \mathbf{T}_t\theta)(x)\,\mathrm{d}t \right\}^{p/2} \\
&\qquad = \int_M \mu(\mathrm{d}x) E_\mu \left[\int_0^N \hat{\Gamma}(\mathbf{T}_{N-t}\theta, \mathbf{T}_{N-t}\theta)(X_t)\,\mathrm{d}t \,\middle|\, X_N = x \right]^{p/2} \\
&\qquad \leqslant \int_M \mu(\mathrm{d}x) E_\mu \left[\left\{ \int_0^N \hat{\Gamma}(\mathbf{T}_{N-t}\theta, \mathbf{T}_{N-t}\theta)(X_t)\,\mathrm{d}t \right\}^{p/2} \,\middle|\, X_N = x \right] \\
&\qquad\qquad \text{(by the Jensen inequality)} \\
&\qquad = E_\mu \left[\left\{ \int_0^N \hat{\Gamma}(\mathbf{T}_{N-t}\theta, \mathbf{T}_{N-t}\theta)(X_t)\,\mathrm{d}t \right\}^{p/2} \right] \\
&\qquad \leqslant E_\mu \left[\left\{ \int_0^N \big\{ \big((\kappa + R)\mathbf{T}_{N-t}\theta(X_s), \mathbf{T}_{N-t}\theta(X_s)\big) \right.\right. \\
&\qquad\qquad \left.\left. + \hat{\Gamma}\big(\mathbf{T}_{N-t}\theta(X_s), \mathbf{T}_{N-t}\theta(X_s)\big) \big\}\,\mathrm{d}t \right\}^{p/2} \right] \\
&\qquad \lesssim E\big[B_N^{p/2}\big] \qquad \text{(thanks to (3.5))} \\
&\qquad \lesssim E\big[Z_N^{p/2}\big] \qquad \text{(thanks to (3.9))} \\
&\qquad = E\big[|\theta(X_N)|^p\big] \\
&\qquad = \|\theta\|_p^p.
\end{aligned}$$

Now $\mathcal{H}\theta$ can be estimated as follows:

$$
\begin{aligned}
\|\mathcal{H}\theta\|_p^p &= \left\| \left\{ \int_0^\infty T_t \hat{\Gamma}(\mathbf{T}_t\theta, \mathbf{T}_t\theta)(x)\,\mathrm{d}t \right\}^{p/2} \right\|_1 \\
&= \lim_{N\to\infty} \int_M \mu(\mathrm{d}x) \left\{ \int_0^N T_t \hat{\Gamma}(\mathbf{T}_t\theta, \mathbf{T}_t\theta)(x)\,\mathrm{d}t \right\}^{p/2} \\
&\lesssim \|\theta\|_p^p.
\end{aligned}
$$

This completes the proof. □

Let us proceed to the estimation of $\mathcal{P}\theta$.

Proposition 3.3. *For $p \geqslant 2$, we have*

$$
\mathcal{P}\theta(x) \leqslant \sqrt{2}\,\mathcal{H}\theta(x) + \frac{\|K\|_\infty}{4\delta^{3/2}}\,\theta^*(x) \tag{3.10}
$$

Proof. We have

$$
\begin{aligned}
\mathcal{P}\theta(x) &= \left\{ \int_0^\infty \hat{\Gamma}(\mathbf{T}_t\theta, \mathbf{T}_t\theta)(x)\,\mathrm{d}t \right\}^{1/2} \\
&= \left\{ \int_0^\infty |D\mathbf{T}_t\theta(x)|_{\widetilde{E}}^2\,\mathrm{d}t \right\}^{1/2} \qquad \text{(thanks to (2.10))} \\
&= \left\{ 2\int_0^\infty |D\mathbf{T}_{2t}\theta(x)|_{\widetilde{E}}^2\,\mathrm{d}t \right\}^{1/2} \\
&= \left\{ 2\int_0^\infty |\mathbf{T}_t D\mathbf{T}_t\theta(x)|_{\widetilde{E}}^2\,\mathrm{d}t \right\}^{1/2} \\
&= \left\{ 2\int_0^\infty \left| \mathbf{S}_t D\mathbf{T}_t\theta(x) + \int_0^t \mathbf{S}_{t-s} K \mathbf{T}_s \mathbf{T}_t\theta(x)\,\mathrm{d}s \right|_{\widetilde{E}}^2 \mathrm{d}t \right\}^{1/2} \qquad \text{(by (2.13))} \\
&\leqslant \sqrt{2} \left\{ \int_0^\infty |\mathbf{S}_t D\mathbf{T}_t\theta(x)|_{\widetilde{E}}^2\,\mathrm{d}t \right\}^{1/2} \\
&\quad + \sqrt{2} \left\{ \int_0^\infty \left\{ \int_0^t |\mathbf{S}_{t-s} K \mathbf{T}_{s+t}\theta(x)|_{\widetilde{E}}\,\mathrm{d}s \right\}^2 \mathrm{d}t \right\}^{1/2} \\
&\leqslant \sqrt{2} \left\{ \int_0^\infty T_t |D\mathbf{T}_t\theta(x)|_{\widetilde{E}}^2\,\mathrm{d}t \right\}^{1/2} \\
&\quad + \sqrt{2} \left\{ \int_0^\infty \left\{ \int_0^t \mathrm{e}^{-\delta(t-s)} T_{t-s} |K\mathbf{T}_{s+t}\theta(x)|_{\widetilde{E}}\,\mathrm{d}s \right\} \right\}^{1/2} \qquad \text{(by (2.8))} \\
&= \sqrt{2}\,\mathcal{H}\theta(x) \\
&\quad + \sqrt{2} \left\{ \int_0^\infty \left\{ \int_0^t \|K\|_\infty \mathrm{e}^{-\delta(t-s)} T_{t-s} \mathrm{e}^{-\delta(s+t)} T_{s+t} |\theta|_E(x)\,\mathrm{d}s \right\}^2 \mathrm{d}t \right\}^{1/2} \\
&= \sqrt{2}\,\mathcal{H}\theta(x) + \sqrt{2} \left\{ \int_0^\infty \|K\|_\infty^2 \mathrm{e}^{-4\delta t} \big(T_{2t} |\theta|_E(x) \big)^2 t^2\,\mathrm{d}t \right\}^{1/2}
\end{aligned}
$$

$$= \sqrt{2}\,\mathcal{H}\theta(x) + \sqrt{2}\,\|K\|_\infty \theta^*(x)\left\{\int_0^\infty t^2 \mathrm{e}^{-4\delta t}\,\mathrm{d}t\right\}^{1/2}$$
$$= \sqrt{2}\,\mathcal{H}\theta(x) + \frac{1}{4\delta^{3/2}}\,\|K\|_\infty \theta^*(x)$$

which is the desired result. □

Combining these two propositions and the maximal ergodic inequality, we easily obtain the following.

Proposition 3.4. *For $p \geqslant 2$, we have*

$$\|\mathcal{P}\theta\|_p \lesssim \|\theta\|_p. \tag{3.11}$$

Before proving the theorem, we give an expression of $\varphi(-\mathbf{L})$. Recall that

$$\varphi(\lambda) = \lambda \int_0^\infty \mathrm{e}^{-2t\lambda} m(t)\,\mathrm{d}t.$$

There exists the following correspondence:

$$\begin{aligned} -\mathbf{L} &\longleftrightarrow \lambda \\ \mathrm{e}^{t\mathbf{L}} &\longleftrightarrow \mathrm{e}^{-t\lambda}. \end{aligned}$$

Therefore $\varphi(-\mathbf{L})$ is expressed as

$$\varphi(-\mathbf{L}) = -\mathbf{L}\int_0^\infty \mathbf{T}_{2t}\, m(t)\,\mathrm{d}t.$$

Proof of Theorem 1.1. Using the expression above, we have

$$\begin{aligned}
\big(\varphi(-\mathbf{L})\theta,\eta\big) &= \left(-\mathbf{L}\int_0^\infty \mathbf{T}_{2t}\, m(t)\,\mathrm{d}t\,\theta,\eta\right) \\
&= \int_0^\infty \int_M (-\mathbf{L}\mathbf{T}_{2t}\theta,\eta)_E\,\mu(\mathrm{d}x) m(t)\,\mathrm{d}t \\
&= \int_0^\infty m(t)\,\mathrm{d}t \int_M \{(-\hat{L}\mathbf{T}_t\theta,\mathbf{T}_t\eta)_E + \big((\kappa+R)\mathbf{T}_t\theta,\mathbf{T}_t\eta\big)_E\}\,\mu(\mathrm{d}x) \\
&= \int_0^\infty m(t)\,\mathrm{d}t \int_M \hat{\Gamma}(\mathbf{T}_t\theta,\mathbf{T}_t\eta)_E \\
&\quad + \int_0^\infty m(t)\,\mathrm{d}t \big((\kappa+R)\mathbf{T}_t\theta,\mathbf{T}_t\eta\big)_E\,\mu(\mathrm{d}x).
\end{aligned}$$

We estimate two terms on the right hand side respectively.

For the first term,

$$\left|\int_0^\infty m(t)\,\mathrm{d}t \int_M \hat{\Gamma}(\mathbf{T}_t\theta,\mathbf{T}_t\eta)\,\mu(\mathrm{d}x)\right|$$

$$
\begin{aligned}
&\leqslant \|m\|_\infty \int_0^\infty \mathrm{d}t \int_M \hat{\Gamma}(\mathbf{T}_t\theta, \mathbf{T}_t\theta)^{1/2} \hat{\Gamma}(\mathbf{T}_t\eta, \mathbf{T}_t\eta)^{1/2}\, \mu(\mathrm{d}x) \\
&\qquad \text{(thanks to the Schwarz inequality for } \hat{\Gamma}) \\
&\leqslant \|m\|_\infty \int_M \left\{ \int_0^\infty \hat{\Gamma}(\mathbf{T}_t\theta, \mathbf{T}_t\theta)\,\mathrm{d}t \right\}^{1/2} \left\{ \int_0^\infty \hat{\Gamma}(\mathbf{T}_t\eta, \mathbf{T}_t\eta)\,\mathrm{d}t \right\}^{1/2} \mu(\mathrm{d}x) \\
&= \|m\|_\infty \int_M \mathcal{P}\theta(x)\mathcal{P}\eta(x)\,\mu(\mathrm{d}x) \\
&\leqslant \|m\|_\infty \, \|\mathcal{P}\theta\|_p \, \|\mathcal{P}\eta\|_q \\
&\lesssim \|m\|_\infty \, \|\theta\|_p \, \|\eta\|_q .
\end{aligned}
$$

For the second term,

$$
\begin{aligned}
&\left| \int_0^\infty m(t)\,\mathrm{d}t \big((\kappa + R)\mathbf{T}_t\theta, \mathbf{T}_t\eta \big)_E \, \mu(\mathrm{d}x) \right| \\
&\qquad \leqslant \|m\|_\infty \int_0^\infty \mathrm{d}t \int_M \|\kappa + R\|_\infty \, |\mathbf{T}_t\theta|_E \, |\mathbf{T}_t\eta|_E \, \mu(\mathrm{d}x) \\
&\qquad \leqslant \|m\|_\infty \int_0^\infty \mathrm{d}t \int_M \|\kappa + R\|_\infty \, \mathrm{e}^{-2\delta t} T_t|\theta|_E \, T_t|\eta|_E \, \mu(\mathrm{d}x) \\
&\qquad \leqslant \|m\|_\infty \, \|\kappa + R\|_\infty \, \frac{1}{2\delta} \, \|\theta\|_p \, \|\eta\|_q .
\end{aligned}
$$

Thus we have shown that

$$
|(\varphi(-\mathbf{L})\theta, \eta)| \lesssim \|\theta\|_p \, \|\eta\|_q
$$

which implies that $\varphi(-\mathbf{L})$ is bounded in L^p. □

4 Hodge–Kodaira operator

In this section we consider the the Hodge–Kodaira operator $-(dd^* + d^*d)$ acting on differential forms. What remains to show is the defective intertwining property. We have to seek for operators $\mathbf{\Lambda}$ and K that satisfy

$$
-\nabla(dd^* + d^*d)\theta = \mathbf{\Lambda}\nabla\theta + K\theta.
$$

Even if θ is a differential form, $\nabla\theta$ is no more a differential form. So we discuss the issue in the framework of tensor fields. Let M be a Riemannian manifold and ∇ be the Levi-Civita connection. The Riemannian curvature tensor is defined by

$$
R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z
$$

where X, Y, $Z \in \Gamma(TM)$. Here Γ denotes the set of all smooth sections of a vector bundle. In this case, $\Gamma(TM)$ is the set of vector fields. Let

$T_nM = \underbrace{T^*M \otimes \cdots \otimes T^*M}_{n}$ be a tensor bundle of type $(0, n)$. Exterior bundle is denoted by $\bigwedge^p T^*M = \underbrace{T^*M \wedge \cdots \wedge T^*M}_{n}$. We define an operator Δ^{HK} on $\Gamma(T_nM)$ as follows. $u_1 \otimes \cdots \otimes u_n$, $u_i \in \Gamma(T^*M)$ is a typical form of an element of $\Gamma(T_nM)$. Any element of $\Gamma(T_nM)$ can be written as a linear combination of them. We are given a Riemmanian metric g and there exists a natural isomophism $\sharp\colon T^*M \to TM$ e.g.,

$$\langle \omega, X\rangle = g(\omega^\sharp, X), \qquad \omega \in T^*M,\ X \in TM.$$

In the sequel, we omit g and denote the inner product $g(X,Y)$ by (X,Y). The inner product in T^*M is also denoted by (ω, η). The natural pairing between T^*M and TM is denoted by $\langle \omega, X\rangle$. We take a local orthonormal basis $\{e_1, \dots, e_n\}$ and let $\{\omega^1, \dots, \omega^n\}$ be its dual basis. We introduce linear opetators $S^{(n)}_{p,q}$ $1 \leqslant p, q \leqslant n$ on $\Gamma(T_n)$ as follows; for $p \neq q$,

$$\begin{aligned} &S^{(n)}_{p,q}(u_1 \otimes \cdots \otimes u_n) \\ &\qquad = \big(R(u_p^\sharp, e_k)u_q^\sharp, e_l\big) u_1 \otimes \cdots \otimes \overset{p}{\breve{\omega^k}} \otimes \cdots \otimes \overset{q}{\breve{\omega^l}} \otimes \cdots \otimes u_n. \end{aligned} \tag{4.1}$$

Here we used the Einstein rule: we omit the summation sign for repeated indices. For example, in the equation above $\sum_{k,l=1}^n$ is omitted. For $p = q$, we define

$$\begin{aligned} S^{(n)}_{p,p}(u_1 \otimes \cdots \otimes u_n) &= \big(\mathrm{Ric}\, u_p^\sharp, e_k\big) u_1 \otimes \cdots \otimes \overset{p}{\breve{\omega^k}} \otimes \cdots \otimes u_n \\ &= \big(R(u_p^\sharp, e_i)e_i, e_k\big) u_1 \otimes \cdots \otimes \overset{p}{\breve{\omega^k}} \otimes \cdots \otimes u_n. \end{aligned} \tag{4.2}$$

Ric denotes the Ricci tensor.

We now define the operator Δ^{HK} by

$$\Delta^{\mathrm{HK}} v = -\nabla^*\nabla v - \sum_{p,q=1}^n S^{(n)}_{p,q} v. \tag{4.3}$$

Here the superscript HK stands for Hodge–Kodaira. This notation is justified by the following proposition.

Proposition 4.1. *For $\theta \in \Gamma(\bigwedge^p T^*M)$, it holds that*

$$\Delta^{\mathrm{HK}}\theta = -(dd^* + d^*d)\theta. \tag{4.4}$$

Proof. We first note the following identity: for $u_1, \dots, u_n \in \Gamma(T^*M)$,

$$u_1 \wedge u_2 \wedge \cdots \wedge u_n := \sum_\sigma \mathrm{sgn}\,\sigma\, u_{\sigma(n)} \otimes \cdots \otimes u_{\sigma(n)}$$

$$= \sum_{\alpha} (-1)^{\alpha-1} u_\alpha \otimes (u_1 \wedge \overset{\alpha}{\check{\cdots}} \wedge u_n).$$

Here σ runs over the set of all permutations of order n, $\operatorname{sgn}\sigma$ is the sign of σ and $\overset{\alpha}{\vee}$ means that u_α is deleted. Similarly we have

$$u_1 \wedge u_2 \wedge \cdots \wedge u_n = \sum_{\alpha<\beta} (-1)^{\alpha+\beta-1} (u_\alpha \otimes u_\beta - u_\beta \otimes u_\alpha)(u_1 \wedge \overset{\alpha\ \beta}{\check{\cdots}\check{}} \wedge u_n).$$

Next let us compute $\sum_{p,q} S^{(n)}_{p,q}$. First, for $\sum_{p\neq q} S^{(n)}_{p,q}$

$$\begin{aligned}
&\sum_{p\neq q} S^{(n)}_{p,q}(u_1 \wedge u_2 \wedge \cdots \wedge u_n) \\
&= \sum_{p\neq q} S^{(n)}_{p,q} \sum_{\sigma} \operatorname{sgn}\sigma\, u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)} \\
&= \sum_{p\neq q} \sum_{\sigma} \operatorname{sgn}\sigma\, (R(u^\sharp_{\sigma(p)}, e_k) u^\sharp_{\sigma(q)}, e_l) u_{\sigma(1)} \otimes \cdots \otimes \overset{p}{\breve{\omega}^k} \otimes \cdots \otimes \overset{q}{\breve{\omega}^l} \otimes \cdots \otimes u_{\sigma(n)}.
\end{aligned}$$

Here p-th $u_{\sigma(p)}$ is replaced by ω^k and q-th $u_{\sigma(q)}$ is replaced by ω^l. By exchanging the order of summation, we have

$$\begin{aligned}
&\sum_{p\neq q} S^{(n)}_{p,q}(u_1 \wedge u_2 \wedge \cdots \wedge u_n) \\
&\quad= \sum_{\alpha\neq\beta} \sum_{\sigma} \operatorname{sgn}\sigma \big(R(u^\sharp_\alpha, e_k) u^\sharp_\beta, e_l\big) u_{\sigma(1)} \otimes \cdots \otimes \overset{\sigma^{-1}(\alpha)}{\breve{\omega}^k} \otimes \cdots \otimes \overset{\sigma^{-1}(\beta)}{\breve{\omega}^l} \otimes \cdots \otimes u_{\sigma(n)} \\
&\quad= \sum_{\alpha\neq\beta} \big(R(u^\sharp_\alpha, e_k) u^\sharp_\beta, e_l\big) \sum_{\sigma} \operatorname{sgn}\sigma\, u_{\sigma(1)} \otimes \cdots \otimes \overset{\sigma^{-1}(\alpha)}{\breve{\omega}^k} \otimes \cdots \otimes \overset{\sigma^{-1}(\beta)}{\breve{\omega}^l} \otimes \cdots \otimes u_{\sigma(n)} \\
&\quad= \sum_{\alpha\neq\beta} \big(R(u^\sharp_\alpha, e_k) u^\sharp_\beta, e_l\big) u_1 \wedge \cdots \wedge \overset{\alpha}{\breve{\omega}^k} \wedge \cdots \wedge \overset{\beta}{\breve{\omega}^l} \wedge \cdots \wedge u_n \\
&\quad= \sum_{\alpha<\beta} \big(R(u^\sharp_\alpha, e_k) u^\sharp_\beta, e_l\big) \omega^k \wedge \omega^l \wedge u_1 \wedge \overset{\alpha\ \beta}{\check{\cdots}\check{}} \wedge u_n \\
&\qquad+ \sum_{\alpha>\beta} \big(R(u^\sharp_\alpha, e_k) u^\sharp_\beta, e_l\big) \omega^l \wedge \omega^k \wedge u_1 \wedge \overset{\alpha\ \beta}{\check{\cdots}\check{}} \wedge u_n \\
&\quad= \sum_{\alpha<\beta} \big(R(u^\sharp_\alpha, e_k) u^\sharp_\beta, e_l\big) \omega^k \wedge \omega^l \wedge u_1 \wedge \overset{\alpha\ \beta}{\check{\cdots}\check{}} \wedge u_n \\
&\qquad+ \sum_{\alpha>\beta} \big(R(u^\sharp_\beta, e_l) u^\sharp_\alpha, e_k\big) \omega^l \wedge \omega^k \wedge u_1 \wedge \overset{\alpha\ \beta}{\check{\cdots}\check{}} \wedge u_n \\
&\quad= 2 \sum_{\alpha<\beta} \big(R(u^\sharp_\alpha, e_k) u^\sharp_\beta, e_l\big) \omega^k \wedge \omega^l \wedge u_1 \wedge \overset{\alpha\ \beta}{\check{\cdots}\check{}} \wedge u_n.
\end{aligned}$$

Similarly we have

$$\begin{aligned}\sum_p S^{(n)}_{p,p}(u_1\wedge u_2\wedge\cdots\wedge u_n) &= \sum_p S^{(n)}_{p,p}\sum_\sigma \operatorname{sgn}\sigma\, u_{\sigma(1)}\otimes\cdots\otimes u_{\sigma(n)}\\ &= \sum_p\sum_\sigma \operatorname{sgn}\sigma\left(\operatorname{Ric} u^\sharp_{\sigma(p)}, e_k\right)u_{\sigma(1)}\otimes\cdots\otimes \overset{p}{\breve{\omega}^k}\otimes\cdots\otimes u_{\sigma(n)}.\end{aligned}$$

Here p-th $u_{\sigma(p)}$ is replaced by ω^k. Exchanging the order of summation, we have

$$\begin{aligned}\sum_p S^{(n)}_{p,p}&(u_1\wedge u_2\wedge\cdots\wedge u_n)\\ &= \sum_\alpha\sum_\sigma \operatorname{sgn}\sigma\,(\operatorname{Ric} u^\sharp_\alpha, e_k)u_{\sigma(1)}\otimes\cdots\otimes \overset{\sigma^{-1}(\alpha)}{\breve{\omega}^k}\otimes\cdots\otimes u_{\sigma(n)}\\ &= \sum_\alpha(\operatorname{Ric} u^\sharp_\alpha, e_k)u_1\wedge\cdots\wedge \overset{\alpha}{\breve{\omega}^k}\wedge\cdots\wedge u_n\\ &= \sum_\alpha(-1)^{\alpha-1}(\operatorname{Ric} u^\sharp_\alpha, e_k)\omega^k\wedge u_1\wedge \overset{\alpha}{\overset{\vee}{\cdots}}\wedge u_n\end{aligned}$$

Using this identity, we can calculate $-(dd^*+d^*d)$. Before that we have to recall the Weitzenböck formula:

$$\begin{aligned}-(dd^*+d^*d) &= -\nabla^*\nabla\\ &\quad+\big(R(e_l,e_j)e_k,e_i\big)\omega^l\wedge\omega^k\wedge i(e_j)i(e_i)-(\operatorname{Ric} e_k,e_i)\omega^k\wedge i(e_i).\end{aligned}$$

Here $i(\,.\,)$ denotes the interior product, i.e., $i(X)\theta=\theta(X,\,.\,,\dots,\,.\,)$. Now we have

$$\begin{aligned}-(dd^*&+d^*d)(u_1\wedge\cdots\wedge u_n)\\ &= -\nabla^*\nabla(u_1\wedge\cdots\wedge u_n)+\big(R(e_l,e_j)e_k,e_i\big)\omega^l\wedge\omega^k\wedge i(e_j)i(e_i)\\ &\qquad\times\sum_{\alpha<\beta}(-1)^{\alpha+\beta-1}(u_\alpha\otimes u_\beta-u_\beta\otimes u_\alpha)(u_1\wedge \overset{\alpha\ \ \beta}{\overset{\vee\ \vee}{\cdots}}\wedge u_n)\\ &\qquad-(\operatorname{Ric} e_k,e_i)\omega^k\wedge i(e_i)\sum_\alpha(-1)^{\alpha-1}u_\alpha\otimes(u_1\wedge \overset{\alpha}{\overset{\vee}{\cdots}}\wedge u_n)\\ &= -\nabla^*\nabla(u_1\wedge\cdots\wedge u_n)\\ &\qquad+\sum_{\alpha<\beta}(-1)^{\alpha+\beta-1}\{\langle u_\alpha,e_i\rangle\langle u_\beta,e_j\rangle-\langle u_\beta,e_i\rangle\langle u_\alpha,e_j\rangle\}\\ &\qquad\quad\times\big(R(e_l,e_j)e_k,e_i\big)\omega^l\wedge\omega^k\wedge u_1\wedge \overset{\alpha\ \ \beta}{\overset{\vee\ \vee}{\cdots}}\wedge u_n\\ &\qquad-\sum_\alpha(-1)^{\alpha-1}(\operatorname{Ric} e_k,e_i)\langle u_\alpha,e_i\rangle i(e_i)\omega^k\wedge u_1\wedge \overset{\alpha}{\overset{\vee}{\cdots}}\wedge u_n\\ &= -\nabla^*\nabla(u_1\wedge\cdots\wedge u_n)+\sum_{\alpha<\beta}(-1)^{\alpha+\beta-1}\end{aligned}$$

$$\times \left\{ \left(R(e_l, u_\beta^\sharp) e_k, u_\alpha^\sharp \right) - \left(R(e_l, u_\alpha^\sharp) e_k, u_\beta^\sharp \right) \right\} \omega^l \wedge \omega^k \wedge u_1 \wedge \overset{\alpha}{\vee} \cdots \overset{\beta}{\vee} \wedge u_n$$
$$- \sum_\alpha (-1)^{\alpha-1} (\mathrm{Ric}\, e_k, u_\alpha) \omega^k \wedge u_1 \wedge \overset{\alpha}{\check{\cdots}} \wedge u_n$$
$$= -\nabla^*\nabla(u_1 \wedge \cdots \wedge u_n)$$
$$+ 2 \sum_{\alpha<\beta} (-1)^{\alpha+\beta-1} \left(R(e_l, u_\beta^\sharp) e_k, u_\alpha^\sharp \right) \omega^l \wedge \omega^k \wedge u_1 \wedge \overset{\alpha}{\vee} \cdots \overset{\beta}{\vee} \wedge u_n$$
$$- \sum_\alpha (-1)^{\alpha-1} (\mathrm{Ric}\, e_k, u_\alpha) \omega^k \wedge u_1 \wedge \overset{\alpha}{\check{\cdots}} \wedge u_n$$
$$= -\nabla^*\nabla(u_1 \wedge \cdots \wedge u_n)$$
$$- 2 \sum_{\alpha<\beta} (-1)^{\alpha+\beta-1} \left(R(u_\alpha^\sharp, e_k) u_\beta^\sharp, e_l \right) \omega^k \wedge \omega^l \wedge u_1 \wedge \overset{\alpha}{\vee} \cdots \overset{\beta}{\vee} \wedge u_n$$
$$- \sum_\alpha (-1)^{\alpha-1} (\mathrm{Ric}\, e_k, u_\alpha) \omega^k \wedge u_1 \wedge \overset{\alpha}{\check{\cdots}} \wedge u_n$$
$$= -\nabla^*\nabla(u_1 \wedge \cdots \wedge u_n) - \sum_{p,q} S_{p,q}^{(n)} (u_1 \wedge \cdots \wedge u_n)$$
$$= \Delta^{\mathrm{HK}} (u_1 \wedge \cdots \wedge u_n)$$

which is the required identity. □

We are interested in the intertwining property for the Hodge–Kodaira operator $-(dd^* + d^*d)$. By the above proposition, it is enough to calculate Δ^{HK}. We first show the intertwining property for $\nabla^*\nabla$.

Proposition 4.2. *It holds that*

$$-\nabla(\nabla^*\nabla)u - (\nabla^*\nabla)\nabla u = \sum_{j=2}^{n+1} \{ S_{1,j}^{(n+1)} \nabla u + S_{j,1}^{(n+1)} \nabla u \} + S_{1,1}^{(n+1)} \nabla u + \omega^k \otimes \nabla_i R^{(n)}(e_i, e_k) u \tag{4.5}$$

Proof. Pick a point $x \in M$ and fix it. We take a normal coordinate at x. Then there exists a local frame $\{e_1, e_2, \ldots, e_n\}$ of TM so that $\nabla_{e_i} e_j(x) = 0$. To avoid complexity, we simply denote ∇_i in place of ∇_{e_i}. Let $\{\omega^1, \omega^2, \ldots, \omega^n\}$ be the dual frame. Due to our choice of a local frame, at the point x it holds that $\nabla_{i,j}^2 = \nabla_i \nabla_j$, $[e_i, e_j] = 0$ and $\nabla_i \omega^k = 0$. Moreover we have the following identity at x:

$$-[e_i, \nabla_j e_k] = \nabla_i \nabla_j e_k, \tag{4.6}$$
$$\nabla_i \nabla_{\nabla_j e_k} = \nabla_{\nabla_i \nabla_j e_k}, \tag{4.7}$$
$$\langle \nabla_i \nabla_i e_k, \omega^l \rangle = -\langle e_k, \nabla_i \nabla_i \omega^l \rangle \tag{4.8}$$

Here (4.7) is the identity for $T_n M$.

To see (4.6) we note that the torsion is free and so we have

$$[e_i, \nabla_j e_k] = \nabla_i \nabla_j e_k - \nabla_{\nabla_j e_k} e_i = \nabla_i \nabla_j e_k.$$

As for (4.7), we use the definition of the curvature $R^{(n)}$.

$$\begin{aligned}\nabla_i \nabla_{\nabla_j e_k} &= R^{(n)}(e_i, \nabla_j e_k) + \nabla_{\nabla_j e_k} \nabla_i + \nabla_{[e_i, \nabla_j e_k]} \\ &= \nabla_{\nabla_i \nabla_j e_k}. \qquad \text{(thanks to (4.6))}\end{aligned}$$

(4.8) can be shown as

$$\begin{aligned}0 &= \nabla_i \nabla_i \langle e_k, \omega^l \rangle \\ &= \langle \nabla_i \nabla_i e_k, \omega^l \rangle + 2\langle \nabla_i e_k, \nabla_i \omega^l \rangle + \langle e_k, \nabla_i \nabla_i \omega^l \rangle \\ &= \langle \nabla_i \nabla_i e_k, \omega^l \rangle + \langle e_k, \nabla_i \nabla_i \omega^l \rangle.\end{aligned}$$

We use these identities freely. From now on all equations are evaluated at the point x. Now

$$\begin{aligned}&-(\nabla^* \nabla) \nabla u + \nabla(\nabla^* \nabla) u \\ &\quad = \nabla_i \nabla_i (\omega^k \otimes \nabla_k u) - \nabla_{\nabla_i e_i}(\omega^k \otimes \nabla_k u) - \nabla(\nabla_i \nabla_i u - \nabla_{\nabla_i e_i} u) \\ &\quad = \nabla_i \nabla_i \omega^k \otimes \nabla_k u + 2\nabla_i \omega^k \otimes \nabla_i \nabla_k u + \omega^k \otimes \nabla_i \nabla_i \nabla_k u \\ &\qquad - \omega^k \otimes \nabla_k \nabla_i \nabla_i u + \omega^k \otimes \nabla_k \nabla_{\nabla_i e_i} u \\ &\quad = \nabla_i \nabla_i \omega^k \otimes \nabla_k u + \omega^k \otimes \nabla_i \nabla_i \nabla_k u - \omega^k \otimes \nabla_k \nabla_i \nabla_i u + \omega^k \otimes \nabla_k \nabla_{\nabla_i e_i} u \\ &\quad = \nabla_i \nabla_i \omega^k \otimes \nabla_k u + \omega^k \otimes \nabla_i \big\{ R^{(n)}(e_i, e_k) u + \nabla_k \nabla_i u + \nabla_{[e_i, e_k]} u \big\} \\ &\qquad - \omega^k \otimes \big\{ R^{(n)}(e_k, e_i) \nabla_i u + \nabla_i \nabla_k \nabla_i u + \nabla_{[e_k, e_i]} \nabla_i u \big\} + \omega^k \otimes \nabla_k \nabla_{\nabla_i e_i} u \\ &\quad = \nabla_i \nabla_i \omega^k \otimes \nabla_k u + \omega^k \otimes \big\{ \nabla_i R^{(n)}(e_i, e_k) u + R^{(n)}(\nabla_i e_i, e_k) u \\ &\qquad + R^{(n)}(e_i, \nabla_i e_k) u + R^{(n)}(e_i, e_k) \nabla_i u + \nabla_i \nabla_{[e_i, e_k]} u \big\} \\ &\qquad - \omega^k \otimes R^{(n)}(e_k, e_i) \nabla_i u + \omega^k \otimes \nabla_k \nabla_{\nabla_i e_i} u \\ &\quad = \nabla_i \nabla_i \omega^k \otimes \nabla_k u + \omega^k \otimes \nabla_i R^{(n)}(e_i, e_k) u + 2\omega^k \otimes R^{(n)}(e_i, e_k) \nabla_i u \\ &\qquad + \omega^k \otimes \big\{ \nabla_i \nabla_{[e_i, e_k]} u + \nabla_k \nabla_{\nabla_i e_i} u \big\}.\end{aligned}$$

On the other hand, using (4.6), (4.7) and (4.8), we have

$$\begin{aligned}&\omega^k \otimes \big\{ \nabla_i \nabla_{[e_i, e_k]} u + \nabla_k \nabla_{\nabla_i e_i} u \big\} \\ &\quad = \omega^k \otimes (\nabla_i \nabla_{\nabla_i e_k} - \nabla_i \nabla_{\nabla_k e_i} + \nabla_k \nabla_{\nabla_i e_i}) u \\ &\quad = \omega^k \otimes (\nabla_{\nabla_i \nabla_i e_k} + \nabla_{R(e_k, e_i) e_i} + \nabla_{\nabla_{[e_k, e_i]} e_i}) u \\ &\quad = \omega^k \otimes (\nabla_{\nabla_i \nabla_i e_k} + \nabla_{R(e_k, e_i) e_i}) u \\ &\quad = \omega^k \otimes (\langle \nabla_i \nabla_i e_k, \omega^l \rangle \nabla_l u + \langle R(e_k, e_i) e_i, \omega^l \rangle \nabla_l u) \\ &\quad = \omega^k \otimes \langle \nabla_i \nabla_i e_k, \omega^l \rangle \nabla_l u + \omega^k \otimes \langle \operatorname{Ric} e_k, \omega^l \rangle \nabla_l u\end{aligned}$$

$$= -\omega^k \otimes \langle e_k, \nabla_i \nabla_i \omega^l \rangle \nabla_l u + \omega^k \otimes \langle \operatorname{Ric} e_k, \omega^l \rangle \nabla_l u \qquad \text{(thanks to (4.8))}$$
$$= -\nabla_i \nabla_i \omega^l \otimes \nabla_l u + \omega^k \otimes \langle \operatorname{Ric} e_k, \omega^l \rangle \nabla_l u.$$

Combining all of them, we have

$$\begin{aligned}
&-(\nabla^*\nabla)\nabla u + \nabla(\nabla^*\nabla)u \\
&\quad = \omega^k \otimes \nabla_i R^{(n)}(e_i, e_k)u + 2\omega^k \otimes R^{(n)}(e_i, e_k)\nabla_i u + \omega^k \otimes \langle \operatorname{Ric} e_k, \omega^l \rangle \nabla_l u \\
&\quad = \omega^k \otimes \nabla_i R^{(n)}(e_i, e_k)u + \sum_{j=2}^{n+1} \big(S_{1,j}^{(n+1)} \nabla u + S_{j,1}^{(n+1)} \nabla u\big) + S_{1,1}^{(n+1)} \nabla u.
\end{aligned}$$

This completes the proof. □

We are now ready to prove the intertwining property for Δ^{HK}.

Proposition 4.3. *Take any local orthonormal frame* $\{e_1, e_2, \dots, e_d\}$ *and its dual frame* $\{\omega^1, \omega^2, \dots, \omega^d\}$. *Then it holds that*

$$\nabla \Delta_n^{\mathrm{HK}} u = \Delta_{n+1}^{\mathrm{HK}} \nabla u - \sum_{p,q=1}^{n} \omega^k \otimes (\nabla_k S_{p,q}^{(n)})u - \omega^k \otimes (\nabla_i R^{(n)}(e_i, e_k))u. \quad (4.9)$$

Proof. We recall (4.3). Then

$$\begin{aligned}
&\nabla \Delta_n^{\mathrm{HK}} u - \Delta_{n+1}^{\mathrm{HK}} \nabla u \\
&\quad = -\nabla \left(\nabla^*\nabla + \sum_{p,q=1}^{n} S_{p,q}^{(n)} \right) + \left(\nabla^*\nabla + \sum_{p,q=1}^{n+1} S_{p,q}^{(n+1)} \right) \nabla u \\
&\quad = -\sum_{j=2}^{n+1} \big(S_{1,j}^{(n+1)} \nabla u + S_{j,1}^{(n+1)} \nabla u\big) - S_{1,1}^{(n+1)} \nabla u - \omega^k \otimes \nabla_i R^{(n)}(e_i, e_k)u \\
&\qquad - \sum_{p,q=1}^{n} \omega^k \otimes \big(\nabla_k S_{p,q}^{(n)}\big)u - \sum_{p,q=1}^{n} \omega^k \otimes S_{p,q}^{(n)} \nabla_k u + \sum_{p,q=1}^{n+1} S_{p,q}^{(n+1)} \nabla u \\
&\quad = -\sum_{j=2}^{n+1} \big(S_{1,j}^{(n+1)} \nabla u + S_{j,1}^{(n+1)} \nabla u\big) - S_{1,1}^{(n+1)} \nabla u - \omega^k \otimes \nabla_i R^{(n)}(e_i, e_k)u \\
&\qquad - \sum_{p,q=1}^{n} \omega^k \otimes \big(\nabla_k S_{p,q}^{(n)}\big)u - \sum_{p,q \geqslant 2}^{n+1} S_{p,q}^{(n+1)} \nabla u + \sum_{p,q=1}^{n+1} S_{p,q}^{(n+1)} \nabla u \\
&\quad = -\sum_{p,q=1}^{n} \omega^k \otimes \big(\nabla_k S_{p,q}^{(n)}\big)u - \omega^k \otimes \nabla_i R^{(n)}(e_i, e_k)u
\end{aligned}$$

which completes the proof. □

The above intertwining property for Δ^{HK} is defective, i.e., it satisfies the identity of the type (2.11). The defective term is removed if we replace ∇ with the exterior derivative d. To define the exterior derivative, we need to introduce alternating operation A as follows. For a tensor u of type $(0,n)$, we define $A^{(n)}$ by

$$A^{(n)}u(X_1,\dots,X_n)=\sum_{\sigma}\operatorname{sgn}\sigma\, u(X_{\sigma(1)},\dots,X_{\sigma(n)}).$$

The exterior derivative is defined by

$$d=A^{(n+1)}\nabla u.$$

This definition is consistent with the usual definition for differential forms. Now we have the following intertwining property.

Proposition 4.4. *For $u\in\Gamma(T_n(M))$, it holds that*

$$d\Delta_n^{\mathrm{HK}}u=\Delta_{n+1}^{\mathrm{HK}}du. \tag{4.10}$$

Proof. By Proposition 4.3, we have

$$\begin{aligned}d\Delta_n^{\mathrm{HK}}u&=\Delta_{n+1}^{\mathrm{HK}}du\\&\quad-\sum_{p,q=1}^{n}A^{(n+1)}\Big(\omega^k\otimes\big(\nabla_k S_{p,q}^{(n)}\big)u\Big)-A^{(n+1)}\Big(\omega^k\otimes\big(\nabla_i R^{(n)}(e_i,e_k)\big)u\Big).\end{aligned}$$

We have to show that the additional terms vanish. Before proving this, we recall the Bianchi identity for the Riemannian curvature:

$$-\mathfrak{S}R(X,Y)Z=0, \tag{4.11}$$

$$\mathfrak{S}\nabla_X R(Y,Z)=0. \tag{4.12}$$

Here $\mathfrak{S}$ stands for the cyclic sum, e.g.,

$$\mathfrak{S}R(X,Y)Z=R(X,Y)Z+R(Y,Z)X+R(Z,X)Y.$$

(4.11) is called the first Bianchi identity and (4.12) is called the second Bianchi identity.

We may assume that $u=u_1\otimes\cdots\otimes u_n$. For $p=q$, we have

$$\begin{aligned}&\sum_{p=1}^{n}A^{(n+1)}\Big(\omega^k\otimes\big(\nabla_k S_{p,p}^{(n)}\big)u\Big)-A^{(n+1)}\Big(\omega^k\otimes\big(\nabla_i R^{(n)}(e_i,e_k)\big)u\Big)\\&\quad=\sum_{p=1}^{n}A^{(n+1)}\big(\omega^k\otimes u_1\otimes\cdots\otimes\nabla_k\operatorname{Ric}u_p^{\sharp}\otimes\cdots\otimes u_n\\&\qquad\qquad+\omega^k\otimes u_1\otimes\cdots\otimes\nabla_i R(e_i,e_k)u_p^{\sharp}\otimes\cdots\otimes u_n\big)\end{aligned}$$

$$= \sum_{p=1}^{n} (-1)^p \big\{ \nabla_k \operatorname{Ric} u_p^\sharp \wedge \omega^k \wedge u_1 \wedge \overset{p}{\check{\cdots}} \wedge u_n + \nabla_i R(e_i, e_k) u_p^\sharp \wedge \omega^k \wedge u_1 \wedge \overset{p}{\check{\cdots}} \wedge u_n \big\}.$$

We need to compute $\nabla_k \operatorname{Ric} u_p^\sharp \wedge \omega^k + \nabla_i R(e_i, e_k) u_p^\sharp \wedge \omega^k$. To do this,

$$\begin{aligned}
&\nabla_k \operatorname{Ric} u_p^\sharp \wedge \omega^k + \nabla_i R(e_i, e_k) u_p^\sharp \wedge \omega^k \\
&\quad = \nabla_k R(u_p^\sharp, e_i) e_i \wedge \omega^k + \nabla_i R(e_i, e_k) u_p^\sharp \wedge \omega^k \\
&\quad = \big(\nabla_k R(u_p^\sharp, e_i) e_i, e_l\big) \omega^l \wedge \omega^k + \big(\nabla_i R(e_i, e_k) u_p^\sharp, e_l\big) \omega^l \wedge \omega^k \\
&\quad = \big\{ -\big(\nabla_{u_p^\sharp} R(e_i, e_k) e_i, e_l\big) - \big(\nabla_i R(e_k, u_p^\sharp) e_i, e_l\big) \\
&\qquad + \big(\nabla_i R(e_i, e_k) u_p^\sharp, e_l\big) \big\} \omega^l \wedge \omega^k \qquad \text{(by the 2nd Bianchi identity)} \\
&\quad = -\big(\nabla_{u_p^\sharp} R(e_i, e_k) e_i, e_l\big) \omega^l \wedge \omega^k \\
&\qquad + \big\{ \big(\nabla_i R(e_i, e_l) u_p^\sharp, e_k\big) + \big(\nabla_i R(e_i, e_k) u_p^\sharp, e_l\big) \big\} \omega^l \wedge \omega^k \\
&\quad = 0.
\end{aligned}$$

Here, in the last line, we used that the coefficients are symmetric with respect to k and l.

For $p \neq q$, we may assume $p < q$.

$$\begin{aligned}
&A^{(n+1)} \Big(\omega^k \otimes \big(\nabla_k S_{p,q}^{(n)}\big) u \Big) \\
&\quad = A^{(n+1)} \Big(\omega^k \otimes \big(\nabla_k R(u_p^\sharp, e_l) u_q^\sharp, e_m\big) u_1 \otimes \cdots \otimes \overset{p}{\breve{w}^l} \otimes \cdots \otimes \overset{q}{\breve{\omega}^m} \otimes \cdots u_n \Big) \\
&\quad = \big(\nabla_k R(u_p^\sharp, e_l) u_q^\sharp, e_m\big) \omega^k \wedge u_1 \wedge \cdots \wedge \overset{p}{\breve{\omega}^l} \wedge \cdots \wedge \overset{q}{\breve{\omega}^m} \wedge \cdots \wedge u_n \\
&\quad = \big(\nabla_k R(u_p^\sharp, e_l) u_q^\sharp, e_m\big) \omega^k \wedge \omega^l \wedge \omega^m \wedge u_1 \wedge \overset{p}{\check{\cdot}} \overset{q}{\check{\cdot}} \wedge u_n.
\end{aligned}$$

To calculate $(\nabla_k R(u_p^\sharp, e_l) u_q^\sharp, e_m) \omega^k \wedge \omega^l \wedge \omega^m$, we have

$$\begin{aligned}
&\big(\nabla_k R(u_p^\sharp, e_l) u_q^\sharp, e_m\big) \omega^k \wedge \omega^l \wedge \omega^m \\
&\quad = \big\{ \big(\nabla_{u_p^\sharp} R(e_l, e_k) u_q^\sharp, e_m\big) - \big(\nabla_l R(e_k, u_p^\sharp) u_q^\sharp, e_m\big) \big\} \omega^k \wedge \omega^l \wedge \omega^m \\
&\qquad\qquad \text{(by the 2nd Bianchi identity)} \\
&\quad = \big(\nabla_{u_p^\sharp} R(e_l, e_k) e_m, u_q^\sharp\big) \omega^k \wedge \omega^l \wedge \omega^m - \big(\nabla_l R(e_k, u_p^\sharp) u_q^\sharp, e_m\big) \omega^k \wedge \omega^l \wedge \omega^m \\
&\quad = \big(\nabla_l R(u_p^\sharp, e_k) u_q^\sharp, e_m\big) \omega^k \wedge \omega^l \wedge \omega^m \qquad \text{(by the first Bianchi identity)} \\
&\quad = \big(\nabla_k R(u_p^\sharp, e_l) u_q^\sharp, e_m\big) \omega^l \wedge \omega^k \wedge \omega^m \qquad \text{(by relabeling)} \\
&\quad = -\big(\nabla_k R(u_p^\sharp, e_l) u_q^\sharp, e_m\big) \omega^k \wedge \omega^l \wedge \omega^m.
\end{aligned}$$

The last term is just the same as the original one with the opposite sign. Thus we have

$$\big(\nabla_k R(u_p^\sharp, e_l) u_q^\sharp, e_m\big) \omega^k \wedge \omega^l \wedge \omega^m = 0$$

as desired. □

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