Generalized entropy, fractional logarithmic Sobolev inequality, and Beckner type inequality

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Abstract

We discuss a generalization of entropy function in terms of Orlicz functional. Orlicz functional is defined by using Young function but we extend a scope of Young function adapted to entropy function. We introduce the ϕ-Sobolev inequality and discuss the relation between the spectral gap and the ϕ-Sobolev inequality. Taking function ϕ to be $\{\log(1+x)\}^\gamma$, we can define the fractional logarihtmic Sobolev inequality. The connection with the Latała-Oleszkiewicz inequality is also discussed.

1. Introduction

In this paper, we try to generalize the notion of entropy. To do this, we use Orlicz spaces. To define an Orlicz space, we need a Young function but our Young function belongs to a wider class than usual one. A Young function is defied as follows:

$$\Phi(x) = \int_0^x \phi(t)dt.$$

Here ϕ is a non-negative right-continuous non-decreasing function with $\phi(0) = 0$. In our definition, we do not assume the positivity of ϕ. By using this Young function, we can define an Orlicz functional and we see that the entropy function is a typical example of Orlicz functional. To be more precise, let (M, m) be a probability space. The entropy function is defied as follows:

$$\mathrm{Ent}(f) = E[f \log f] - E[f] \log E[f].$$

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Here E stands for the integration with respect to m. We will define an Orlicz functional $\| \|_\Phi$ (for the definition, see Section 2) so that if we take $\phi(x) = \log x$, then

$$\|f\|_\Phi = \mathrm{Ent}(f).$$

Using this Orlicz functional, we define a ϕ-Sobolev inequality and discuss its properties. Under suitable conditions, we can show that the ϕ-Sobolev inequality is in between the Poincaré inequality and the logarithmic Sobolev inequality. Such kind of inequality was discussed by, e.g., F.-Y. Wang [9], D. Chafaï[3], and we give a different formulation. We also discuss the fractional logarithmic Sobolev inequality. In particular, we discuss these inequalities in L^p setting. A typical inequality is the following:

$$E\left[|f|^p \log_+^{p\beta}(|f|^p/\|f\|_p^p)\right] \le K_1\|\nabla f\|_p^p + K_2\|f\|_p^p.$$

Further we consider the following Beckner type inequality:

$$\frac{E[|f|^p] - E[|f|^q]^{p/q}}{(p-q)^\delta} \le L_1\|\nabla f\|_p^p + L_2\|f\|_p^p.$$

Latała-Oleszkiewicz [5] discussed this type of inequality in the case $p = 2$. We generalize it to the L^p case.

The organization of the paper is as follows. In Section 2, we discuss the Orlicz functional for a generalized Young function. We define ϕ-Sobolev inequality and defective ϕ-Sobolev inequality and show that ϕ-Sobolev inequality is stronger than the Poincaré inequality. In Section 3, we introduce the fractional logarithmic Sobolev inequality and discuss it in L^p setting. Last, in Section 4, we discuss the relation between the fractional logarithmic Sobolev inequality and the Beckner type inequality,

2. Generalized entropy and Poincaré inequality

Young functions play a fundamental role in the Orlicz space theory. In this section, we treat a little wider class of functions. Usually, we are given a function ϕ satisfying $\phi(0) = 0$, $\phi(t) > 0$ if $t > 0$ and $\phi(\infty) = \infty$. Φ is defined to be an integral of ϕ. Here, we generalize ϕ so that

1. ϕ: $[0,\infty) \to \mathbb{R} \cup \{-\infty\}$ is continuous and strictly increasing.
2. $\phi(\infty) = \infty$.
3. ϕ is integrable on any finite interval of $[0,\infty)$.
4. ϕ is of class C^1.

In particular, ϕ can be negative and unbounded from below. Therefore it may happen that $\phi(0) = \lim_{t\to 0} \phi(t) = -\infty$. Typical example is $\phi(t) = \log t$. We will define a functional, which is an entropy when $\phi(t) = \log t$. Φ is defined as

$$\Phi(x) = \int_0^x \phi(t)\,dt$$

and satisfies the following:

1. Φ is convex and $\Phi(0) = 0$.

2. $\dfrac{\Phi(x)}{x}$ is strictly increasing, $\lim_{x\to 0} \dfrac{\Phi(x)}{x} = \phi(0)$, and $\lim_{x\to\infty} \dfrac{\Phi(x)}{x} = \phi(\infty)$.

The inverse function ψ is defined by

$$\psi(u) = \inf\{t; \phi(t) > u\}.$$

Note that $\psi(u)$ is defined for all $u \in \mathbb{R}$ and $\psi(u) = 0$ for $u \leq \phi(0)$. The complementary function Ψ is defined by

$$\Psi(y) = \int_{-\infty}^{y} \psi(u)\, du.$$

Then the following holds:

$$xy \leq \Phi(x) + \Psi(y), \tag{2.1}$$

$$x\phi(x) = \Phi(x) + \Psi(\phi(x)). \tag{2.2}$$

(2.1) is called the Young inequality. We also have

$$\Psi(y) = \sup_{x\geq 0}\{xy - \Phi(x)\}. \tag{2.3}$$

Suppose we are given a measure space (M, m). We assume that a reference measure m is a probability measure. We denote the integration with respect to m by $E[\ ]$. From the Young inequality, we have the following proposition.

Proposition 2.1. Let U, $V \geq 0$ be functions on the measure space and let E denote a integration with respect to a finite measure. Suppose $-\infty < E[U\phi(U)] < \infty$. If

$$E[U\phi(U)] \leq E[V\phi(U)] + C, \tag{2.4}$$

then we have

$$E[\Phi(U)] \leq E[\Phi(V)] + C. \tag{2.5}$$

Here we use the convention $x\phi(x) = 0$ for $x = 0$.

Proof. (2.4) means that $P[V > 0, U = 0] = 0$ when $\phi(0) = -\infty$. By the Young inequality, we have

$$E[V\phi(U)] \leq E[\Phi(V)] + E[\Psi(\phi(U))],$$
$$E[U\phi(U)] = E[\Phi(U)] + E[\Psi(\phi(U))].$$

Since $E[U\phi(U)] < \infty$, it follows that $E[\Psi(\phi(U))] < \infty$ and

$$\begin{aligned} E[\Phi(U)] &= E[U\phi(U)] - E[\Psi(\phi(U))] \\ &\leq E[V\phi(U)] + C - E[\Psi(\phi(U))] \\ &\leq E[\Phi(V)] + C, \end{aligned}$$

which is the desired inequality. □

We introduce some functionals in connection to Φ. Usually, these functions become norms but they do not in our case because ϕ is not positive.

First define N_Φ by

$$N_\Phi(f) = \inf\{\lambda > 0;\ E[\Phi(|f|/\lambda)] \le 1\}. \tag{2.6}$$

We denote by L^Φ the set of all functions with $N_\Phi(f) < \infty$ and call it an Orlicz space. For $f \in L^\Phi$, $F(t) = E[\Phi(t|f|)]$ is a convex function. It is easy to see that $F(T) = 1$ if and only if $T = \dfrac{1}{N_\Phi(f)}$. We denote the minimum of Φ by $-m$. N_Φ satisfies the following.

Proposition 2.2. We have

(1) $N_\Phi(f) \ge 0$.

(2) $N_\Phi(f) = 0$ if and only if $f = 0$.

(3) $N_\Phi(cf) = |c| N_\Phi(f)$.

(4) If $f_1,\ f_2 \ge 0$, then $N_\Phi(f_1 + f_2) \le N_\Phi(f_1) + N_\Phi(f_2)$.

(5) For general $f_1,\ f_2$, $N_\Phi(f_1 + f_2) \le (1+m)(N_\Phi(f_1) + N_\Phi(f_2))$.

Proof. (1), (2), (3) are easy.

To show (4), set $a_i = N_\Phi(f_i)$, $b = a_1 + a_2$. Then

$$\begin{aligned}
E\left[\Phi\left(\frac{f_1+f_2}{b}\right)\right] &= E\left[\Phi\left(\frac{a_1}{b}\frac{f_1}{a_1} + \frac{a_2}{b}\frac{f_2}{a_2}\right)\right] \\
&\le E\left[\frac{a_1}{b}\Phi\left(\frac{f_1}{a_1}\right) + \frac{a_2}{b}\Phi\left(\frac{f_2}{a_2}\right)\right] \\
&\le \frac{a_1}{b} + \frac{a_2}{b} = 1.
\end{aligned}$$

Now $N_\Phi(f_1 + f_2) \le b$ follows.

To show (5), set $\Phi_+ = \max\{\Phi, 0\}$. Φ_+ is a non-negative, increasing and convex function satisfying $\Phi_+(0) = 0$. In particular, $\Phi_+(\theta x) \le \theta \Phi_+(x)$, $0 < \theta < 1$. We also note that $\Phi \le \Phi_+ \le \Phi + m$. Now, setting $a_i = N_\Phi(f_i)$, $b = a_1 + a_2$, we have

$$\begin{aligned}
E\left[\Phi\left(\frac{|f_1+f_2|}{(1+m)b}\right)\right] &\le E\left[\Phi_+\left(\frac{|f_1+f_2|}{(1+m)b}\right)\right] \\
&\le E\left[\Phi_+\left(\frac{|f_1|+|f_2|}{(1+m)b}\right)\right] \\
&\le E\left[\Phi_+\left(\frac{a_1}{b}\frac{|f_1|}{(1+m)a_1} + (\frac{a_2}{b}\frac{|f_2|}{(1+m)a_2}\right)\right] \\
&\le \frac{a_1}{b}E\left[\Phi_+\left(\frac{|f_1|}{(1+m)a_1}\right)\right] + \frac{a_2}{b}E\left[\Phi_+\left(\frac{|f_2|}{(1+m)a_2}\right)\right] \\
&\le \frac{a_1}{b}\frac{1}{1+m}E\left[\Phi_+\left(\frac{|f_1|}{a_1}\right)\right] + \frac{a_2}{b}\frac{1}{1+m}E\left[\Phi_+\left(\frac{|f_2|}{a_2}\right)\right]
\end{aligned}$$

$$\leq \frac{a_1}{b}\frac{1}{1+m}E\left[\Phi\left(\frac{|f_1|}{a_1}\right)+m\right]+\frac{a_2}{b}\frac{1}{1+m}E\left[\Phi\left(\frac{|f_2|}{a_2}\right)+m\right]$$
$$=1,$$

which yields (5). □

The property (5) shows that N_Φ is a quasi-norm and so L^Φ is a topological linear space. This topology is stronger than L^1 as will be shown below.

Proposition 2.3. The following inequality holds:

$$\|f\|_1 \leq (1+\Psi(1))N_\Phi(f). \tag{2.7}$$

Proof. If $N_\Phi(f)=1$, then the Young inequality $xy \leq \Phi(x)+\Psi(y)$ yields

$$E[|f|y] \leq E[\Phi(|f|)+\Psi(y)] = 1+\Psi(y).$$

Setting $y=1$,

$$\|f\|_1 \leq 1+\Psi(1).$$

Now (2.7) follows easily. □

We introduce another functional $\| \ \|_\Phi$. Define

$$\|f\|_\Phi = \sup\{E[|f|g];\ E[\Psi(g)] \leq 1\}. \tag{2.8}$$

We call $\| \ \|_\Phi$ the *Orlicz functional*. If Φ is a nice Young function, then $\| \ \|_\Phi$ becomes a norm. But it does not in our case as we remarked.

Proposition 2.4. Let $f \geq 0$ be a non-trivial function. Choose $\lambda > 0$ so that

$$E[\Psi(\phi(\lambda f))] = 1. \tag{2.9}$$

Then we have

$$\|f\|_\Phi = E[f\phi(\lambda f)] = \frac{1}{\lambda}\{E[\Phi(\lambda f)]+1\}. \tag{2.10}$$

For general f, we have

$$\|f\|_\Phi = \inf_{\kappa>0}\frac{1}{\kappa}\{E[\Phi(\kappa|f|)]+1\}. \tag{2.11}$$

Proof. First we assume $E[\Psi(\phi(f))]=1$, i.e., in the case of $\lambda=1$. If g satisfies $E[\Psi(g)] \leq 1$, then, by the Young inequality, we have

$$E[fg] \leq E[\Phi(f)]+E[\Psi(g)] \leq E[\Phi(f)]+1.$$

If we take $g=\phi(f)$, then $E[\Psi(g)] \leq 1$ and

$$E[fg] = E[f\phi(f)] = E[\Phi(f)]+E[\Psi(\phi(f))] = E[\Phi(f)]+1.$$

This shows that $\|f\|_\Phi = E[f\phi(f)] = E[\Phi(f)] + 1$.

Next, for non-negative f, we take λ satisfying (2.9). The above result implies

$$\|\lambda f\|_\Phi = E[\lambda f \phi(\lambda f)] = E[\Phi(\lambda f)] + 1$$

and (2.10) follows because of $\|\lambda f\|_\Phi = \lambda \|f\|_\Phi$.

Last we see (2.11). If $E[\Psi(g)] \leq 1$, then, for $\kappa > 0$,

$$\begin{aligned} E[|f|g] &= \frac{1}{\kappa} E[\kappa |f| g] \\ &\leq \frac{1}{\kappa}\{E[\Phi(\kappa|f|)] + E[\Psi(g)]\} \\ &\leq \frac{1}{\kappa}\{E[\Phi(\kappa|f|)] + 1\}. \end{aligned}$$

Hence

$$E[|f|g] \leq \inf_{\kappa>0} \frac{1}{\kappa}\{E[\Phi(\kappa|f|)] + 1\}.$$

Since g is arbitrary, we have

$$\|f\|_\Phi \leq \inf_{\kappa>0} \frac{1}{\kappa}\{E[\Phi(\kappa|f|)] + 1\}.$$

We have already seen that the equality holds if $\kappa = \lambda$. Now the proof is complete. □

We see some properties of the Orlicz functional $\|f\|_\Phi$.

Proposition 2.5. The Orlicz functional $\| \ \|_\Phi$ satisfies the following

(1) For $f \not\equiv 0$, $\Psi^{-1}(1) \leq \dfrac{\|f\|_\Phi}{\|f\|_1}$. The equality holds if and only if $|f|$ is constant.

(2) $\|cf\|_\Phi = |c|\|f\|_\Phi$.

(3) For $f_1,\ f_2 \geq 0$, $\|f_1 + f_2\|_\Phi \leq \|f_1\|_\Phi + \|f_2\|_\Phi$.

Proof. Take g so that $g \equiv \Psi^{-1}(1)$. Then $E[\Psi(g)] = 1$ and so

$$E[|f|\Psi^{-1}(1)] \leq \|f\|_\Phi.$$

This means that

$$\Psi^{-1}(1) \leq \frac{\|f\|_\Phi}{\|f\|_1}.$$

It is easy to see that the equality holds when $|f|$ is constant.

We will show the converse: if the equality holds, then $|f|$ is constant. We take λ so that $E[\Psi(\phi(\lambda|f|)] = 1$. From Proposition 2.4, we have

$$\|f\|_\Phi = \frac{1}{\lambda}\{E[\Phi(\lambda|f|)] + 1\}.$$

Therefore

$$\begin{aligned}\|\lambda f\|_\Phi &= E[\Phi(\lambda|f|)] + 1\\ &\geq \Phi(E[\lambda|f|]) + 1 \qquad (\because\ \Phi \text{ is convex})\\ &= \Phi(E[\lambda|f|]) + \Psi(\Psi^{-1}(1))\\ &\geq E[\lambda|f|]\Psi^{-1}(1) \qquad (\because\ \text{the Young inequality})\\ &= \|\lambda f\|_\Phi.\end{aligned}$$

Hence, all inequalities above should be equalities. In particular, $E[\Phi(\lambda f)] = \Phi(E[\lambda f])$ and so f is constant since Φ is strictly convex.

(3) is easy. □

Let us see the relation between N_Φ and $\| \ \|_\Phi$.

Proposition 2.6. The following inequality holds:

$$\|f\|_\Phi \leq 2N_\Phi(f). \tag{2.12}$$

Proof. For any λ, we have

$$\|f\|_\Phi \leq \frac{1}{\lambda}\{E[\Phi(\lambda f) + 1\}.$$

Taking $\lambda = N_\Phi(f)^{-1}$,

$$\|f\|_\Phi \leq N_\Phi(f)\left\{E\left[\Phi\left(\frac{f}{N_\Phi(f)}\right)\right] + 1\right\} \leq 2N_\Phi(f),$$

which is the desired result. □

When we are given two Young functions Φ_1, Φ_2, we can have the following comparisons. If there exists a constant a so that $\Phi_1(x) \leq \Phi_2(ax)$, $x \geq 0$, then

(1) $N_{\Phi_1}(f) \leq aN_{\Phi_2}(f)$.

(2) $\|f\|_{\Phi_1} \leq a\|f\|_{\Phi_2}$.

The proof is standard (e.g., see [8, II.2.2]) and will be omitted.

To get the comparison, we only need the inequality near ∞, i.e., $\Phi_1(x) \leq \Phi_2(ax)$, $x \geq x_0$ for some constant x_0. In this case, we can find a constant C so that $N_{\Phi_1}(f) \leq CN_{\Phi_2}(f)$.

From now on, we add the following assumption to ϕ:

$$\sup_{x>0} x\phi'(x) = l < \infty. \tag{2.13}$$

This means that the growth order of ϕ is less than that of log.

Proposition 2.7. Under the assumption (2.13), we have

$$N_\Phi(f) \leq \max\{\|f\|_\Phi, l\|f\|_1\}. \tag{2.14}$$

Proof. Using $x\phi'(x) \le l$,

$$\frac{d}{dx}\Psi \circ \phi(x) = \Psi'(\phi(x))\phi'(x) = \phi^{-1} \circ \phi(x)\phi'(x) = x\phi'(x) \le l.$$

Since $\Psi \circ \phi(0) = 0$, we have$\Psi \circ \phi(x) \le lx$. Therefore

$$E\left[\Psi\left(\phi\left(\frac{f}{l\|f\|_1}\right)\right)\right] \le E\left[l\frac{f}{l\|f\|_1}\right] = 1.$$

Now we have

$$\begin{aligned}\|f\|_\Phi &= \sup\{E\,[fg]\,;\, E\,[\Psi(g)] \le 1\} \\ &\ge E\left[f\phi\left(\frac{f}{\max\{\|f\|_\Phi, l\|f\|_1\}}\right)\right] \\ &= \max\{\|f\|_\Phi, l\|f\|_1\}E\left[\frac{f}{\max\{\|f\|_\Phi, l\|f\|_1\}}\phi\left(\frac{f}{\max\{\|f\|_\Phi, l\|f\|_1\}}\right)\right] \\ &= \max\{\|f\|_\Phi, l\|f\|_1\}E\left[\Phi\left(\frac{f}{\max\{\|f\|_\Phi, l\|f\|_1\}}\right) + \Psi\left(\phi\left(\frac{f}{\max\{\|f\|_\Phi, l\|f\|_1\}}\right)\right)\right].\end{aligned}$$

This brings

$$\begin{aligned}E\left[\Phi\left(\frac{f}{\max\{\|f\|_\Phi, l\|f\|_1\}}\right)\right] &\le \frac{\|f\|_\Phi}{\max\{\|f\|_\Phi, l\|f\|_1\}} - E\left[\Psi\left(\phi\left(\frac{f}{\max\{\|f\|_\Phi, l\|f\|_1\}}\right)\right)\right] \\ &\le 1.\end{aligned}$$

This shows (2.14). $\square$

Under the assumption (2.14), let us further investigate $\|\ \|_\Phi$. Our aim is to generalize the inequality in Deuschel-Stroock [4, Chapter VI (6.1.26)]. Define a constant α as

$$\alpha = \phi^{-1} \circ \Psi^{-1}(1). \tag{2.15}$$

Proposition 2.8. Let f be a bounded function and set $\hat{f} = f - E\,[f]$. Then we have

$$\|f^2\|_\Phi \le \|\hat{f}^2\|_\Phi + 2l\|\hat{f}\|_2^2 + \Psi^{-1}(1)E\,[f]^2\,, \tag{2.16}$$

$$\|\hat{f}^2\|_\Phi \le \|f^2\|_\Phi + 2l\|f\|_2^2 - \Psi^{-1}(1)E\,[f]^2 \tag{2.17}$$

and further

$$\lim_{t\downarrow 0}\frac{\|(1+tf)^2\|_\Phi - \Psi^{-1}(1) - 2\Psi^{-1}(1)E\,[f]\,t}{t^2} = 2\alpha\phi'(\alpha)\|\hat{f}\|_2^2 + \Psi^{-1}(1)E\left[f^2\right]. \tag{2.18}$$

Proof. We first show

$$\|(1+tf)^2\|_\Phi \le \Psi^{-1}(1) + \Psi^{-1}(1)E\,[f]\,t + t^2\|f^2\|_\Phi + 2lt^2E\left[f^2\right]. \tag{2.19}$$

To do this, take any $\delta > 0$ and set

$$g_t = \phi(\lambda_t\{(1+tf)^2 + \delta\}).$$

Here λ_t is taken so that $E[\Psi(g_t)] = 1$. By Proposition 2.4, we have $\|(1+tf)^2+\delta\|_\Phi = E[\{(1+tf)^2+\delta\}g_t]$. Since $E[\Psi(g_t)] = 1$, by differentiating in t, we have

$$0 = E[\Psi'(g_t)g_t'] = E\left[\psi\circ\phi(\lambda_t\{(1+tf)^2+\delta\})g_t'\right] = \lambda_t E\left[\{(1+tf)^2+\delta\}g_t'\right],$$

which yields

$$E\left[\{(1+tf)^2+\delta\}g_t'\right] = 0. \tag{2.20}$$

On the other hand, g_t' is computed as

$$g_t' = \phi'\lambda_t'\{(1+tf)^2+\delta\} + \phi' 2\lambda_t(1+tf)f.$$

Here we have omitted the variable $\lambda_t\{(1+tf)^2+\delta\}$ of ϕ'. Substituting this into (2.20),

$$\begin{aligned} 0 &= E\left[\{(1+tf)^2+\delta\}\phi'\{\lambda_t'\{(1+tf)^2+\delta\}+2\lambda_t(1+tf)f\}\right] \\ &= \lambda_t' E\left[\{(1+tf)^2+\delta\}^2\phi'\right] + 2\lambda_t E\left[\phi'\{(1+tf)^2+\delta\}(1+tf)f\right] \end{aligned}$$

and therefore

$$\lambda_t' = -\frac{2\lambda_t E[\phi'\{(1+tf)^2+\delta\}(1+tf)f]}{E[\{(1+tf)^2+\delta\}^2\phi']}. \tag{2.21}$$

Now we set

$$K_t = \|(1+tf)^2+\delta\|_\Phi - t^2\|f^2\|_\Phi = E\left[\{(1+tf)^2+\delta\}g_t\right] - t^2\|f^2\|_\Phi.$$

Differentiating this, we have

$$\begin{aligned} K_t' &= E\left[\{(1+tf)^2+\delta\}g_t'\right] + E[2(1+tf)fg_t] - 2t\|f^2\|_\Phi \\ &= E[2(1+tf)fg_t] - 2t\|f^2\|_\Phi. \quad (\because (2.20)) \end{aligned}$$

Further

$$\begin{aligned} K_t'' &= 2E\left[f^2 g_t\right] + E[2(1+tf)fg_t'] - 2\|f^2\|_\Phi \\ &\le E[2(1+tf)fg_t'] \\ &= E\left[2(1+tf)f\left\{\phi'\lambda_t'\{(1+tf)^2+\delta\} + 2\phi'\lambda_t(1+tf)f\right\}\right] \\ &= 2\lambda_t' E\left[\phi'\{(1+tf)^2+\delta\}(1+tf)f\right] + 4E\left[\phi'\lambda_t(1+tf)^2 f^2\right] \\ &= -\frac{4\lambda_t E[\phi'\{(1+tf)^2+\delta\}(1+tf)f]^2}{E[\{(1+tf)^2+\delta\}\phi']} \\ &\quad + 4E\left[\phi'\lambda_t\{(1+tf)^2+\delta\}f^2\right] - 4E\left[\phi'\lambda_t\delta f^2\right] \quad (\because (2.21)) \\ &\le 4E\left[\phi'\lambda_t\{(1+tf)^2+\delta\}f^2\right]. \end{aligned}$$

The variable of ϕ' above is omitted. Writing it explicitly, we have

$$\phi'\lambda_t\{(1+tf)^2+\delta\}f^2 = \phi'(\lambda_t\{(1+tf)^2+\delta\})\lambda_t\{(1+tf)^2+\delta\}f^2 \le lf^2.$$

We now obtain

$$K_t'' \leq 4lE\left[f^2\right].$$

g_0 satisfies $\Psi(g_0) = 1$ and so $g_0 = \Psi^{-1}(1)$. By noting that $K_0 = E[(1+\delta)g_0] = (1+\delta)\Psi^{-1}(1)$, $K_0' = E[2fg_0] = 2\Psi^{-1}(1)E[f]$, we have

$$K_t \leq (1+\delta)\Psi^{-1}(1) + 2\Psi^{-1}(1)E[f]\,t + 2lt^2E\left[f^2\right].$$

Combining all of them,

$$\|(1+tf)^2+\delta\|_\Phi \leq (1+\delta)\Psi^{-1}(1) + 2\Psi^{-1}(1)E[f]\,t + t^2\|f^2\|_\Phi + 2lt^2E\left[f^2\right].$$

Letting $\delta \downarrow 0$, we get (2.19).

Next, we show (2.16), (2.17). If $E[f] = 0$, then (2.16), (2.17) are trivial since $f = \hat{f}$.

If $E[f] \neq 0$, then we take $\hat{f}/E[f]$ instead of f. Then

$$\left\|\left(\frac{f}{E[f]}\right)^2\right\|_\Phi = \left\|\left(1+\frac{\hat{f}}{E[f]}\right)^2\right\|_\Phi \leq \Psi^{-1}(1) + \left\|\left(\frac{\hat{f}}{E[f]}\right)^2\right\|_\Phi + 2l\left\|\frac{\hat{f}}{E[f]}\right\|_2^2$$

and (2.16) follows. If we take $-f/E[f]$ instead of f, then

$$\left\|\left(\frac{\hat{f}}{E[f]}\right)^2\right\|_\Phi = \left\|\left(1-\frac{f}{E[f]}\right)^2\right\|_\Phi \leq \Psi^{-1}(1) - 2\Psi^{-1}(1) + \left\|\left(\frac{f}{E[f]}\right)^2\right\|_\Phi + 2l\left\|\frac{f}{E[f]}\right\|_2^2.$$

which shows (2.17).

Lastly we show (2.18). Since t is small, we may assume that $1+tf$ is positive. Hence we can put $\delta = 0$ in the above argument. We set $g_t = \phi(\lambda_t(1+tf)^2)$ and choose λ_t so that $E[\Psi(g_t)] = 1$. Set

$$K_t = \|(1+tf)^2\|_\Phi = E\left[(1+tf)^2g_t\right].$$

By differentiating in t,

$$K_t' = E[2(1+tf)fg_t] + E\left[(1+tf)^2g_t'\right] = E[2(1+tf)fg_t]$$

and

$$K_t'' = E\left[2f^2g_t\right] + E[2(1+tf)fg_t']$$

Since $g_0 = \Psi^{-1}(1)$, $K_0 = \Psi^{-1}(1)$ and $K_0' = 2\Psi^{-1}(1)E[f]$. Let us compute K_0''.

$$g_t' = \phi'\lambda_t'(1+tf)^2 + \phi' 2\lambda_t(1+tf)f$$

and

$$\lambda_t' = -\frac{2\lambda_tE[\phi'(1+tf)^3f]}{E[(1+tf)^4\phi']}.$$

Noting that $\lambda_0 = \phi^{-1} \circ \Psi^{-1}(1) = \alpha$, we have

$$\lambda_0' = -\frac{2\lambda_0 E\left[\phi'(\lambda_0) f\right]}{E\left[\phi'(\lambda_0)\right]} = -2\alpha E\left[f\right],$$
$$g_0' = \phi'(\lambda_0)\lambda_0' + \phi'(\lambda_o)2\lambda_0 f = -2\alpha\phi'(\alpha)E\left[f\right] + 2\alpha\phi'(\alpha)f = 2\alpha\phi'(\alpha)\hat{f}.$$

Thus

$$\begin{aligned} K_0'' = E\left[2f^2 g_0\right] + E\left[2f g_0'\right] &= 2\Psi^{-1}(1)E\left[f^2\right] + 2E\left[f 2\alpha\phi'(\alpha)\hat{f}\right] \\ &= 2\Psi^{-1}(1)E\left[f^2\right] + 4\alpha\phi'(\alpha)E\left[\hat{f}^2\right]. \end{aligned}$$

Now we have

$$\lim_{t\downarrow 0}\frac{K_t - \Psi^{-1}(1) - 2\Psi^{-1}(1)E\left[f\right]t}{t^2} = \frac{K_0''}{2} = \Psi^{-1}(1)E\left[f^2\right] + 2\alpha\phi'(\alpha)E\left[\hat{f}^2\right],$$

which shows (2.18). □

The *entropy function* is defined by

$$\text{Ent}(f) = E[f\log f] - E[f]\log E[f]. \tag{2.22}$$

Here f is a non-negative function. Usually f is assumed to satisfy $E[f] = 1$ but we do not assume it. If we take $\phi(t) = \log t$, then

$$\Phi(x) = \int_0^x \log t\, dt = x\log x - x,$$
$$\Psi(y) = \int_{-\infty}^y e^u du = e^y$$

$$\text{Ent}(f) = \|f\|_\Phi.$$

In fact, by Proposition 2.4

$$\|f\|_\Phi = \inf_{\lambda>0} E\left[f\log f - f\log\lambda - f + \lambda\right].$$

The right hand side is nothing but the entropy. So our Orlicz functional is regarded as a generalization of the entropy.

We call the following inequality a *ϕ-Sobolev inequality*:

$$\|f^2\|_\Phi \le 2\lambda\mathcal{E}(f,f) + \Psi^{-1}(1)\|f\|_2^2. \tag{2.23}$$

Here $\mathcal{E}$ is a Dirichlet form on a probability space (M,m). λ is a non-negative constant and (2.23) holds for all $f \in \text{Dom}(\mathcal{E})$. We assume that $1 \in \text{Dom}(\mathcal{E})$ and $\mathcal{E}(1,1) = 0$. So the associated generator has the maximal eigenvalue 0 and its eigenfunction 1. The term

$\Psi^{-1}(1)\|f\|_2^2$ is indispensable since the equality holds when $f = 1$. The following little weakened inequality is called a *defective ϕ-Sobolev inequality*:

$$\|f^2\|_\Phi \leq 2\lambda\mathcal{E}(f,f) + \Psi^{-1}(1)\|f\|_2^2 + 2\mu\|f\|_2^2. \tag{2.24}$$

When $\phi(t) = \log t$, $\| \ \|_\Phi$ is the entropy and (2.23) is a logarithmic Sobolev inequality. So our terminology is consistent with this.

On the other hand the following inequality is called the Poincaré inequality:

$$\|\hat{f}\|_2^2 \leq \nu\mathcal{E}(f,f). \tag{2.25}$$

Here, $\hat{f} = f - E[f]$. Now we have the following theorem:

Theorem 2.9. We have the following:

(1) If the ϕ-Sobolev inequality (2.23) holds, then, setting $\nu = \lambda/\alpha\phi'(\alpha)$, the Poincaré inequality (2.25) holds.

(2) If we assume the defective ϕ-Sobolev inequality (2.24) and, in addition, the Poincaré inequality, the ϕ-Sobolev inequality (2.23) holds for $\lambda + (\mu + l)\nu$ in place of λ.

Proof. To show (1), from (2.23), we note

$$\begin{aligned}\|(1+tf)^2\|_\Phi &\leq 2\lambda t^2\mathcal{E}(f,f) + \Psi^{-1}(1)\|1+tf\|_2^2 \\ &= 2\lambda t^2\mathcal{E}(f,f) + \Psi^{-1}(1)(1 + 2tE[f] + t^2E\left[f^2\right]).\end{aligned}$$

But, from Proposition 2.8 and (2.18), we have

$$\lim_{t\downarrow 0} \frac{\|(1+tf)^2\|_\Phi - \Psi^{-1}(1)(1+2tE[f])}{t^2} = 2\alpha\phi'(\alpha)\|\hat{f}\|_2^2 + \Psi^{-1}(1)E\left[f^2\right]$$

and hence

$$2\alpha\phi'(\alpha)\|\hat{f}\|_2^2 \leq 2\lambda\mathcal{E}(f,f).$$

This is what we wanted.

As for (2), by Proposition 2.8 and (2.16), we have

$$\begin{aligned}\|f^2\|_\Phi &\leq \Psi^{-1}(1)E[f]^2 + \|\hat{f}^2\|_\Phi + 2l\|\hat{f}\|_2^2 \\ &\leq \Psi^{-1}(1)E[f]^2 + 2\lambda\mathcal{E}(\hat{f},\hat{f}) + \Psi^{-1}(1)E\left[\hat{f}^2\right] + 2\mu\|\hat{f}\|_2^2 + 2l\|\hat{f}\|_2^2 \\ &\leq 2\lambda\mathcal{E}(\hat{f},\hat{f}) + 2(\mu+l)\nu\mathcal{E}(f,f) + \Psi^{-1}(1)E\left[f^2\right] \\ &= 2(\lambda + (\mu+l)\nu)\mathcal{E}(f,f) + \Psi^{-1}(1)E\left[f^2\right],\end{aligned}$$

which is the desired result. □

3. Fractional Logarithmic Sobolev inequality

When we take $\phi = \log^{\gamma}(1+x)$, the inequality

$$\|f^2\|_{\Phi} \leq 2\lambda\mathcal{E}(f,f) + \Psi^{-1}(1)\|f\|_2^2 \tag{3.1}$$

is called a *logarithmic Sobolev inequality of the fractional order* γ. Here $\gamma > 0$ and when $\gamma = 1$, it is a usual logarithmic Sobolev inequality and it is natural to take $\phi(x) = \log x$ in this case. Anyway, we call these inequalities fractional logarithmic Sobolev inequalities in general. We are interested in inequalities which is stronger than the Poincaré inequality and is weaker than the logarithmic Sobolev inequality. So, in the sequel, we always assume

$$0 < \gamma \leq 1. \tag{3.2}$$

We mention here some recent related results. Latała-Oleszkiewicz [5] proposed the intermediate inequality in between the Poincaré inequality and the logarithmic Sobolev inequality. F.-Y. Wang [9] proved the Latała-Oleszkiewicz inequality is equivalent to the strong Poincaré inequality and, at the same time, he proved that it is equivalent to the F-Sobolev inequality. Therefore, inequalities we have discussed are equivalent to the F-Sobolev inequality and we formulated them in the framework of Orlicz space. We have seen that the ϕ-Sobolev inequality is stronger than the Poincaré inequality. On the other hand, the defective logarithmic Sobolev inequality is not necessarily stronger than the Poincaré inequality. So, to be more precise, the F-Sobolev inequality corresponds to the defective logarithmic Sobolev inequality.

So far, our discussions are in L^2 setting. We are now turning to L^p setting. To do so, we change the formulation. We assume that the Dirichlet form is of the gradient form:

$$\mathcal{E}(f,g) = \int_M (\nabla f, \nabla g)dm. \tag{3.3}$$

We consider the following inequality in L^p setting:

$$\|f\|_{\Phi} \leq K(\|\nabla f\|_p + \|f\|_p). \tag{3.4}$$

We are interested in this kind of L^p inequality. To investigate these inequalities, we prepare a general theory. So we consider a general norm $\| \ \|$ for a while. We also need a class of functions Φ. For $\alpha > 0$, we define $\Phi \in U(\alpha)$ if and only if

$$x\Phi'(x) \leq \alpha\Phi(x), \quad x \geq 0.$$

Proposition 3.1. Assume $\Phi \in U(\alpha)$. If the following inequality

$$\|f\|_{\Phi} \leq K\|f\|$$

holds, then

$$E[\Phi(f)] \leq K^{\alpha}\|f\|^{\alpha} + 1. \tag{3.5}$$

Proof. Note

$$N_\Phi(f) \le \|f\|_\Phi \le K\|f\|.$$

Using $\Phi \in U(\alpha)$, if $K\|f\| \ge 1$, then we have

$$\begin{aligned} E[\Phi(f)] &= E\left[\Phi(K\|f\|\frac{f}{K\|f\|})\right] \\ &\le E\left[K^\alpha\|f\|^\alpha\Phi(\frac{f}{K\|f\|})\right] \\ &\le K^\alpha\|f\|^\alpha. \end{aligned}$$

If $K\|f\| \le 1$, then

$$E[\Phi(f)] \le E\left[\Phi(\frac{f}{K\|f\|})\right] \le 1.$$

In all cases, (3.5) holds. □

We can also show the converse.

Proposition 3.2. If

$$E[\Phi(f)] \le K\|f\|^\alpha + C$$

for $\alpha \ge 1$, then

$$\|f\|_\Phi \le \alpha K^{1/\alpha}\left(\frac{C+1}{\alpha-1}\right)^{1-1/\alpha}\|f\|. \tag{3.6}$$

In case of $\alpha = 1$, (3.6) should be read as

$$\|f\|_\Phi \le K\|f\|.$$

Proof. By Proposition 2.4, we have

$$\|f\|_\Phi \le \frac{1}{\lambda}\{E[\Phi(\lambda f)] + 1\}.$$

From the assumption,

$$\begin{aligned} \frac{1}{\lambda}\{E[\Phi(\lambda f)] + 1\} &\le \frac{1}{\lambda}\{K\lambda^\alpha\|f\|^\alpha + C + 1\} \\ &\le \lambda^{\alpha-1}K\|f\|^\alpha + \frac{1}{\lambda}(C+1) \\ &=: g(\lambda). \end{aligned}$$

It is enough to minimize the $g(\lambda)$. By an easy computation, $g(\lambda)$ takes its minimum at $\lambda = \|f\|^{-1}(\frac{C+1}{K(\alpha-1)})^{1/\alpha}$ and the minimum is

$$g\left(\|f\|^{-1}\left(\frac{C+1}{K(\alpha-1)}\right)^{1/\alpha}\right) = \|f\|^{1-\alpha}\left(\frac{C+1}{K(\alpha-1)}\right)^{(\alpha-1)/\alpha}K\|f\|^\alpha$$

$$+\|f\|\left(\frac{K(\alpha-1)}{C+1}\right)^{1/\alpha}(C+1)$$
$$=\|f\|\left(\frac{C+1}{\alpha-1}\right)^{1-1/\alpha}(K^{-1+1/\alpha}\cdot K+K^{1/\alpha}\alpha)$$
$$=\alpha K^{1/\alpha}\left(\frac{C+1}{\alpha-1}\right)^{1-1/\alpha}\|f\|,$$

which is (3.6).

When $\alpha=1$, we have

$$\frac{1}{\lambda}\{E\left[\Phi(\lambda f)\right]+1\}\leq\frac{1}{\lambda}\{K\lambda\|f\|+C+1\}\leq K\|f\|+\frac{1}{\lambda}(C+1).$$

Now letting $\lambda\to\infty$,

$$\|f\|_\Phi\leq K\|f\|.$$

This is the desired result. □

We now return to the inequality

$$\|f\|_\Phi\leq K(\|\nabla f\|_p+\|f\|_p).$$

We assume that ϕ is of the following form:

$$\phi(x)=\phi_{p,\beta}(x)=x^{p-1}\log^{p\beta}(e+x).$$

Here $p>1$ and $\beta\in\mathbb{R}$. We denote its integration by $\Phi=\Phi_{p,\beta}$:

$$\Phi_{p,\beta}(x)=\int_0^x\phi_{p,\beta}(y)dy.$$

It is not difficult to see that $\Phi\in U(\alpha)$ for sufficiently large α. From now on, to avoid the change of constant, we use the following convention: we denote $A\lesssim B$ if there exists a constant c so that $A\leq cB$.

Proposition 3.3. Assume $\beta\geq 0$. Then the inequality

$$\|f\|_{\Phi_{p,\beta}}\leq K(\|\nabla f\|_p+\|f\|_p) \tag{3.7}$$

is equivalent to

$$E\left[|f|^p\log_+^{p\beta}(|f|^p/\|f\|_p^p)\right]\leq K_1\|\nabla f\|_p^p+K_2\|f\|_p^p. \tag{3.8}$$

Proof. Assume (3.7). From Proposition 3.1, there exists a constant α such that

$$E\left[\Phi_{p,\beta}(f)\right]\lesssim(\|\nabla f\|_p+\|f\|_p)^\alpha+1.$$

Using $x^p \log_+^{p\beta} x \le K\Phi_{p,\beta}(x)$, we have

$$E\left[|f|^p \log_+^{p\beta} |f|^p\right] \lesssim (\|\nabla f\|_p + \|f\|_p)^\alpha + 1.$$

We show that the left hand side dominates $E\left[|f|^p \log_+^{p\beta}(|f|^p/\|f\|_p^p)\right]$. If $\|f\|_p \ge 1$, then $\log_+^{p\beta}(|f|^p/\|f\|_p^p) \le \log_+^{p\beta} |f|^p$ and the result is clear. In the case of $\|f\|_p < 1$, we have

$$\begin{aligned}
E\left[|f|^p \log_+^{p\beta}(|f|^p/\|f\|_p^p)\right] &\le E\left[|f|^p \left\{\log_+ |f|^p + \log_+ \frac{1}{\|f\|_p^p}\right\}^{p\beta}\right] \\
&\le 2^{p\beta-1}\left\{E\left[|f|^p \log_+^{p\beta} |f|^p\right] + E\left[|f|^p \log_+^{p\beta} \frac{1}{\|f\|_p^p}\right]\right\} \\
&\le 2^{p\beta-1}\left\{E\left[|f|^p \log_+^{p\beta} |f|^p\right] + \|f\|_p^p \log_+^{p\beta} \frac{1}{\|f\|_p^p}\right\}.
\end{aligned}$$

Since $f(x) = x \log^{p\beta} \frac{1}{x}$ is bounded on $(0, 1]$, we have

$$E\left[|f|^p \log_+^{p\beta}(|f|^p/\|f\|_p^p)\right] \lesssim E\left[|f|^p \log_+^{p\beta} |f|^p\right] + 1$$

which leads to

$$E\left[|f|^p \log_+^{p\beta}(|f|^p/\|f\|_p^p)\right] \lesssim (\|\nabla f\|_p + \|f\|_p)^\alpha + 1.$$

Substitute λf in place of f,

$$E\left[|f|^p \log_+^{p\beta}(|f|^p/\|f\|_p^p)\right] \lesssim \lambda^{\alpha-p}(\|\nabla f\|_p + \|f\|_p)^\alpha + \lambda^{-p}.$$

Now we take $\lambda = \frac{1}{\|\nabla f\|_p + \|f\|_p}$. Then

$$\begin{aligned}
E\left[|f|^p \log_+^{p\beta}(|f|^p/\|f\|_p^p)\right] &\lesssim (\|\nabla f\|_p + \|f\|_p)^{p-\alpha}(\|\nabla f\|_p + \|f\|_p)^\alpha + (\|\nabla f\|_p + \|f\|_p)^p \\
&\lesssim (\|\nabla f\|_p + \|f\|_p)^p \\
&\lesssim \|\nabla f\|_p^p + \|f\|_p^p.
\end{aligned}$$

Next let us show the reversed inequality. So we assume (3.8). Then

$$\begin{aligned}
E\left[|f|^p \log_+^{p\beta} |f|^p\right] &= E\left[|f|^p \log_+^{p\beta}(|f|^p/\|f\|_p^p) \cdot \|f\|_p^p\right] \\
&\lesssim E\left[|f|^p(\log_+^{p\beta} |f|^p/\|f\|_p^p + \log_+^{p\beta} \|f\|_p^p)\right] \\
&\lesssim \|\nabla f\|_p^p + \|f\|_p^p + \|f\|_p^p \log_+^{p\beta} \|f\|_p^p.
\end{aligned}$$

Using $\Phi_{p,\beta}(x) \lesssim x^p \log_+^{p\beta} x + 1$, we have

$$\begin{aligned}
E\left[\Phi_{p,\beta}(f)\right] &\lesssim \|\nabla f\|_p^p + \|f\|_p^p + \|f\|_p^p \log_+^{p\beta} \|f\|_p^p + 1 \\
&\lesssim (\|\nabla f\|_p + \|f\|_p)^q + 1.
\end{aligned}$$

Here q is any number satisfying $q > p$. The rest is easy by Proposition 3.2. □

Now we assume the fractional logarithmic Sobolev inequality of the following form:

$$E\left[|f|^2\log_+^\gamma(|f|^2/\|f\|_2^2)\right] \le C_1\|\nabla f\|_2^2 + C_2\|f\|_2^2. \tag{3.9}$$

This means that we only assume the defective inequality. From now on, until the end of this section, we always *assume* (3.9) and we will deduce L^p inequalities of the type (3.9).

We introduce a new Young function as follows. Setting

$$\theta(x) = \{x^2\log^\gamma(e+x^2)\}^{(p-2)/4}\log^{p\beta/4}(k + x^2\log^\gamma(e+x^2)), \tag{3.10}$$

we define

$$\Theta(x) = \int_0^x \theta(y)dy. \tag{3.11}$$

Then we have

Proposition 3.4. There exist $k \ge 1$ and $K > 0$ so that

$$x^p\log^{p\gamma/2}(e+x^2)\log^{p\beta/2}(k + x^2\log^\gamma(e+x^2)) \le K\Theta(x)^2\log^\gamma(e+\Theta(x)^2). \tag{3.12}$$

Proof. We divide into two cases.
(i) In the case of $\beta \ge 0$, $k = 1$.

We investigate the asymptotic behavior as $x \to 0$.

$$\text{LHS} \sim x^p \cdot x^{(p\beta/2)2} = x^{p(1+\beta)}.$$

Here $A \sim B$ means that $\lim \frac{A}{B} = 1$. Further

$$\theta(x) \sim x^{(p-2)/2+(p\beta)/2} = x^{p(\beta+1)/2-1}$$

and hence

$$\Theta(x) \sim \frac{2}{p(\beta+1)}x^{p(\beta+1)/2},$$
$$\Theta(x)^2 \sim \frac{4}{p^2(\beta+1)^2}x^{p(\beta+1)}.$$

We can see that they have the same asymptotics up to constant.

Next, when $x \to \infty$,

$$\text{LHS} \sim x^p 2^{p\gamma/2}\log^{p\gamma/2}x\, 2^{p\beta/2}\log^{p\beta/2}x = 2^{p(\beta+\gamma)/2}x^p\log^{p(\beta+\gamma)/2}x.$$

On the other hand

$$\begin{aligned}\theta(x) &\sim x^{(p-2)/2}2^{(p-2)\gamma/4}(\log^{(p-2)\gamma/4}x)2^{p\beta/4}\log^{p\beta/4}x \\ &= 2^{(p\beta+p\gamma-2\gamma)/4}x^{(p-2)/2}\log^{(p\beta+p\gamma-2\gamma)/4}x, \\ \Theta(x) &\sim (2/p)2^{(p\gamma+p\gamma-2\gamma)/4}x^{p/2}\log^{(p\beta+p\gamma-2\gamma)/4}x \\ &= p^{-1}2^{(p\beta+p\gamma-2\gamma+4)/4}x^{p/2}\log^{(p\beta+p\gamma-2\gamma)/4}x,\end{aligned}$$

$$\Theta(x)^2\log^\gamma(e+\Theta(x)^2)\sim p^{-2}2^{(p\beta+p\gamma-2\gamma+4)/2}x^p(\log^{(p\beta+p\gamma-2\gamma)/2}x)p^\gamma\log^\gamma x$$
$$=p^{-2+\gamma}2^{(p\beta+p\gamma-2\gamma+4)/2}x^p\log^{(p\beta+p\gamma)/2}x.$$

Again, they have the same asymptotics.
(ii) In the case of $\beta<0$.

This time, k is taken to be sufficiently large. The behavior near $x=\infty$ can be checked in a similar manner as in the case $\beta\geq 0$. When $x\to 0$, it is clear that LHS $\sim\log^{p\beta/2}kx^p$. Further

$$\theta(x)\sim(\log^{p\beta/4}k)x^{(p-2)/2}$$
$$\Theta(x)\sim\frac{2\log^{p\beta/4}k}{p}x^{p/2}$$
$$\Theta(x)^2\log^\gamma(e+\Theta(x)^2)\sim\frac{4\log^{p\beta/2}k}{p^2}x^p$$

and we can see that they have the same asymptotics. □

Using this proposition, we have the following

Theorem 3.5. For $p>2$, $\beta\in\mathbb{R}$, we have

(3.13)

$$E[\Phi_{p,(\beta+\gamma)/2}(|f|)]\lesssim E[e+\Phi_{p,(\beta+\gamma-(2\gamma/p))/2}(|f|)]\{1+\log^\gamma(e+E[\Phi_{p,(\beta+\gamma-(2\gamma/p))/2}(|f|)])\}$$
$$+E[\Phi_{p,\beta/2}(|\nabla f|)].$$

Proof. First we set $g=\sqrt{e+\Theta(|f|)^2}$. Then, since $\nabla g=\frac{2\Theta(|f|)\Theta'(|f|)\nabla|f|}{2\sqrt{\Theta(|f|)^2+e}}$, we have $|\nabla g|\leq\theta(|f|)|\nabla f|$. Combining this with the fractional logarithmic Sobolev inequality

$$E[g^2\log_+^\gamma g^2/\|g\|_2^2]\leq K_1E[|\nabla g|^2]+K_2E[g^2],$$

we have

$$E[g^2\log_+^\gamma g^2]\lesssim E[g^2\log_+^\gamma\|g\|_2^2]+E[|\nabla g|^2]+E[g^2]$$

and hence

$$E[\{e+\Theta(|f|)^2\}\log^\gamma(e+\Theta(|f|)^2)]$$
$$\lesssim E[e+\Theta(|f|)^2]\log^\gamma E[e+\Theta(|f|)^2]+E[|\nabla f|^2\theta(|f|)^2]+E[e+\Theta(|f|)^2].$$

Thus we have

$$E[\Theta(|f|)^2\log^\gamma(e+\Theta(|f|)^2)]$$
$$\lesssim E[e+\Theta(|f|)^2](1+\log^\gamma E[e+\Theta(|f|)^2])+2\lambda E[|\nabla f|^2\theta(|f|)^2].$$

We set $\phi(x)=x^{(p/2)-1}\log^{p\beta/2}(k+x)$ and $U=|f|^2\log^\gamma(e+|f|^2)$. Here k is taken to be large enough. Note that θ has been defined to be $\theta(|f|)^2=\phi(U)$.

$$U\phi(U)=|f|^2\log^\gamma(e+|f|^2)\{|f|^2\log^\gamma(e+|f|^2)\}^{(p/2)-1}\log^{p\beta/2}(k+|f|^2\log^\gamma(e+|f|^2))$$

$$
\begin{aligned}
&= |f|^p \log^{p\gamma/2}(e+|f|^2)\log^{p\beta/2}(k+|f|^2\log^\gamma(e+|f|^2)) \\
&\lesssim \Theta(|f|)^2 \log^\gamma(e+\Theta(|f|)^2). \quad (\because\ (3.12))
\end{aligned}
$$

Hence

$$
E[U\phi(U)] \lesssim E[e+\Theta(|f|)^2](1+\log^\gamma E[e+\Theta(|f|)^2]) + E[|\nabla f|^2\phi(U)].
$$

Now, by Proposition 2.1, we have

$$
E[\Phi(U)] \lesssim E[e+\Theta(|f|)^2](1+\log^\gamma E[e+\Theta(|f|)^2]) + E[\Phi(C|\nabla f|^2)].
$$

Since Φ is an integral of $\phi(x) = x^{(p/2)-1}\log^{p\beta/2}(k+x)$, we have

$$
\begin{aligned}
\Phi(Cx^2) &\lesssim x^2\phi(x^2) \\
&= x^2(x^2)^{(p-2)/2}\log^{p\beta/2}(k+x^2) \\
&= x^p \log^{p\beta/2}(k+x^2) \\
&\lesssim \Phi_{p,\beta/2}(x).
\end{aligned}
$$

Further

$$
\begin{aligned}
\Phi(x^2\log^\gamma(e+x^2)) &\gtrsim x^2\log^\gamma(e+x^2)\phi(x^2\log^\gamma(e+x^2)) \\
&= x^2\log^\gamma(e+x^2)\{x^2\log^\gamma(e+x^2)\}^{(p-2)/2}\log^{p\beta/2}(k+x^2\log^\gamma(e+x^2)) \\
&= x^p\log^{p\gamma/2}(e+x^2)\log^{p\beta/2}(k+x^2\log^\gamma(e+x^2)) \\
&\gtrsim x^p\log^{p\gamma/2}(e+x)\log^{p\beta/2}(k+x) \\
&\gtrsim x^p\log^{p(\beta+\gamma)/2}(k+x) \\
&\gtrsim \Phi_{p,(\beta+\gamma)/2}(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\Theta(x)^2 &\lesssim x^2\theta(x)^2 \\
&= x^2\{x^2\log^\gamma(e+x^2)\}^{(p-2)/2}\log^{p\beta/2}(k+x^2\log^\gamma(e+x^2)) \\
&\lesssim x^p\log^{(p-2)\gamma/2}(e+x)\log^{p\beta/2}(k+x) \\
&\lesssim x^p\log^{(p\beta+p\gamma-2\gamma)/2}(k+x) \\
&= x^p\log^{p(\beta+\gamma-(2\gamma/p))/2}(k+x) \\
&\lesssim \Phi_{p,(\beta+\gamma-(2\gamma/p))/2-(1/p)}(x).
\end{aligned}
$$

Combining all of them, we eventually have

$$
\begin{aligned}
E[\Phi_{p,(\beta+\gamma)/2}(|f|)] &\lesssim E[e+\Phi_{p,(\beta+\gamma-(2\gamma/p))/2}(|f|)](1+\log^\gamma E[e+\Phi_{p,(\beta+\gamma-(2\gamma/p))/2}(|f|)]) \\
&\quad + E[\Phi_{p,\beta/2}(|\nabla f|)].
\end{aligned}
$$

This is what we wanted. □

By the above theorem, it follows that

$$\|f\|_{\Phi_{p,(\beta+\gamma)/2}} \lesssim \|f\|_{\Phi_{p,(\beta+\gamma-(2\gamma/p))/2}} + \|\nabla f\|_{\Phi_{p,\beta/2}}.$$

This implies that differentiability improves the integrability by the logarithmic order $p\gamma/2$.

When $p = 2$, the same result holds if we impose the additional condition $\beta \geq 0$. We give a proof for completeness. Set

$$\theta(x) = \log^{\beta/2}(1 + x^2 \log(e + x^2)) \tag{3.14}$$

and

$$\Theta(x) = \int_0^x \theta(y)dy. \tag{3.15}$$

We need the monotonicity of θ, which forces $\beta \geq 0$. Moreover, in the proof of Theorem 3.5, ϕ becomes $\phi = \log^{\beta}(k + x)$, which also forces $\beta \geq 0$. By being aware of this, we just repeat the same proof. So we have the following

Proposition 3.6. There exists a constant $K > 0$ so that

$$x^2 \log^{\gamma}(e + x^2) \log^{\beta}(1 + x^2 \log^{\gamma}(e + x^2)) \leq K\Theta(x)^2 \log^{\gamma}(e + \Theta(x)^2). \tag{3.16}$$

Proof. We investigate the asymptotic behavior.

Near $x = 0$, noting $\Theta(x) \sim x^{\beta+1}$, we have $\Theta(x)^2 \log(e + \Theta(x)^2) \sim x^{2\beta+2}$. The left hand side is $\sim x^2 \cdot x^{2\beta} = x^{2\beta+2}$ and so they have same asymptotics.

Next we consider when $x \to \infty$. This time,

$$\begin{aligned}
\lim_{x\to\infty} \frac{\Theta(x)}{x \log^{\beta/2} x^2} &= \lim_{x\to\infty} \frac{\Theta'(x)}{\log^{\beta/2} x^2 + \beta \log^{(\beta/2)-1} x^2} \\
&= \lim_{x\to\infty} \frac{\log^{\beta/2}(k + x^2 \log(e + x))}{\log^{\beta/2} x^2} \\
&= \lim_{x\to\infty} \frac{\log^{\beta/2} x^2}{\log^{\beta/2} x^2} \\
&= 1,
\end{aligned}$$

which yields $\log^{\gamma}(e + \Theta(x)^2) \sim \log^{\gamma} x^2$. Combining them, we have

$$\Theta(x)^2 \log^{\gamma}(e + \Theta(x)^2) \sim x^2(\log^{\beta} x^2) \log^{\gamma} x^2 = x^2 \log^{\beta+\gamma} x^2.$$

It is easy to see that the left hand side has the same asymptotics. Thus we have obtained (3.16). □

To proceed further, it is subtle to see $x\theta(x) \lesssim \Theta(x) \lesssim x\theta(x)$. In the case of $p > 2$, $\theta(x) \asymp x\theta'(x)$ holds but in the case of $p = 2$, this does not hold. In fact

$$\theta'(x) = \frac{\beta x\{(e + x^2)^{\gamma} \log(e + x^2) + \gamma x^2 \log^{\gamma-1}(e + x^2)\} \log^{\beta/2}(1 + x^2 \log^{\gamma}(e + x^2))}{(e + x^2)(1 + x^2 \log^{\gamma}(e + x^2)) \log(1 + x^2 \log^{\gamma}(e + x^2))}.$$

Hence

$$\lim_{x\to\infty}\frac{x\theta'(x)}{\theta(x)}=0,\qquad \lim_{x\to 0}\frac{x\theta'(x)}{\theta(x)}=\beta$$

and we can show $x\theta'(x)\lesssim\theta(x)$. But the reversed estimate does not hold. We should notice that the reversed estimate does hold for Θ. To see this,

$$\begin{aligned}\lim_{x\to\infty}\frac{x\theta(x)}{\Theta(x)}&=\lim_{x\to\infty}\frac{\theta(x)+x\theta'(x)}{\theta(x)}=1,\quad (\because x\theta'(x)/\theta(x)\to 0),\\ \lim_{x\to 0}\frac{x\theta(x)}{\Theta(x)}&=\lim_{x\to 0}\frac{\theta(x)+x\theta'(x)}{\theta(x)}=1+\beta,\end{aligned}$$

which shows $x\theta(x)\lesssim\Theta(x)\lesssim x\theta(x)$. Noticing this, we have the following.

Theorem 3.7. For $p=2$ and $\beta\geq 0$, there exists a constant $C>$ so that

$$\begin{aligned}E[\Phi_{2,(\beta+\gamma)/2}(|f|)]\leq CE[e+\Phi_{p,\beta/2}(|f|)](2\mu+\log^{\gamma}(e+E[\Phi_{p,\beta/2}(|f|)]))\\ +CE[\Phi_{p,\beta/2}(|\nabla f|)].\end{aligned}\tag{3.17}$$

Proof. Set $g=\sqrt{e+\Theta(|f|)^2}$. Then, by noting that $|\nabla g|\leq\theta(|f|)|\nabla f|$ since $\nabla g=\frac{2\Theta(|f|)\Theta'(|f|)\nabla|f|}{2\sqrt{\Theta(|f|)^2+e}}$, and by using the defective logarithmic Sobolev inequality

$$E[g^2\log^{\gamma}(g^2/\|g\|_2^2)]\leq 2\lambda E[|\nabla g|^2]+2\mu E[g^2],$$

we have

$$\begin{aligned}&E[\{e+\Theta(|f|)^2\}\log^{\gamma}(e+\Theta(|f|)^2)]\\ &\quad\lesssim E[e+\Theta(|f|)^2]\log^{\gamma}E[e+\Theta(|f|)^2]+E[|\nabla f|^2\theta(|f|)^2]+E[e+\Theta(|f|)^2]\end{aligned}$$

and hence

$$\begin{aligned}&E[\Theta(|f|)^2\log^{\gamma}(e+\Theta(|f|)^2)]\\ &\qquad\leq E[e+\Theta(|f|)^2](1+\log^{\gamma}(e+E[\Theta(|f|)^2]))+E[|\nabla f|^2\theta(|f|)^2].\end{aligned}$$

Now we set $\phi(x)=\log^{\beta}(1+x)$ and $U=|f|^2\log^{\gamma}(e+|f|^2)$. Then, since θ satisfies $\theta(|f|)^2=\phi(U)$, we have

$$\begin{aligned}U\phi(U)&=|f|^2\log^{\gamma}(e+|f|^2)\log^{\beta}(1+|f|^2\log^{\gamma}(e+|f|^2))\\ &\leq K\Theta(|f|)^2\log^{\gamma}(e+\Theta(|f|)^2).\quad(\because\ (3.16))\end{aligned}$$

Combining this with the previous result, we obtain

$$E[U\phi(U)]\lesssim E[e+\Theta(|f|)^2](1+\log^{\gamma}(e+E[\Theta(|f|)^2]))+E[|\nabla f|^2\phi(U)].$$

From Proposition 2.1,

$$E[\Phi(U)]\lesssim E[e+\Theta(|f|)^2](1+\log^{\gamma}(e+E[\Theta(|f|)^2]))+E[\Phi(C|\nabla f|^2)].$$

Since Φ is an integral of $\phi(x) = \log^\beta(1+x)$, we have

$$\Phi(Cx^2) \lesssim x^2\phi(x^2) = x^2\log^\beta(1+x^2) \lesssim \Phi_{2,\beta/2}(x).$$

Further

$$\begin{aligned}\Phi(x^2\log^\gamma(e+x^2)) &\gtrsim x^2\log^\gamma(e+x^2)\phi(x^2\log^\gamma(e+x^2))\\ &= x^2\log^\gamma(e+x^2)\log^\beta(1+x^2\log^\gamma(e+x^2))\\ &\gtrsim x^2\log^\gamma(e+x)\log^\beta(e+x)-1\\ &= x^2\log^{\beta+\gamma}(e+x)-1\\ &\geq \Phi_{2,(\beta+1)/2}(x)-1\end{aligned}$$

and

$$\begin{aligned}\Theta(x)^2 &\lesssim x^2\theta(x)^2\\ &= x^2\log^\beta(1+x^2\log^\gamma(e+x^2))\\ &\lesssim x^2\log^\beta(1+x^2)\\ &\lesssim \Phi_{2,\beta/2}(x).\end{aligned}$$

Combining all of them, we eventually obtain

$$E[\Phi_{2,(\beta+\gamma)/2}(|f|)] \lesssim E[e+\Phi_{2,\beta/2}(|f|)](1+\log^\gamma(e+E[\Phi_{2,\beta/2}(|f|)]) + E[\Phi_{2,\beta/2}(|\nabla f|)].$$

This is what we wanted. $\square$

By the above theorem, we can get

$$\|f\|_{\Phi_{2,(\beta+\gamma)/2}} \lesssim \|f\|_{\Phi_{2,\beta/2}} + \|\nabla f\|_{\Phi_{2,\beta/2}}. \tag{3.18}$$

The term $\|f\|_{\Phi_{2,\beta/2}}$ in the right hand side is of no importance. To see this, we need the following proposition.

Proposition 3.8. Take any $p \geq 1$, $\beta > 0$, $\alpha \in \mathbb{R}$. Then, for any $\varepsilon > 0$, there exists a constant K which depends on ε, p, β and α so that

$$\|f\|_{\Phi_{p,\alpha}} \leq \varepsilon\|f\|_{\Phi_{p,\alpha+\beta}} + K_\varepsilon\|f\|_1. \tag{3.19}$$

Proof. Since

$$\lim_{x\to\infty}\frac{\Phi_{p,\alpha+\beta}(\varepsilon x)}{\Phi_{p,\alpha}(x)} = \infty,$$

there exists a constant $C > 0$ so that $\Phi_{p,\alpha+\beta}(\varepsilon x) \geq \Phi_{p,\alpha}(x)$ for $x \geq C$. When $x \leq C$, we can take a constant $K > 0$ so that $\Phi_{p,\alpha}(x) \leq Kx$.Therefore

$$\begin{aligned}E[\Phi_{p,\alpha}(|f|)] &= E[\Phi_{p,\alpha}(|f|)\,;|f|\geq C] + E[\Phi_{p,\alpha}(|f|)\,;|f|<C]\\ &\leq E[\Phi_{p,\alpha+\beta}(\varepsilon|f|)] + E[K|f|].\end{aligned}$$

For any $\lambda > 0$, we have

$$
\begin{aligned}
\|f\|_{\Phi_{p,\alpha}} &\le \frac{1}{\lambda}\{E[\Phi_{p,\alpha}(\lambda|f|)] + 1\} \\
&\le \frac{1}{\lambda}\{E[\Phi_{p,\alpha+\beta}(\varepsilon\lambda|f|)] + E[K\lambda|f|] + 1\} \\
&\le \frac{\varepsilon}{\lambda\varepsilon}\{E[\Phi_{p,\alpha+\beta}(\lambda\varepsilon|f|)] + 1\} + KE[|f|].
\end{aligned}
$$

Letting λ run over positive numbers and taking the infimum of the right hand side, we have

$$\|f\|_{\Phi_{p,\alpha}} \le \varepsilon\|f\|_{\Phi_{p,\alpha+\beta}} + K\|f\|_1,$$

which is the desired result. □

Taking into account the above result, we eventually obtain the following

Theorem 3.9. We assume $p > 2$, $\beta \in \mathbb{R}$ or $p = 2$, $\beta \ge 0$. Then we have

$$\|f\|_{\Phi_{p,(\beta+\gamma)/2}} \lesssim \|\nabla f\|_{\Phi_{p,\beta/2}} + \|f\|_1. \tag{3.20}$$

4. Beckner type inequality

Throughout this section, we again *assume* the following fractional logarithmic Sobolev inequality:

$$E\left[|f|^2 \log_+^\gamma(|f|^2/\|f\|_2^2)\right] \le C_1\|\nabla f\|_2^2 + C_2\|f\|_2^2. \tag{4.1}$$

In Theorem 3.9, we take p so that $p\gamma/2 = 1$ and use Proposition 3.3 to obtain

$$E\left[|f|^p \log_+(|f|^p/\|f\|_p^p)\right] \le K_1\|\nabla f\|_p^p + K_2\|f\|_p^p.$$

We do not need to take positive part of log, so we formulate it in the following form:

$$E\left[|f|^p \log(|f|^p/\|f\|_p^p)\right] \le K_1\|\nabla f\|_p^p + K_2\|f\|_p^p. \tag{4.2}$$

The left hand side is an entropy, i.e., $E\left[|f|^p \log(|f|^p/\|f\|_p^p)\right] = \mathrm{Ent}(|f|^p)$.

Let p be as above and take $q \in [1, p)$. We are now interested in the following inequality:

$$\frac{E[|f|^p] - E[|f|^q]^{p/q}}{(p-q)^\delta} \le L_1\|\nabla f\|_p^p + L_2\|f\|_p^p. \tag{4.3}$$

The inequality of this type was discussed by Latała-Oleszkiewicz [5] in the case of $p = 2$. This inequality with $p = 2$ and $\delta = 1$ holds for the Ornstein-Uhlenbeck process, which was first proved by Beckner [2]. So we call it the Beckner type inequality.

We first consider the case of $\delta = 1$ for general p. It is just a slight modification of Latała-Oleszkiewicz' argument.

Set $\alpha(t) = \log \|f\|_{1/t}$. Clearly $\alpha(t)$ is convex. Take any $p > 1$ and fix it. $\beta(t) = e^{p\alpha(t)} = E[f^{1/t}]^{pt}$ is also convex. Therefore

$$\frac{\beta(t) - \beta(1/p)}{t - 1/p}$$

is non-decreasing on $(1/p, 1]$ and

$$\varphi(q) = \frac{\beta(1/p) - \beta(1/q)}{1/q - 1/p}$$

is non-decreasing on $[1, p)$. We now set

$$V_p(q) = \frac{\beta(1/p) - \beta(1/q)}{p - q} = \frac{E[|f|^p] - E[|f|^q]^{p/q}}{p - q}$$

and prove

$$\lim_{q \uparrow p} V_p(q) = \frac{1}{p} \operatorname{Ent}(|f|^p), \tag{4.4}$$

$$V_p(q) \leq \operatorname{Ent}(|f|^p), \quad q \in [1, p). \tag{4.5}$$

To show (4.4), by noting $\frac{d}{dq}\beta(1/q)|_{q=p}$, we get

$$\frac{d}{dq} E[|f|^q]^{p/q} = E[|f|^q]^{p/q} \log E[|f|^q]^{p/q} \Big(-\frac{p}{q^2}\Big) + \frac{p}{q} E[|f|^q]^{(p/q)-1} E[|f|^p \log |f|]$$

and if, in particular, $q = p$, then

$$\begin{aligned} \frac{d}{dq} E[|f|^q]^{p/q}\Big|_{q=p} &= -\frac{1}{p} E[|f|^p] \log E[|f|^p] + E[|f|^p \log |f|] \\ &= \frac{1}{p} E[|f|^p \log |f|^p / \|f\|_p^p] \\ &= \frac{1}{p} \operatorname{Ent}(|f|^p), \end{aligned}$$

which is (4.4).

Next let us see (4.5). Note

$$\begin{aligned} V_p(q) &= \frac{1}{pq} \frac{\beta(1/p) - \beta(1/p)}{1/q - 1/p} \\ &= \frac{1}{pq} \varphi(q) \leq \frac{1}{p} \lim_{q \uparrow p} \varphi(q). \quad (\because\ \varphi \text{ is non-decreasing and } q \geq 1.) \end{aligned}$$

Here

$$\lim_{q \uparrow p} \varphi(q) = \lim_{q \uparrow p} pq V_p(q) = p^2 \lim_{q \uparrow p} V_p(q) = p \operatorname{Ent}(|f|^p).$$

Combining these, we can easily show (4.5).

From these facts, we have the following

Proposition 4.1. Take any $p > 1$ and fix it. If

$$\mathrm{Ent}(|f|^p) \le K_1 \|\nabla f\|_p^p + K_2 \|f\|_p^p$$

holds, then it follows that

$$\frac{E[|f|^p] - E[|f|^q]^{p/q}}{p-q} \le K_1 \|\nabla f\|_p^p + K_2 \|f\|_p^p.$$

Conversely, if we assume

$$\frac{E[|f|^p] - E[|f|^q]^{p/q}}{p-q} \le L_1 \|\nabla f\|_p^p + L_2 \|f\|_p^p,$$

then

$$\mathrm{Ent}(|f|^p) \le pL_1 \|\nabla f\|_p^p + pL_2 \|f\|_p^p$$

follows.

Now we return to the Beckner type inequality.

Theorem 4.2. We assume the fractional logarithmic Sobolev inequality (4.1). Then, putting $p = 2/\gamma$, there exist constants K_1, K_2 so that for any $q \in [1, p)$

$$\frac{E[|f|^p] - E[|f|^q]^{p/q}}{p-q} \le K_1 \|\nabla f\|_p^p + K_2 \|f\|_p^p, \quad q \in [1, p). \tag{4.6}$$

Proof. From the assumption, the inequality (4.2) holds. Then Proposition 4.1 bears the result. □

In the above theorem, we assumed $p = \frac{2}{\gamma}$. We now consider the case $2 \le p \le \frac{2}{\gamma}$. The inequality (4.5) was crucial. Making use of (4.5), we will prove a little modified inequality to show (4.3).

Proposition 4.3. For $1 \le q < p$ and $0 < \delta < 1$,

$$\frac{E[|f|^p] - E[|f|^q]^{p/q}}{(p-q)^\delta} \le E\left[|f|^p \log_+^\delta(|f|^p/\|f\|_p^p)\right] + \frac{(p-q)^{1-\delta}}{e}\|f\|_p^p \tag{4.7}$$

Proof. From (4.5), it follows that

$$E\left[|f|^p\right] \le E\left[|f|^q\right]^{p/q} + E\left[|f|^p (p-q) \log(|f|^p/\|f\|_p^p)\right]. \tag{4.8}$$

Define a set A by

$$A = \{x;\ (p-q)\log(|f|^p/\|f\|_p^p) \le 1\}.$$

Then

$$|f|^p (p-q)^\delta \log_+^\delta(|f|^p/\|f\|_p^p) \ge |f|^p (p-q) \log(|f|^p/\|f\|_p^p) \quad \text{on } A$$

$$|f|^p(p-q)^\delta \log_+^\delta(|f|^p/\|f\|_p^p) \geq |f|^p \quad \text{on } A^c.$$

Writing the indicator function of A by χ, we can see that

$$\begin{aligned}|f\chi|^p(p-q)^\delta \log_+^\delta(|f\chi|^p/\|f\|_p^p) &\geq |f\chi|^p(p-q)\log(|f\chi|^p/\|f\|_p^p),\\ |f(1-\chi)|^p(p-q)^\delta \log_+^\delta(|f(1-\chi)|^p/\|f\|_p^p) &\geq |f(1-\chi)|^p.\end{aligned}$$

Adding them and then integrating them, we get

$$\begin{aligned}&E[|f|^p(p-q)^\delta \log_+^\delta(|f|^p/\|f\|_p^p)]\\ &\quad \geq E\left[|f\chi|^p(p-q)\log(|f\chi|^p/\|f\|_p^p)\right] + E\left[|f(1-\chi)|^p\right]\\ &\quad = E\left[|f\chi|^p(p-q)\log(|f\chi|^p/\|f\chi\|_p^p)\right]\\ &\qquad + E\left[|f\chi|^p(p-q)\log(\|f\chi\|^p/\|f\|_p^p)\right] + E\left[|f(1-\chi)|^p\right]\\ &\quad \geq E\left[|f\chi|^p\right] - E\left[|f\chi|^q\right]^{p/q} + (p-q)\|f\chi\|_p^p\log(\|f\chi\|^p/\|f\|_p^p) + E\left[|f(1-\chi)|^p\right]\\ &\quad = E\left[|f|^p\right] - E\left[|f\chi|^q\right]^{p/q} + (p-q)\|f\chi\|_p^p\log(\|f\chi\|^p/\|f\|_p^p)\\ &\quad \geq E\left[|f|^p\right] - E\left[|f|^q\right]^{p/q} + (p-q)\|f\chi\|_p^p\log(\|f\chi\|^p/\|f\|_p^p).\end{aligned}$$

Thus we have

$$E\left[|f|^p\right] - E\left[|f|^q\right]^{p/q} \leq (p-q)^\delta E\left[|f|^p \log_+^\delta(|f|^p/\|f\|_p^p)\right] + (p-q)\|f\chi\|_p^p\log(\|f\|^p/\|f\chi\|_p^p)$$

and, by dividing the both hands by $(p-q)^\delta$,

$$\frac{E\left[|f|^p\right] - E\left[|f|^q\right]^{p/q}}{(p-q)^\delta} \leq E\left[|f|^p \log_+^\delta(|f|^p/\|f\|_p^p)\right] + (p-q)^{1-\delta}\|f\chi\|_p^p\log(\|f\|^p/\|f\chi\|_p^p).$$

Now we assume $\|f\|_p = 1$. Then $\|f\chi\|_p^p \leq 1$ and hence

$$\frac{E\left[|f|^p\right] - E\left[|f|^q\right]^{p/q}}{(p-q)^\delta} \leq E\left[|f|^p \log_+^\delta(|f|^p)\right] + (p-q)^{1-\delta}\|f\chi\|_p^p\log(1/\|f\chi\|_p^p).$$

Noting that the function $-x\log x$ $(x>0)$ takes its maximum $\frac{1}{e}$ at $x=\frac{1}{e}$, we have

$$\frac{E\left[|f|^p\right] - E\left[|f|^q\right]^{p/q}}{(p-q)^\delta} \leq E\left[|f|^p \log_+^\delta(|f|^p)\right] + \frac{(p-q)^{1-\delta}}{e}.$$

Taking $|f|/\|f\|_p$, which satisfies $\|(|f|/\|f\|_p)\|_p = 1$, we obtain

$$\frac{E[|f|^p] - E[|f|^q]^{p/q}}{(p-q)^\delta} \leq E\left[|f|^p \log_+^\delta(|f|^p/\|f\|_p^p)\right] + \frac{(p-q)^{1-\delta}}{e}\|f\|_p^p.$$

This is what we wanted. □

Now we recall Theorem 3.9. Then, assuming the fractional logarithmic Sobolev inequality (4.1), we have, for $p \geq 2$,

$$E\left[|f|^p \log_+^{p\gamma/2}(|f|^p/\|f\|_p^p)\right] \leq K_1\|\nabla f\|_p^p + K_2\|f\|_p^p. \tag{4.9}$$

Thus, from the above proposition, the following Beckner type inequality easily follows:

Theorem 4.4. Assume the fractional logarithmic Sobolev inequality (4.1). Then, for $2 \le p \le 2/\gamma$, there exist constants L_1 and L_2 so that

$$\frac{E[|f|^p] - E[|f|^q]^{p/q}}{(p-q)^{p\gamma/2}} \le L_1 \|\nabla f\|_p^p + L_2 \|f\|_p^p, \quad q \in [1, p). \tag{4.10}$$

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