Defective intertwining property and generator domain

Ichiro Shigekawa*

Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan
E-mail: ichiro@math.kyoto-u.ac.jp

We discuss the defective intertwining property of generators of semigroups. We give some equivalent conditions in terms of generator, resolvent and semigroup. As an application, using this property, we give an example in which we can determine the exact generator domain of a Schrödinger operator.

Key Words: semigroup, intertwining property, generator domain

1. INTRODUCTION

The intertwining property plays an important role in dealing with semigroups. The intertwining property takes the following form:

$$DA = \hat{A}D$$

where A and $\hat{A}$ are generators of semigroups and D is a closed operator. In the previous paper [12], we discussed the intertwining property and applied it to the issue of the domain of a generator. The intertwining property was used e.g., in Bakry's paper [2] to discuss the Riesz transformation. But there are many issues which are not within the scope of (complete) intertwining property. In this paper, we extend it to the following defective intertwining property:

$$DA = \hat{A}D + R.$$

Here, an additional term R appears. If D is the identity map, the relation above is noting but a perturbation of operators, in which there are many

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results (see e.g., [9, Chapter 3]). And so we can say that this work is a generalization of perturbation theory to some extent. Such a relation appeared in Yoshida's paper [15] in connection with the Littlewood-Paley theory. Yoshida noticed the importance of this relation but he treated only bounded R. One of our motivations is to remove this restriction.

We discuss equivalent conditions in terms of resolvents and semigroups. We formulate the issue in the framework of Banach space. In the case of Hilbert space, the admissible class of R can be slightly extended. This extension is useful when we deal with Schrödinger operators. In fact, as an application, we discuss the Schrödinger operator of the form $\Delta - V$ on $\mathbb{R}^d$ where V is a scalar potential. We give a characterization of the domain of this operator. Further applications are discussed in the papers [13, 8] where the Littlewood-Paley theory is developed for the Schrödinger operators on a Riemannian manifold. In this case, the defective term R is unbounded and our extension in this paper is crucial.

The organization of this paper is as follows. In §2, we give a precise definition of the intertwining property and discuss the relationship with resolvents and semigroups. We use the Hille-Yosida theory of semigroups. In §3, we discuss the same problem in the case of Hilbert space setting. We deal with the generators that satisfy the sector condition. Lastly we consider a Schrödinger operator in §4. We give an example in which we can exactly determine the domain of the operator.

2. DEFECTIVE INTERTWINING PROPERTY

In this section, we discuss the intertwining property of the generators of semigroups. Suppose we are given two strongly continuous semigroups $\{T_t\}$ and $\{\hat{T}_t\}$ on Banach spaces B and $\hat{B}$. Let D be a closed operator from B into $\hat{B}$ with the domain $\mathrm{Dom}(D)$. We always denote by Dom the domain of an operator or, later, the domain of quadratic form. The following property is called the intertwining property:

$$DT_t = \hat{T}_t D. \tag{1}$$

We denote the generator of $\{T_t\}$ and $\{\hat{T}_t\}$ by A and $\hat{A}$, respectively. Then the intertwining property above is (at least formally) equivalent to

$$DA = \hat{A}D.$$

For the moment, we use this notation formally. This property is sometimes too restrictive and so we will relax it as follows.

$$DA = \hat{A}D + R. \tag{2}$$

Here R is an appropriate operator. If A and $\hat{A}$ satisfy this identity, we say that the defective intertwining property holds. We have to precisely give the subspace where the equation (2) holds because our operators are unbounded in general.

We will give the precise meaning of (2) and also give equivalent conditions in terms of semigroups and resolvents. We denote the Resolvent set of A by $\rho(A)$. For $\lambda \in \rho(A)$, the resolvent of A is denoted by $G_\lambda = (\lambda - A)^{-1}$. Similarly $\hat{G}_\lambda = (\lambda - \hat{A})^{-1}$ denotes the resolvent of $\hat{A}$. We regard $\mathrm{Dom}(D)$ as a Banach space equipped with the graph norm of D. In the sequel, we always assume that the domain of a closed operator is regarded as a Banach space equipped with the graph norm. Following this convention, we assume

(A.1) R is a bounded operator from $\mathrm{Dom}(D)$ into $\hat{B}$.

Here we regard $\mathrm{Dom}(D)$ to be equipped with the graph norm of D.

For later use, we introduce some notations. We denote the set of all bounded linear operators from B_1 into B_2 by $\mathcal{L}(B_1, B_2)$. The operator norm is denoted by $\| \ \|_{\mathcal{L}(B_1,B_2)}$. When $B_1 = B_2$, we use $\mathcal{L}(B_1)$ in place of $\mathcal{L}(B_1, B_1)$. Hence the condition (A.1) can be written as $R \in \mathcal{L}(\mathrm{Dom}(D), \hat{B})$.

Now we can give a characterization of defective intertwining property.

Theorem 2.1. *Assume the condition* (A.1). *Then the following three statements are equivalent to each other.*

(a) *There exists a subspace $\mathcal{D} \subseteq B$ satisfying the following conditions:*

i) $\mathcal{D} \subseteq \mathrm{Dom}(A) \cap \mathrm{Dom}(D)$.

ii) $A\mathcal{D} \subseteq \mathrm{Dom}(D)$, $D\mathcal{D} \subseteq \mathrm{Dom}(\hat{A})$.

iii) For sufficiently large λ, $(\lambda - A)\mathcal{D}$ is dense in $\mathrm{Dom}(D)$.

iv) The following equality holds.

$$DAx = \hat{A}Dx + Rx, \quad \forall x \in \mathcal{D}. \tag{3}$$

(b) *For sufficiently large λ, $G_\lambda \,\mathrm{Dom}(D) \subseteq \mathrm{Dom}(D)$ and*

$$DG_\lambda x = \hat{G}_\lambda Dx + \hat{G}_\lambda RG_\lambda x, \quad \forall x \in \mathrm{Dom}(D). \tag{4}$$

(c) *For any $t \geq 0$, $\{T_t\}$ is a (C_0)-semigroup not only on B but also on* $\mathrm{Dom}(D)$ *and the following holds:*

$$DT_t x = \hat{T}_t Dx + \int_0^t \hat{T}_{t-s} RT_s x \, ds, \quad \forall x \in \mathrm{Dom}(D). \tag{5}$$

Proof. We first show (a)$\Rightarrow$(b). Take any $y \in (\lambda - A)\mathcal{D}$ and set $x = G_\lambda y \in \mathcal{D}$. By the assumption (3),

$$D(A-\lambda)x = (\hat{A}-\lambda)Dx + Rx.$$

Applying $\hat{G}_\lambda$ to both sides of the preceding equality, we have

$$\hat{G}_\lambda D(A-\lambda)x = -Dx + \hat{G}_\lambda Rx.$$

We recall that $(A-\lambda)x = -y$ and hence

$$\hat{G}_\lambda Dy = DG_\lambda y - \hat{G}_\lambda RG_\lambda y$$

which yields

$$DG_\lambda y = \hat{G}_\lambda Dy + \hat{G}_\lambda RG_\lambda y, \quad \forall y \in (\lambda - A)\mathcal{D}. \tag{6}$$

We have to show that the identity above holds for all $y \in \mathrm{Dom}(D)$.

We recall that there exist $M > 0$ and $\omega \geq 0$ such that

$$\begin{aligned}
\|T_t\|_{\mathcal{L}(B)} &\leq Me^{\omega t}, \\
\|\hat{T}_t\|_{\mathcal{L}(\hat{B})} &\leq Me^{\omega t}
\end{aligned}$$

and hence, for $\lambda > \omega$,

$$\begin{aligned}
\|\lambda G_\lambda\|_{\mathcal{L}(B)} &\leq M/(\lambda - \omega), \\
\|\lambda \hat{G}_\lambda\|_{\mathcal{L}(\hat{B})} &\leq M/(\lambda - \omega).
\end{aligned}$$

Since G_λ is defined on $\mathcal{D}(\subseteq \mathrm{Dom}(D))$, we can consider the graph norm $\|G_\lambda y\|_D := \|DG_\lambda y\|_{\hat{B}} + \|G_\lambda y\|_B$ and we have, by (6),

$$\begin{aligned}
\|G_\lambda y\|_D &= \|\hat{G}_\lambda Dy + \hat{G}_\lambda RG_\lambda y\|_{\hat{B}} + \|G_\lambda y\|_B \\
&\leq \|\hat{G}_\lambda\|_{\mathcal{L}(\hat{B})}\|Dy\|_{\hat{B}} + \|\hat{G}_\lambda\|_{\mathcal{L}(\hat{B})}\|R\|_{\mathcal{L}(\mathrm{Dom}(D),\hat{B})}\|G_\lambda y\|_D \\
&\quad + \|G_\lambda\|_{\mathcal{L}(B)}\|y\|_B \\
&\leq \frac{M}{\lambda - \omega}\|Dy\|_{\hat{B}} + \frac{M}{\lambda - \omega}\|R\|_{\mathcal{L}(\mathrm{Dom}(D),\hat{B})}\|G_\lambda y\|_D + \frac{M}{\lambda - \omega}\|y\|_B.
\end{aligned}$$

Therefore

$$\left(1 - \frac{M}{\lambda - \omega}\|R\|_{\mathcal{L}(\mathrm{Dom}(D),\hat{B})}\right)\|G_\lambda y\|_D \leq \frac{M}{\lambda - \omega}\|y\|_D.$$

Taking λ to be large enough, we can see that G_λ is bounded in $\mathrm{Dom}(D)$. Here G_λ is defined on $(\lambda - A)\mathcal{D}$. Due to the density of $(\lambda - A)\mathcal{D}$ in $\mathrm{Dom}(D)$, we can see that G_λ is a bounded operator from $\mathrm{Dom}(D)$ into $\mathrm{Dom}(D)$ and (4) holds for all $x \in \mathrm{Dom}(D)$.

Secondly we show (b)$\Rightarrow$(c). From the assumption, G_λ is a bounded operator on $\mathrm{Dom}(D)$ for sufficiently large λ. Moreover $\{G_\lambda\}_\lambda$ satisfies the resolvent equation on $\mathrm{Dom}(D)$. It remains to show that there exists a strongly continuous semigroup on $\mathrm{Dom}(D)$ associated to $\{G_\lambda\}_\lambda$. To show this, we first show the strong continuity of $\{G_\lambda\}_\lambda$. By the same argument as above, $\|\lambda G_\lambda\|_{\mathcal{L}(\mathrm{Dom}(D))}$ is uniformly bounded for large λ. By (4), we have, for $x \in \mathrm{Dom}(D)$,

$$\begin{aligned}
\|\lambda G_\lambda x - x\|_D &= \|D\lambda G_\lambda x - Dx\|_{\hat{B}} + \|\lambda G_\lambda x - x\|_B \\
&= \|\lambda \hat{G}_\lambda Dx + \lambda \hat{G}_\lambda R G_\lambda x - Dx\|_{\hat{B}} + \|\lambda G_\lambda x - x\|_B \\
&\leq \frac{1}{\lambda}\|\lambda \hat{G}_\lambda\|_{\mathcal{L}(\hat{B})}\|R\|_{\mathcal{L}(\mathrm{Dom}(D),\hat{B})}\|\lambda G_\lambda\|_{\mathcal{L}(\mathrm{Dom}(D))}\|x\|_D \\
&\quad + \|\lambda \hat{G}_\lambda Dx - Dx\|_{\hat{B}} + \|\lambda G_\lambda x - x\|_B.
\end{aligned}$$

Clearly the right hand side of the equation above converges to 0 as $\lambda \to \infty$ and hence the strong continuity of $\{G_\lambda\}$ in $\mathrm{Dom}(D)$ follows.

We set $A_\mu = \mu A G_\mu = \mu^2 G_\mu - \mu$. Then A_μ is a bounded operator not only on B but also on $\mathrm{Dom}(D)$. We also set $\hat{A}_\mu = \mu \hat{A} \hat{G}_\mu$. Then (4) yields

$$\begin{aligned}
DA_\mu x - \hat{A}_\mu Dx &= D(\mu^2 G_\mu - \mu x) - (\mu^2 \hat{G}_\mu Dx - \mu Dx) \\
&= \mu^2 (DG_\mu - \hat{G}_\mu Dx) \\
&= \mu^2 \hat{G}_\mu R G_\mu x \\
&= R_\mu x
\end{aligned} \tag{7}$$

Here $R_\mu = \mu^2 \hat{G}_\mu R G_\mu$. It is easy to see that $R_\mu \in \mathcal{L}(\mathrm{Dom}(D), \hat{B})$ and the operator norm of R_μ is uniformly bounded for large μ. Now we claim the following identity.

$$De^{tA_\mu}x - e^{t\hat{A}_\mu}Dx = \int_0^t e^{(t-s)\hat{A}_\mu} R_\mu e^{sA_\mu} x\, ds, \quad \forall x \in \mathrm{Dom}(D). \tag{8}$$

To see this, set

$$u(t) = De^{tA_\mu}x - e^{t\hat{A}_\mu}Dx - \int_0^t e^{(t-s)\hat{A}_\mu} R_\mu e^{sA_\mu} x\, ds.$$

Then, using (7)

$$\begin{aligned}
\frac{d}{dt}u(t) &= DA_\mu e^{tA_\mu}x - \hat{A}_\mu e^{t\hat{A}_\mu}Dx - R_\mu e^{tA_\mu}x - \int_0^t \hat{A}_\mu e^{(t-s)\hat{A}_\mu}R_\mu e^{sA_\mu}x\,ds \\
&= \hat{A}_\mu De^{tA_\mu}x + R_\mu e^{tA_\mu}x - \hat{A}_\mu e^{t\hat{A}_\mu}Dx - R_\mu e^{tA_\mu}x \\
&\quad - \hat{A}_\mu \int_0^t e^{(t-s)\hat{A}_\mu}R_\mu e^{sA_\mu}x\,ds \\
&= \hat{A}_\mu\Big\{De^{tA_\mu}x - e^{t\hat{A}_\mu}Dx - \int_0^t e^{(t-s)\hat{A}_\mu}R_\mu e^{sA_\mu}x\,ds\Big\} \\
&= \hat{A}_\mu u(t).
\end{aligned}$$

This means that $u(t)$ satisfies the following differential equation.

$$\begin{cases} \dfrac{d}{dt}u(t) &= \hat{A}_\mu u(t), \\ \quad u(0) &= 0. \end{cases}$$

The uniqueness of the solution deduces $u(t) \equiv 0$ which proves (8). We estimate the operator norm of e^{tA_μ} on $\mathrm{Dom}(D)$. We first recall that $\|e^{t\hat{A}_\mu}\|_{\mathcal{L}(\hat{B})} \leq Me^{2t\omega}$ (see, e.g., [3, §2.3 (2.13)]) and hence

$$\begin{aligned}
\|De^{tA_\mu}x\|_{\hat{B}} &\leq \|e^{t\hat{A}_\mu}Dx\|_{\hat{B}} + \left\|\int_0^t e^{(t-s)\hat{A}_\mu}R_\mu e^{sA_\mu}xds\right\|_{\hat{B}} \\
&\leq Me^{2t\omega}\|Dx\|_{\hat{B}} \\
&\quad + \int_0^t Me^{2(t-s)\omega}\|R_\mu\|_{\mathcal{L}(\mathrm{Dom}(D),\hat{B})}\|e^{sA_\mu}\|_{\mathcal{L}(\mathrm{Dom}(D))}\|x\|_D ds \\
&\leq Me^{2t\omega}\|Dx\|_{\hat{B}} + Me^{2t\omega}\|R_\mu\|_{\mathcal{L}(\mathrm{Dom}(D),\hat{B})} \\
&\quad \times \int_0^t e^{-2s\omega}\|e^{sA_\mu}\|_{\mathcal{L}(\mathrm{Dom}(D))}\|x\|_D ds.
\end{aligned}$$

Combining this with $\|e^{tA_\mu}x\|_B \leq Me^{2t\omega}\|x\|_B$, we have

$$\begin{aligned}
\|e^{tA_\mu}x\|_D &= \|De^{tA_\mu}x\|_{\hat{B}} + \|e^{tA_\mu}x\|_B \\
&\leq Me^{2t\omega}\|x\|_D + Me^{2t\omega}\|R_\mu\|_{\mathcal{L}(\mathrm{Dom}(D),\hat{B})} \\
&\quad \times \int_0^t e^{-2s\omega}\|e^{sA_\mu}\|_{\mathcal{L}(\mathrm{Dom}(D))}\|x\|_D ds.
\end{aligned}$$

Then

$$\begin{aligned} &e^{-2t\omega}\|e^{tA_\mu}\|_{\mathcal{L}(\mathrm{Dom}(D))} \\ &\qquad \leq M + M\|R_\mu\|_{\mathcal{L}(\mathrm{Dom}(D),\hat{B})}\int_0^t e^{-2s\omega}\|e^{sA_\mu}\|_{\mathcal{L}(\mathrm{Dom}(D))}ds. \end{aligned}$$

Now by the Gronwall lemma, we have

$$e^{-2t\omega}\|e^{tA_\mu}\|_{\mathcal{L}(\mathrm{Dom}(D))} \leq M\exp\{tM\|R_\mu\|_{\mathcal{L}(\mathrm{Dom}(D),\hat{B})}\}.$$

The right hand side is independent of μ since $\|R_\mu\|_{\mathcal{L}(\mathrm{Dom}(D),\hat{B})}$ is uniformly bounded with respect to μ. It is easy to see that there exists $\tilde{M} > 0$, $\tilde{\omega} \geq 0$ such that

$$\|e^{tA_\mu}\|_{\mathcal{L}(\mathrm{Dom}(D))} \leq \tilde{M}e^{\tilde{\omega}t}.$$

Denote the resolvent of A_μ by $R(\lambda; A_\mu)$. We also set $R(\lambda; A) = G_\lambda$. Now we have, for $\lambda \geq \tilde{\omega}$,

$$\|R(\lambda; A_\mu)^n\|_{\mathcal{L}(\mathrm{Dom}(D))} \leq \frac{\tilde{M}}{(\lambda - \tilde{\omega})^n}. \tag{9}$$

On the other hand, it holds that (see e.g., [9, §1.7 (7.7)])

$$\begin{aligned} R(\lambda; A_\mu) &= (\lambda+\mu)^{-1}(\mu - A)R\Big(\frac{\mu\lambda}{\mu+\lambda}; A\Big) \\ &= \frac{\mu^2}{(\mu+\lambda)^2}R\Big(\frac{\mu\lambda}{\mu+\lambda}; A\Big) - \frac{1}{\mu+\lambda}. \end{aligned}$$

Set $\kappa = \frac{\mu\lambda}{\mu+\lambda}$. Then $\lambda = \frac{\mu\kappa}{\mu-\kappa}$. (9) implies

$$\left\|\left\{\left(\frac{\mu}{\mu+\lambda}\right)^2 R(\kappa; A) - \frac{1}{\mu+\lambda}\right\}^n\right\|_{\mathcal{L}(\mathrm{Dom}(D))} \leq \frac{\tilde{M}}{(\lambda-\tilde{\omega})^n}.$$

We fix κ and let $\mu \to \infty$. Then $\lambda \to \kappa$ and we have

$$\|R(\kappa; A)^n\|_{\mathcal{L}(\mathrm{Dom}(D))} \leq \frac{\tilde{M}}{(\kappa-\tilde{\omega})^n}.$$

By Hille-Yosida's theorem, this leads that A generates (C_0)-semigroup on $\mathrm{Dom}(D)$ and further e^{tA_μ} converges to the semigroup strongly as $\mu \to \infty$. In addition, the convergence is uniform on a compact interval of t. The limit of $\{e^{tA_\mu}\}$ in $\mathrm{Dom}(D)$ clearly coincides with $\{T_t\}$. We also note that R_μ converges to R strongly as $\mu \to \infty$. Now, taking limit in (8), we have

$$DT_t - \hat{T}_t Dx = \int_0^t \hat{T}_{t-s} R T_s x ds, \quad \forall x \in \mathrm{Dom}(D)$$

which shows (c).

Lastly we show the implication (c)⇒(a). Let A^D be the generator of $\{T_t\}$ on $\mathrm{Dom}(D)$. Clearly A is an extension of A^D. We set $\mathcal{D} = \mathrm{Dom}(A^D)$. Take $x \in \mathrm{Dom}(A^D)$ and differentiate (5) in t at $t = 0$, and we have

$$DAx = \hat{A} Dx + Rx.$$

All properties in (a) are now clear. ∎

We say that the *defective intertwining property* holds when one of (and hence all of) conditions of the theorem above is fulfilled.

We remark that the form (5) has already appeared in Yoshida [15]. Statement Theorem 2.1 (a) is complicated. Imposing additional conditions on semigroups, we give a little simpler condition of the generator. To do this, we suppose that $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(D)$ and there exists a dual (C_0)-semigroup $\{\hat{T}_t^*\}$ of $\{T_t\}$. We denote the generator of $\{\hat{T}_t^*\}$ by $\hat{A}^*$.

Theorem 2.2. *Assume that* $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(D)$ *and there exists a dual* (C_0)*-semigroup* $\{\hat{T}_t^*\}$*. Then the following conditions are equivalent.*

(1) $\mathrm{Dom}(\hat{A}^*) \subseteq \mathrm{Dom}(D^*)$ *and*

$$\begin{aligned} {}_B\langle Ax, D^*\theta\rangle_{B^*} &= {}_{\hat{B}}\langle Dx, \hat{A}^*\theta\rangle_{\hat{B}^*} + {}_{\hat{B}}\langle Rx,\ \theta\rangle_{\hat{B}^*}, \\ &\qquad \forall x \in \mathrm{Dom}(A), \theta \in \mathrm{Dom}(\hat{A}^*). \end{aligned} \tag{10}$$

(2) *For sufficiently large* λ*,*

$$DG_\lambda x = \hat{G}_\lambda Dx + \hat{G}_\lambda R G_\lambda x, \quad \forall x \in \mathrm{Dom}(D).$$

Proof. We first show (1)⇒(2). For any $x \in \mathrm{Dom}(A)$ and $\theta \in \mathrm{Dom}(\hat{A}^*)$, we have

$${}_B\langle (\lambda - A)x, D^*\theta\rangle_{B^*} = {}_{\hat{B}}\langle Dx, (\lambda - \hat{A}^*)\theta\rangle_{\hat{B}^*} - {}_{\hat{B}}\langle Rx, \theta\rangle_{\hat{B}^*}.$$

Set $x = G_\lambda y$ and $\theta = \hat{G}_\lambda^* \xi$ for $y \in \mathrm{Dom}(D)$ and $\xi \in \hat{B}^*$. Then we have

$$\begin{aligned}
{}_B\langle \hat{G}_\lambda D y, \xi\rangle_{B^*} &= {}_B\langle y, D^* \hat{G}_\lambda^* \xi\rangle_{B^*} \\
&= {}_B\langle D G_\lambda y, \xi\rangle_{B^*} - {}_B\langle R G_\lambda y, \hat{G}_\lambda^* \xi\rangle_{B^*} \\
&= {}_B\langle D G_\lambda y, \xi\rangle_{B^*} - {}_B\langle \hat{G}_\lambda R G_\lambda y, \xi\rangle_{B^*}.
\end{aligned}$$

Since ξ is arbitrary, we obtain

$$\hat{G}_\lambda D y = D G_\lambda y - \hat{G}_\lambda R G_\lambda y$$

which is the desired result.

We proceed to prove the converse (2)⇒(1). We first show $\mathrm{Dom}(\hat{A}^*) \subseteq \mathrm{Dom}(D^*)$. Since $G_\lambda(B) \subseteq \mathrm{Dom}(D)$, the closed operator $S = DG_\lambda \colon B \to \hat{B}$ is bounded by the closed graph theorem. Similarly $V = RG_\lambda$ is a bounded operator from B into $\hat{B}$. Hence, for $x \in \mathrm{Dom}(D)$, $\theta \in \hat{B}^*$,

$$\begin{aligned}
{}_B\langle x, S^*\theta\rangle_{B^*} &= {}_{\hat{B}}\langle Sx, \theta\rangle_{\hat{B}^*} \\
&= {}_{\hat{B}}\langle D G_\lambda x, \theta\rangle_{\hat{B}^*} \\
&= {}_{\hat{B}}\langle \hat{G}_\lambda D x, \theta\rangle_{\hat{B}^*} + {}_{\hat{B}}\langle \hat{G}_\lambda R G_\lambda x, \theta\rangle_{\hat{B}^*} \\
&= {}_{\hat{B}}\langle D x, \hat{G}_\lambda^*\theta\rangle_{\hat{B}^*} + {}_B\langle x, V^* \hat{G}_\lambda^* \theta\rangle_{B^*}.
\end{aligned}$$

Setting $\theta = (\lambda - \hat{A}^*)\xi$ for $\xi \in \mathrm{Dom}(\hat{A}^*)$, we get

$${}_{\hat{B}}\langle Dx, \xi\rangle_{\hat{B}^*} = \langle x, S^*(\lambda - \hat{A}^*)\xi\rangle_{B^*} - {}_B\langle x, V^*\xi\rangle_{B^*} \quad \forall x \in \mathrm{Dom}(D) \tag{11}$$

which implies $\xi \in \mathrm{Dom}(D^*)$ and

$$D^*\xi = S^*(\lambda - \hat{A}^*)\xi - V^*\xi.$$

Further, by putting $x = (\lambda - A)y$, $y \in \mathrm{Dom}(A)$ in (11),

$$\begin{aligned}
{}_B\langle (\lambda - A)y, D^*\xi\rangle_{B^*} &= {}_{\hat{B}}\langle S(\lambda - A)y, (\lambda - \hat{A}^*)\xi\rangle_{\hat{B}^*} - {}_{\hat{B}}\langle V(\lambda - A)y, \xi\rangle_{\hat{B}^*} \\
&= {}_{\hat{B}}\langle D G_\lambda(\lambda - A)y, (\lambda - \hat{A}^*)\xi\rangle_{\hat{B}^*} \\
&\quad - {}_{\hat{B}}\langle R G_\lambda(\lambda - A)y, \xi\rangle_{\hat{B}^*} \\
&= {}_{\hat{B}}\langle Dy, (\lambda - \hat{A}^*)\xi\rangle - \langle Ry, \xi\rangle_{\hat{B}^*}
\end{aligned}$$

which is (10). This completes the proof. ▮

In the theorem above, the equation (10) is required to hold on the whole spaces of $\mathrm{Dom}(A)$ and $\mathrm{Dom}(\hat{A}^*)$ but it is enough to assume it on dense subspaces as follows:

Theorem 2.3. *Under the same assumption of Theorem 2.2, the following two conditions are equivalent.*

(a) *There exists dense subspaces* $\mathcal{D} \subseteq \mathrm{Dom}(A)$ *and* $\hat{\mathcal{D}} \subseteq \mathrm{Dom}(\hat{A}^*)$ *such that* $\hat{\mathcal{D}} \subseteq \mathrm{Dom}(D^*)$ *and*

$$ {}_B\langle Ax, D^*\theta\rangle_{B^*} = {}_{\hat{B}}\langle Dx, \hat{A}^*\theta\rangle_{\hat{B}^*} + {}_{\hat{B}}\langle Rx, \theta\rangle_{\hat{B}^*}, \quad \forall x \in \mathcal{D}, \theta \in \hat{\mathcal{D}}. \tag{12} $$

(b) *For sufficiently large* λ*,*

$$ DG_\lambda x = \hat{G}_\lambda Dx + \hat{G}_\lambda RG_\lambda x, \quad \forall x \in \mathrm{Dom}(D). $$

Proof. (b)⇒(a) is clear from Theorem 2.2. We show the converse (a)⇒(b). From (12),

$$ \begin{aligned} {}_B\langle(\lambda - A)x, D^*\theta\rangle_{B^*} &= {}_{\hat{B}}\langle Dx, (\lambda - \hat{A}^*)\theta\rangle_{\hat{B}^*} - {}_{\hat{B}}\langle Rx, \theta\rangle_{\hat{B}^*}, \\ &\qquad \forall x \in \mathcal{D}, \forall \theta \in \hat{\mathcal{D}}. \end{aligned} \tag{13} $$

Since $\mathcal{D}$ is dense in $\mathrm{Dom}(A)$, the identity above holds for all $x \in \mathrm{Dom}(A)$. In particular, putting $x = G_\lambda y$, $y \in \mathrm{Dom}(D)$,

$$ {}_{\hat{B}}\langle Dy, \theta\rangle_{\hat{B}^*} = {}_B\langle y, D^*\theta\rangle_{B^*} = {}_{\hat{B}}\langle DG_\lambda y, (\lambda - \hat{A}^*)\theta\rangle_{\hat{B}^*} - {}_{\hat{B}}\langle RG_\lambda y, \theta\rangle_{\hat{B}^*}. $$

Again, by the density of $\hat{\mathcal{D}}$ in $\mathrm{Dom}(\hat{A}^*)$, we have for any $\theta \in \mathrm{Dom}(\hat{A}^*)$,

$$ \begin{aligned} {}_{\hat{B}}\langle Dy, \theta\rangle_{\hat{B}^*} &= {}_{\hat{B}}\langle DG_\lambda y, (\lambda - \hat{A}^*)\theta\rangle_{\hat{B}^*} - {}_{\hat{B}}\langle RG_\lambda y, \theta\rangle_{\hat{B}^*} \\ &= {}_B\langle y, (DG_\lambda)^*(\lambda - \hat{A}^*)\theta\rangle_{B^*} - {}_B\langle y, (RG_\lambda)^*\theta\rangle_{B^*}, \\ &\quad \forall y \in \mathrm{Dom}(D). \end{aligned} $$

This implies $\theta \in \mathrm{Dom}(D^*)$. Now the rest is the same as in Theorem 2.2. ∎

3. DEFECTIVE INTERTWINING PROPERTY II: HILBERT SPACE CASE

In this section, we discuss semigroups on Hilbert spaces. Let $\{T_t\}$ and $\{\hat{T}_t\}$ be (C_0)-semigroups on Hilbert spaces H and $\hat{H}$. The generators of $\{T_t\}$ and $\{\hat{T}_t\}$ are denoted by A and $\hat{A}$, respectively. We assume that they are bounded from below in the following sense: there exists $\omega \geq 0$ such that

$$ \begin{aligned} (Ax, x)_H &\geq -\omega |x|_H^2, \\ (\hat{A}\theta, \theta)_{\hat{H}} &\geq -\omega |\theta|_{\hat{H}}^2 \end{aligned} $$

Hence $A-\omega$ and $\hat{A}-\omega$ generate contraction semigroups. We further assume that they satisfy the weak sector condition. We denote the associated quadratic form by $\mathcal{E}$ and $\hat{\mathcal{E}}$, e.g.,

$$\mathcal{E}(x,y) = -(Ax,y)_H \quad \forall x \in \mathrm{Dom}(A),\ \forall y \in \mathrm{Dom}(\mathcal{E}).$$

We fix $\delta > \omega$ and set

$$\mathcal{E}_\delta(x,y) = \mathcal{E}(x,y) + \delta(x,y)_H.$$

Then $\mathcal{F} = \mathrm{Dom}(\mathcal{E})$ is a Hilbert space with the inner product

$$(x,y)_{\mathcal{F}} = \frac{1}{2}\Big\{\mathcal{E}_\delta(x,y) + \overline{\mathcal{E}_\delta(y,x)}\Big\}. \tag{14}$$

Here $^{-}$ denotes the complex conjugation. By the weak sector condition, $\mathcal{E}$ is a bounded sesqui-linear form on $\mathcal{F} \times \mathcal{F}$, i.e., there exists a constant $C > 0$ such that

$$|\mathcal{E}(x,y)| \le C(x,x)_{\mathcal{F}}^{1/2}(y,y)_{\mathcal{F}}^{1/2}.$$

Similarly we define

$$\hat{\mathcal{E}}_\delta(\theta,\eta) = \hat{\mathcal{E}}(\theta,\eta) + \delta(\theta,\eta)_{\hat{H}}$$

and a Hilbert space $\hat{\mathcal{F}} = \mathrm{Dom}(\hat{\mathcal{E}})$ with the inner product

$$(\theta,\eta)_{\hat{\mathcal{F}}} = \frac{1}{2}\Big\{\hat{\mathcal{E}}_\delta(\theta,\eta) + \overline{\hat{\mathcal{E}}_\delta(\eta,\theta)}\Big\}. \tag{15}$$

H^* denotes the set of all conjugate linear continuous functional on H. $\mathcal{F}^*$ can be defined similarly, i.e., the set of all conjugate linear continuous functional on $\mathcal{F}$. Clearly $H^* \subseteq \mathcal{F}^*$ and, by the Riesz theorem, we can identify H^* with H. Hence we have a triplet $\mathcal{F} \subseteq H \subseteq \mathcal{F}^*$. Moreover A can be extended to a bounded linear operator from $\mathcal{F}$ onto $\mathcal{F}^*$ and generates a (C_0)-semigroup $\{\tilde{T}_t\}$ on $\mathcal{F}^*$ (see Tanabe [14, §2.2]). We denote the generator by $A^{\sim}$.

Similarly we can define a triplet $\hat{\mathcal{F}} \subseteq \hat{H} \subseteq \hat{\mathcal{F}}^*$ and $\hat{A}^{\sim} : \hat{\mathcal{F}} \to \hat{\mathcal{F}}^*$ which is an extension of $\hat{A}$. The semigroup generated by $\hat{A}^{\sim}$ is denoted by $\{\hat{T}_t^{\sim}\}$. The associated resolvent is denoted by $\hat{G}_\lambda^{\sim}$.

Suppose also that we are given a closed operator D from H into $\hat{H}$ satisfying $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(D)$. Now we consider the following defective intertwining property

$$DA = \hat{A}D + R. \tag{16}$$

Contrary to the previous section, we assume that

(B.1) R is a bounded linear operator from $\mathrm{Dom}(D)$ into $\hat{\mathcal{F}}^*$.

We have the following theorem.

THEOREM 3.1. *We assume that* $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(D)$ *and* (B.1). *Then the following three statements are equivalent to each other.*

(a) $\mathrm{Dom}(\hat{A}^*) \subseteq \mathrm{Dom}(D^*)$ *and*

$$(Ax, D^*\theta)_H = (Dx, \hat{A}^*\theta)_{\hat{H}} + {}_{\hat{\mathcal{F}}^*}(Rx, \theta)_{\hat{\mathcal{F}}} \qquad \forall x \in \mathrm{Dom}(A), \forall \theta \in \mathrm{Dom}(\hat{A}^*). \tag{17}$$

(b) *For sufficiently large* λ,

$$DG_\lambda x = \hat{G}_\lambda Dx + \hat{G}_\lambda^{\sim} RG_\lambda x, \quad \forall x \in \mathrm{Dom}(D). \tag{18}$$

(c) $\{T_t\}$ *is a* (C_0)*-semigroup on* $\mathrm{Dom}(D)$ *and the following holds:*

$$DT_t = \hat{T}_t Dx + \int_0^t \hat{T}_{t-s}^{\sim} RT_s x\, ds, \quad \forall x \in \mathrm{Dom}(D).$$

Here the integral is the limit of Riemann sum in $\hat{\mathcal{F}}^*$.

Proof. We do not need to prove this theorem since $\{\hat{T}_t^{\sim}\}$ is a (C_0)-semigroup in $\hat{\mathcal{F}}^*$. $\{T_t\}$ and $\{\hat{T}_t^{\sim}\}$ satisfy the conditions of Theorem 2.1. The difference is that we have to show (17) for $x \in \mathrm{Dom}(A)$ and $\theta \in \mathrm{Dom}((\hat{A}^{\sim})^*) = \hat{\mathcal{F}}$. But this follows from Theorem 2.3 (a) because $\mathrm{Dom}(\hat{A}^*)$ is dense in $\hat{\mathcal{F}}$. ∎

As in Theorem 2.3, it is sufficient to assume that the equation (17) holds on a dense domain of $\mathrm{Dom}(A)$ and $\mathrm{Dom}(\hat{A})$. In fact, if we assume that $\hat{\mathcal{F}} \subseteq \mathrm{Dom}(D^*)$, we can relax the condition (1). Before that, we prepare the following proposition. This proposition also plays an essential role in the next section.

PROPOSITION 3.1. *Assume that the defective intertwining property holds and* $\mathcal{F} \subseteq \mathrm{Dom}(D)$. *Then the following two statements are equivalent to each other.*

(a) $D^* \colon \hat{\mathcal{F}} \to H$ *is bounded, i.e.,* $\hat{\mathcal{F}} \subseteq \mathrm{Dom}(D^*)$.
(b) $D\colon \mathrm{Dom}(A) \to \hat{\mathcal{F}}$ *is bounded, i.e.,* $D\,\mathrm{Dom}(A) \subseteq \hat{\mathcal{F}}$.

Proof. We first show (a)⇒(b). Take any $x \in \mathrm{Dom}(A)$ and set

$$\Phi(\theta) = -(Ax, D^*\theta)_H + \lambda(Dx,\theta)_{\hat{H}} + {}_{\hat{\mathcal{F}}^*}(Rx,\theta)_{\hat{\mathcal{F}}}, \quad \theta \in \hat{\mathcal{F}}.$$

From the assumption (a), $D^*\colon \hat{\mathcal{F}} \to H$ is bounded and hence $\Phi\colon \hat{\mathcal{F}} \to \mathbb{C}$ is also bounded. Then the Lax-Milgram theorem yields that there exists $\eta \in \hat{\mathcal{F}}$ such that

$$\Phi(\theta) = \hat{\mathcal{E}}_\lambda(\eta,\theta).$$

If $\theta \in \mathrm{Dom}(\hat{A})$, then

$$\begin{aligned}
(\eta, (\lambda - \hat{A})^*\theta)_{\hat{H}} &= \hat{\mathcal{E}}_\lambda(\eta,\theta) \\
&= \Phi(\theta) \\
&= -(Ax, D^*\theta)_H + \lambda(Dx,\theta)_{\hat{H}} + {}_{\hat{\mathcal{F}}^*}(Rx,\theta)_{\hat{\mathcal{F}}} \\
&= -(Dx, \hat{A}^*\theta)_{\hat{H}} - {}_{\hat{\mathcal{F}}^*}(Rx,\theta)_{\hat{\mathcal{F}}} \\
&\quad + \lambda(Dx,\theta)_{\hat{H}} + {}_{\hat{\mathcal{F}}^*}(Rx,\theta)_{\hat{\mathcal{F}}} \quad (\because\ (17)) \\
&= (Dx, (\lambda - \hat{A}^*)\theta)_{\hat{H}}.
\end{aligned}$$

Since $(\lambda - \hat{A}^*)\,\mathrm{Dom}(\hat{A}^*) = \hat{H}$, we have $\eta = Dx$. This means $Dx \in \hat{\mathcal{F}}$ and hence (b) follows.

Conversely we assume (b). We note that operators $G_\lambda\colon H \to \mathrm{Dom}(A)$ and $\hat{G}_\lambda^{\sim}\colon \hat{\mathcal{F}}^* \to \hat{\mathcal{F}}$ are bounded. Combining this with (b), we see that $S = DG_\lambda - \hat{G}_\lambda^{\sim} RG_\lambda\colon H \to \hat{\mathcal{F}}$ is bounded. Take any $x \in \mathrm{Dom}(D)$ and $\theta \in \hat{\mathcal{F}}$.

$$\begin{aligned}
(Dx,\theta)_{\hat{H}} &= {}_{\hat{\mathcal{F}}^*}(Dx, \hat{G}_\lambda^*(\lambda - \hat{A}^*)\theta)_{\hat{\mathcal{F}}} \\
&= {}_{\hat{\mathcal{F}}}(\hat{G}_\lambda Dx, (\lambda - \hat{A}^*)\theta)_{\hat{\mathcal{F}}^*} \\
&= {}_{\hat{\mathcal{F}}}((DG_\lambda + \hat{G}_\lambda^{\sim} R\hat{G}_\lambda)x, (\lambda - \hat{A}^*)\theta)_{\hat{\mathcal{F}}^*} \\
&= {}_{\hat{\mathcal{F}}}(Sx, (\lambda - \hat{A}^*)\theta)_{\hat{\mathcal{F}}^*} \\
&= (x, S^*(\lambda - \hat{A}^*)\theta)_H \quad (\because S^*\colon \hat{\mathcal{F}}^* \to H^* \text{ is bounded})
\end{aligned}$$

which implies $\theta \in \mathrm{Dom}(D^*)$. Thus we have $\hat{\mathcal{F}} \subseteq \mathrm{Dom}(D^*)$. ∎

Now we are ready to prove the following theorem.

THEOREM 3.2. *We assume the condition of Theorem 3.1 and* $\hat{\mathcal{F}} \subseteq \mathrm{Dom}(D^*)$. *Then the following statements are equivalent to each other.*

(a) *There exist a dense subspace* $\mathcal{D} \subseteq \mathrm{Dom}(A)$ *and a dense subspace* $\hat{\mathcal{D}} \subseteq \mathrm{Dom}(\hat{\mathcal{E}})$ *such that* $D\mathcal{D} \subseteq \mathrm{Dom}(\hat{\mathcal{E}})$ *and*

$$(Ax, D^*\theta)_H = -\hat{\mathcal{E}}(Dx,\theta) + {}_{\hat{\mathcal{F}}^*}(Rx,\theta)_{\hat{\mathcal{F}}}, \quad \forall x \in \mathcal{D}, \forall\theta \in \hat{\mathcal{D}}. \tag{19}$$

(b) *For sufficiently large* λ,

$$DG_\lambda x = \hat{G}_\lambda Dx + \hat{G}_\lambda^{\sim} RG_\lambda x, \quad \forall x \in \mathrm{Dom}(D). \tag{20}$$

Proof. To show (b)$\Rightarrow$(a), take $\mathcal{D} = \mathrm{Dom}(A)$, $\hat{\mathcal{D}} = \mathrm{Dom}(\hat{\mathcal{E}})$. $D\mathcal{D} \subseteq \mathrm{Dom}(\hat{\mathcal{E}})$ follows from the previous proposition.

Conversely we assume (a). Since $\hat{\mathcal{D}}$ is dense in $\hat{\mathcal{E}}$, (19) holds for $x \in \mathcal{D}$ and $\theta \in \hat{\mathcal{F}}$. If, in particular, $\theta \in \mathrm{Dom}(\hat{A})$, then it follows that

$$(Ax, D^*\theta)_H = (Dx, \hat{A}^*\theta)_{\hat{H}} + {}_{\hat{\mathcal{F}}^*}(Rx, \theta)_{\hat{\mathcal{F}}}, \quad x \in \mathcal{D}.$$

The density of $\mathcal{D}$ in $\mathrm{Dom}(A)$ deduces that the equation above holds for all $x \in \mathrm{Dom}(A)$. Now, by Theorem 3.1, we have (b) ∎

A natural expression of the defective intertwining property is of the form (16). But it is rather difficult to give the definite region. When R is bounded, we can give a region where (16) holds as follows.

Proposition 3.2. *Under the assumptions of Proposition 3.1, we additionally suppose one of (and hence both of) properties (1) and (2) of Proposition 3.1. We further assume that R is bounded. Then, for any $x \in \mathrm{Dom}(A^2)$, we have $Dx \in \mathrm{Dom}(\hat{A})$, $\hat{A}Dx \in \mathrm{Dom}(\hat{\mathcal{E}})$ and the following identity holds.*

$$\hat{A}Dx = DAx - Rx. \tag{21}$$

Proof. Take any $x \in \mathrm{Dom}(A^2)$. We first show $Dx \in \mathrm{Dom}(\hat{A})$. It follows from Proposition 3.1 that $Dx \in \mathrm{Dom}(\hat{\mathcal{E}})$. Hence, for $\theta \in \mathrm{Dom}(\hat{A}^*)$,

$$\begin{aligned}
\hat{\mathcal{E}}(Dx, \theta) &= -(Dx, \hat{A}^*\theta) \\
&= -(Ax, D^*\theta) + (Rx, \theta) \quad (\because \text{Theorem 3.1 (a)}) \\
&= -(DAx, \theta) + (Rx, \theta).
\end{aligned}$$

Clearly this identity holds for all $\theta \in \mathrm{Dom}(\hat{\mathcal{E}})$ and the right hand side is continuous in θ with respect to the $\hat{H}$-norm since R is bounded. This yields that $Dx \in \mathrm{Dom}(\hat{A})$ and $\hat{A}Dx = DAx - Rx$. ∎

In the case $R = 0$, the proposition above can be extended to the higher order case.

Proposition 3.3. *Assume assumptions of Proposition 3.2 and $R = 0$. Then, for any $x \in \mathrm{Dom}(A^n)$, we have $Dx \in \mathrm{Dom}(\hat{A}^{n-1})$, $\hat{A}^{n-1}Dx \in \mathrm{Dom}(\hat{\mathcal{E}})$ and the following identities hold:*

$$\begin{aligned}
&\hat{A}^{n-1}Dx = DA^{n-1}x,\\
&\hat{\mathcal{E}}(\hat{A}^{n-1}Dx, \theta) = -(A^n x, D^*\theta), \quad \forall \theta \in \mathrm{Dom}(\hat{\mathcal{E}}).
\end{aligned}$$

Proof. We prove them by the induction on n. The case $n = 1$ is nothing but Proposition 3.1.

Assuming the case n, we will prove them for $n+1$. So let us suppose $x \in \mathrm{Dom}(A^{n+1})$. Set $y = Ax$. We can use the assumption of induction since $y \in \mathrm{Dom}(A^n)$. Hence we have $Dy \in \mathrm{Dom}(\hat{A}^{n-1})$, $\hat{A}^{n-1}Dy \in \mathrm{Dom}(\hat{\mathcal{E}})$ and it holds that

$$\hat{A}^{n-1}Dy = DA^{n-1}y. \tag{22}$$

Since $x \in \mathrm{Dom}(A^2)$, we have by virtue of Proposition 3.2,

$$\hat{A}Dx = DAx = Dy \in \mathrm{Dom}(\hat{A}^{n-1})$$

which implies $Dx \in \mathrm{Dom}(\hat{A}^n)$. Further

$$\hat{A}^n Dx = \hat{A}^{n-1}\hat{A}Dx = \hat{A}^{n-1}Dy \in \mathrm{Dom}(\hat{\mathcal{E}}).$$

Thus we have obtained $\hat{A}^n Dx \in \mathrm{Dom}(\hat{\mathcal{E}})$. Therefore

$$\begin{aligned}
\hat{A}^n Dx &= \hat{A}^{n-1}Dy\\
&= DA^{n-1}y \quad (\because (22))\\
&= DA^n x.
\end{aligned}$$

Now, for any $\theta \in \mathrm{Dom}(\hat{\mathcal{E}})$,

$$\begin{aligned}
\hat{\mathcal{E}}(\hat{A}^n Dx, \theta) &= \hat{\mathcal{E}}(\hat{A}^{n-1}Dy, \theta)\\
&= -(A^n y, D^*\theta) \quad \text{(the assumption of induction)}\\
&= -(A^{n+1}x, D^*\theta).
\end{aligned}$$

Thus we have obtained the case $n+1$. ∎

4. GENERATOR DOMAIN

In this section, we see that we can determine the generator domain using the defective intertwining property.

Let the notations be the same as in the previous section. We further assume

(B.2) $\quad \mathcal{E}(x,y) = (Dx, Dy)_{\hat{H}}.$

Therefore $\mathcal{E}$ is symmetric and the generator A is given by

$$A = -D^*D. \tag{23}$$

But the symmetry of $\hat{\mathcal{E}}$ is not required.

Theorem 4.1. *Assume* (B.1), (B.2) *and the defective intertwining property between A and $\hat{A}$. We further assume $D^* \in \mathcal{L}(\hat{\mathcal{F}}; H)$. Then $x \in$* $\mathrm{Dom}(A)$ *if and only if $x \in \mathcal{F}$ and $Dx \in \hat{\mathcal{F}}$. Moreover the following equality holds:*

$$(Ax, Ax)_H = \hat{\mathcal{E}}(Dx, Dx) - {}_{\hat{\mathcal{F}}^*}(Rx, Dx)_{\hat{\mathcal{F}}} \quad \forall x \in \mathrm{Dom}(A). \tag{24}$$

Proof. Suppose $x \in \mathcal{F}$ and $Dx \in \hat{\mathcal{F}}$. Since we have assumed $\hat{\mathcal{F}} \subseteq \mathrm{Dom}(D^*)$ and $A = -D^*D$, we have $x \in \mathrm{Dom}(A)$. The reversed implication is nothing but the Proposition 3.1. The equation (24) is easily obtained by setting $\theta = Dx$ in (19). ∎

We can also extend the theorem above to the higher order case as follows.

Theorem 4.2. *Assume all assumptions of Theorem 4.1 and $R = 0$. Then $x \in \mathrm{Dom}(A^n)$ if and only if $Dx \in \mathrm{Dom}(\hat{A}^{n-1})$ and $\hat{A}^{n-1}Dx \in \mathrm{Dom}(\hat{\mathcal{E}})$. In this case, the following equality holds:*

$$\hat{A}^{n-1}Dx = DA^{n-1}x. \tag{25}$$

Proof. The sufficiency of $x \in \mathrm{Dom}(A^n)$ is already proved in Proposition 3.3. We prove the necessity by the induction on n.

Assuming the case n, we will show it for $n+1$. So we suppose $Dx \in \mathrm{Dom}(\hat{A}^n)$ and $\hat{A}^n Dx \in \mathrm{Dom}(\hat{\mathcal{E}})$. Since these relations holds for $n-1$, we have $x \in \mathrm{Dom}(A^n)$ by the assumption of induction. Therefore $D^*\hat{A}^n Dx$ is well-defined since $\mathrm{Dom}(\hat{\mathcal{E}}) \subseteq \mathrm{Dom}(D^*)$.

Now let us take any $z \in \mathrm{Dom}(A)$.

$$\begin{aligned}(D^*\hat{A}^n Dx, z) &= (\hat{A}^n Dx, Dz)\\ &= (\hat{A}\hat{A}^{n-1}Dx, Dz)\\ &= (D^*(\hat{A}^{n-1}Dx), Az) \quad (\because \text{Theorem 3.1 (a)})\\ &= (D^*(DA^{n-1}x), Az)\\ &= (A^n x, Az).\end{aligned}$$

Since the left hand side is continuous in z with respect to the H-norm, this yields $A^n x \in \mathrm{Dom}(A)$. Thus we have proved the result for $n+1$. ▮

As an example, we consider a Schrödinger operator of the form $A = \Delta - V$ on $\mathbb{R}^d$. Here V is a scalar potential. We assume that V is bounded from below. Our aim is to give a characterization of the domain $\mathrm{Dom}(\Delta - V)$. Here we regard $\Delta - V$ as a self-adjoint operator on $L^2(\mathbb{R}^d)$. It is well-known that $f \in L^2(\mathbb{R}^d)$ belongs to $\mathrm{Dom}(\Delta - V)$ if and only if $(\Delta - V)f \in L^2(\mathbb{R}^d)$ in the sense of distribution.

We give a different characterization. To apply Theorem 4.1, we have to introduce another semigroup acting on 1-forms on $\mathbb{R}^d$. Let $T^*\mathbb{R}^d$ be the cotangent bundle of $\mathbb{R}^d$ and we denote the all L^2-sections of $T^*\mathbb{R}^d$ by $L^2\Gamma(T^*\mathbb{R}^d)$, i.e., $L^2\Gamma(T^*\mathbb{R}^d)$ is the set of all square integrable 1-forms. We define an operator $\hat{A}$ on $L^2\Gamma(T^*\mathbb{R}^d)$ by

$$\hat{A} = \Delta - V.$$

$\hat{A}$ has the same form as $A = \Delta - V$ but it acts on 1-forms: a 1-form θ is regarded as an $\mathbb{R}^d$-valued function $\theta = (\theta_1, \dots, \theta_d)$ and $\hat{A}$ acts componentwisely, i.e.,

$$\hat{A}\theta = (A\theta_1, \dots, A\theta_d).$$

So we have the following defective intertwining property:

$$\begin{aligned}\nabla A f &= \nabla(\Delta f - V f)\\ &= \nabla\Delta f - V\nabla f - f\nabla V\\ &= \hat{A}\nabla f + Rf\end{aligned}$$

where R is defined by

$$Rf = -f\nabla V. \tag{26}$$

Here the identity above holds for $f \in C_0^\infty(\mathbb{R}^d)$. But $C_0^\infty(\mathbb{R}^d)$ is a core for the operator A and we can apply Theorem 3.2; the defective intertwining property in our sense holds.

The associated quadratic forms with A and $\hat{A}$ are given by

$$\begin{aligned}
\mathcal{E}(f,g) &= \int_{\mathbb{R}^d} (\nabla f, \nabla g)\,dx + \int_{\mathbb{R}^d} V f g\,dx, \\
\hat{\mathcal{E}}(\theta,\eta) &= \int_{\mathbb{R}^d} (\nabla \theta, \nabla \eta)\,dx + \int_{\mathbb{R}^d} V(\theta,\eta)\,dx.
\end{aligned}$$

Since V is bounded from below, there exists $\delta > 0$ such that $\mathcal{E}_\delta = \mathcal{E} + \delta(\ ,\ )$ and $\hat{\mathcal{E}}_\delta = \hat{\mathcal{E}} + \delta(\ ,\ )$ are non-negative definite. We take $\omega > \delta$ and fix it. We denote the domain of $\mathcal{E}$ by $\mathcal{F}$ and the domain of $\hat{\mathcal{E}}$ by $\hat{\mathcal{F}}$. To ensure the boundedness of $R\colon \mathcal{F} \to \hat{\mathcal{F}}^*$, we assume the following condition for the potential V:

$$|\nabla V| \le C(V_+ + 1). \tag{27}$$

Here V_+ is the positive part of V. The boundedness of R can be seen as

$$\begin{aligned}
|(Rf,\theta)| &= \int_{\mathbb{R}^d} (f \nabla V, \theta) dx \\
&\le C \int_{\mathbb{R}^d} (V_+ + 1)|f||\theta| dx \\
&\le C \Big\{ \int_{\mathbb{R}^d} (V_+ + 1)|f|^2 dx \Big\}^{1/2} \Big\{ \int_{\mathbb{R}^d} (V_+ + 1)|\theta|^2 dx \Big\}^{1/2} \\
&\le C' \mathcal{E}_\omega(f,f)^{1/2} \hat{\mathcal{E}}_\omega(\theta,\theta)^{1/2}.
\end{aligned}$$

This means that R is a bounded operator from $\mathcal{F}$ into $\hat{\mathcal{F}}^*$. Now we can apply Theorem 4.1 to obtain the following theorem.

Theorem 4.3. *Assume that V is continuously differentiable, bounded from below and satisfies* (27). *Then*

$$\operatorname{Dom}(\Delta - V) = \{ f \in L^2; \nabla f, \nabla^2 f, V f \in L^2 \}. \tag{28}$$

Proof. First suppose $f \in \operatorname{Dom}(\Delta - V)$. We notice that ∇^* is a minus divergence operator and hence $\operatorname{Dom}(\nabla^*) \subseteq \operatorname{Dom}(\hat{\mathcal{E}})$. Now, applying Theorem 4.1, we have $\nabla f \in \operatorname{Dom}(\hat{\mathcal{E}})$ and hence $\nabla^2 f \in L^2$ which leads $\Delta f \in L^2$. On the other hand, it holds that $(\Delta - V) f \in L^2$ and it follows that $V f \in L^2$.

Conversely, suppose that $f, \nabla^2 f, Vf \in L^2$. Then, clearly $(\Delta - V)f \in L^2$ which implies $f \in \mathrm{Dom}(\Delta - V)$. This completes the proof. ∎

Remark 4. 1. This result is known when V is a polynomial (see, Guibourg [4], Shen [10].)[1]

(28) is equivalent to $\mathrm{Dom}(\Delta - V) = \mathrm{Dom}(\Delta) \cap \mathrm{Dom}(V)$. Under this condition, Ichinose-Tamura [5] proved the norm convergence of Trotter-Kato product formula. Our case include, e.g., the case $V(x) = e^x$ as a special case.

We can discuss similar problem on a Riemannian manifold. In this case, the space is curved and so an effect of curvature comes in. Let M be a complete Riemannian manifold and we denote the Laplace-Beltrami operator by Δ and the Ricci curvature by Ric. We assume that Ric is bounded from below and define a bilinear form Q_{Ric} on the space of square integrable differential 1-forms by

$$Q_{\mathrm{Ric}}(\theta, \eta) = \int_M (\mathrm{Ric}\,\theta, \eta)dx$$

where dx denote the Riemannian volume. Suppose we are given a scalar function V which satisfies the same condition (27). Then we have the following theorem.

THEOREM 4.4. *The following equalities hold:*

$$\begin{aligned}\mathrm{Dom}(\Delta - V) &= \mathrm{Dom}(\Delta) \cap \mathrm{Dom}(V) \\ &= \mathrm{Dom}(\nabla^2) \cap \mathrm{Dom}(V) \\ &= \{f \in L^2; \sqrt{|V|+1}\nabla f, \nabla^2 f, Vf \in L^2\} \cap \mathrm{Dom}(Q_{\mathrm{Ric}}).\end{aligned}$$

Proof. Set $A = \Delta - V$ and $\hat{A} = -dd^* - d^*d - V$. Here d is the exterior differentiation and d^* is its dual. $\hat{A}$ is acting on square integrable differential 1-forms. Using the Weitzenböck formula, the associated bilinear forms are given by

$$\mathcal{E}(f,g) = \int_M (\nabla f, \nabla g)dx, \hat{\mathcal{E}}(\theta, \eta) \quad = \int_M (\nabla\theta, \nabla\eta)dx + \int_M (\mathrm{Ric}\,\theta, \eta)dx + \int_M (V\theta, \eta)dx.$$

In this case, the defective intertwining property takes the following form:

$$\nabla A = \hat{A}\nabla - \nabla V.$$

[1] The author thanks Professor T. Ichinose who taught the author the references.

Take any $f \in \mathrm{Dom}(\Delta)$. Then, by Theorem 4.1, it holds that $\nabla f \in \mathrm{Dom}(\hat{\mathcal{E}})$, i.e., $\nabla^2 f \in L^2$, $\sqrt{|V|+1}\nabla f \in L^2$ and $\nabla f \in \mathrm{Dom}(Q_{\mathrm{Ric}})$.

Conversely, for any $f \in \mathrm{Dom}(\nabla^2) \cap \mathrm{Dom}(V)$, it is easy to see that $f \in \mathrm{Dom}(\Delta)$. In fact, Δf is nothing but the trace of $\nabla^2 f$. Therefore it follows that $f \in \mathrm{Dom}(\Delta - V)$.

The remaining equalities are easy. ▌

The interesting point of the theorem above is that for any $f \in \mathrm{Dom}(\Delta)$, it holds that $\int_M (\mathrm{Ric}\,\nabla f, \nabla f)dx < \infty$ no matter how large Ric is.

Remark 4. 2. The argument above works even in infinite dimensional space, e.g., an abstract Wiener space. In this case, we replace the Laplacian with the Ornstein-Uhlenbeck operator.

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