DIRICHLET FORMS ON SEPARABLE METRIC SPACES

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0. Introduction

The general theory of Dirichlet forms on locally compact state spaces has its origin in the classical work by Beurling and Deny [6, 7] and has been developed deeply by Fukushima [11] and Silverstein [20]. Recently various investigations on Dirichlet forms on infinite dimensional, and hence non-locally compact, topological vector spaces, and many attempts to extend the general theory on locally compact spaces to such spaces have been made by several authors. See [1, 2] and the references therein. In particular, on many topological vector spaces, diffusion processes associated with Dirichlet forms have been constructed ([1, 16]). Moreover, symmetric Markov processes corresponding to Dirichlet forms on separable metric spaces have been deeply studied in [2]. Once one obtains a symmetric Markov process, the recent development of general theory of right processes (cf. [15, 19]) brings us to the world where the machineries of stochastic calculus and probabilistic potential theory work. In this sense, most works on Dirichlet forms on non-locally compact state spaces correspond to their probabilistic aspects. On the contrary, we devote a half of the paper to the investigation of analytic aspects of Dirichlet forms on non-locally compact spaces and we aim at unifying their analytic and probabilistic aspects. Our goal will be to present a general theory of Dirichlet forms with non-locally compact state spaces, following the celebrated work by Fukushima [11].

In the paper, we consider a Dirichlet form on a Lusinian separable metric space. A key assumption we make is that the corresponding 1-capacity is tight. See Assumption (A.3) in Section 1. Roughly speaking, this assumption means that the state space may be thought of as a locally compact space from the point of view of Dirichlet forms. Moreover, since most measures appearing in the study of Dirichlet forms (like measures of finite energy integral, killing measures and so on) are dominated by the capacity, this assumption also implies the tightness of families of such measures, which is a substitute for the fact that on a locally compact space every positive linear functional on a space of bounded continuous functions is realized by a measure. Hence we need not to identify the Dirichlet form with the one on a compact metric space by using the compactification argument as in [1, 16].

In Section 1, we will fix the situation to deal with and make some preliminary observations on the domain of the Dirichlet form. Section 2 will be devoted to the study of measures of finite energy integral, smooth measures, and α-potentials. In Section 3, 1-equilibrium potentials of Borel sets and the spectral synthesis will be studied. The Beurling-Deny formula for Dirichlet forms on non-locally compact spaces will be established in Section 4. The existence of the associated Hunt process, a brief review on the associated probabilistic potential theory, and the unification of analytic and probabilistic potential theoretical notions will be discussed in Section 5. In Section 6, we study the local property and give a probabilistic interpretation of the Beurling-Deny formula. That the stochastic calculus developed in [11] in the case of locally compact state space remains valid in our situation will be seen in Section 7. After a long course of presenting a general theory of Dirichlet forms with non-locally compact metric space, we will see in

Section 8 that the result in [13] on closable parts of pre-Dirichlet forms remains valid in our situation. This gives rise to a lot of Dirichlet forms on non-locally compact spaces.

Another systematic study of a general theory of Dirichlet forms on non-locally compact spaces will be found in the book [17] which Ma and Röckner are now preparing.

1. Preliminaries

Let X be a Lusinian separable metric space and $\mathcal{B}(X)$ be its topological Borel field. We fix a probability measure m on $(X, \mathcal{B}(X))$ such that $supp[m] = X$ and a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X; m)$ with $1 \in \mathcal{F}$.

For open $G \subset X$ and any $A \subset X$, we define

$$
\begin{aligned}
Cap(G) &= \inf\{\mathcal{E}_1(u,u) : u \in \mathcal{F} \text{ and } u \geq 1 \ m\text{-a.e. on } G\}, && (1.1)\\
Cap(A) &= \inf\{Cap(G) : G \text{ is open and } A \subset G\}, && (1.2)
\end{aligned}
$$

where $\mathcal{E}_1(u,u) = \mathcal{E}(u,u) + (u,u)_m$ and $(u,v)_m = \int_X uvdm$. Then Cap is a Choquet capacity :

$$
Cap(\cup A_n) = \sup_n Cap(A_n) \qquad \text{and} \qquad Cap(\cap K_n) = \inf_n Cap(K_n), \tag{1.3}
$$

for any increasing sequence $\{A_n\}$ of subsets of X and any decreasing sequence $\{K_n\}$ of compact subsets. Moreover Cap enjoys that

$$
Cap(\cup A_n) \leq \sum_n Cap(A_n), \tag{1.4}
$$

and, for any $A \in \mathcal{B}(X)$,

$$
Cap(A) = \sup\{Cap(K) : K \subset A,\ K \text{ is compact}\}. \tag{1.5}
$$

See [1, 11, 12].

Now we introduce the hypotheses assumed throughout the paper :

(A.1) $\mathcal{F} \cap C_b(X)$ is dense in $(\mathcal{F}, \mathcal{E}_1)$, where $C_b(X)$ is the space of bounded continuous functions on X,

(A.2) $\mathcal{F} \cap C_b(X)$ separates the points of X,

(A.3) $Cap(\cdot)$ is tight: for any $\varepsilon > 0$, there exists a compact set $K \subset X$ such that $Cap(X \setminus K) < \varepsilon$.

A statement depending on $x \in A$ is said to hold "q.e." on A if it holds on A except for a set of zero capacity with respect to Cap. A function $u : X \to \mathbf{R}$ is said to be quasi-continuous if there is a decreasing sequence $\{G_n\}$ of open sets such that $Cap(G_n) \downarrow 0$ and u is continuous on each $X \setminus G_n$. As in [11, §3.1], we see that each $u \in \mathcal{F}$ possesses a quasi-continuous m-version $\tilde{u}$. Moreover,

Theorem 1.1. *(i) For each open $G \subset X$, there is a unique $e_G \in \mathcal{F}$ such that $\mathcal{E}_1(e_G, e_G) = Cap(G)$, $0 \leq \tilde{e}_G \leq 1$, and $\tilde{e}_G = 1$ q.e. on G. Moreover, if $w \in \mathcal{F}$ satisfies $w = 1$ m-a.e. on G, then $\mathcal{E}_1(w, e_G) = Cap(G)$.*
(ii) If $\{u_n\}$ is a Cauchy sequence in $(\mathcal{F}, \mathcal{E}_1)$, then there is a subsequence $\{u_{n_k}\}$ and quasi-continuous $u \in \mathcal{F}$ such that $u_{n_k} \to u$ q.e. and in $(\mathcal{F}, \mathcal{E}_1)$.
(iii) If $\{u_n\}$ is a Cauchy sequence in $(\mathcal{F}, \mathcal{E}_1)$ and if quasi-continuous versions $\tilde{u}_n$ of u_n converges to $\tilde{u}$, then $\tilde{u} \in \mathcal{F}$ and $u_n \to \tilde{u}$ in $(\mathcal{F}, \mathcal{E}_1)$.

In the remainder of this section, we investigate several properties of $\mathcal{F}$ following from the assumptions. We first see that $\mathcal{F} \cap C_b(X)$ separates the compact sets and finite measures in X;

Lemma 1.2. *(i) Let K_i, $i = 1, 2$, be disjoint compact sets in X. Then there exists an $f \in \mathcal{F} \cap C_b(X)$ such that (a) $f = 1$ on K_1 and $= 0$ on K_2, and (b) $0 \leq f \leq 1$.*
(ii) If μ and ν are finite measures on $(X, \mathcal{B}(X))$ such that $\int_X f d\mu = \int_X f d\nu$, $f \in \mathcal{F} \cap C_b(X)$, then $\mu = \nu$.

Proof. Let K_1 and K_2 be disjoint compact sets. Choose $g \in C_b(X)$ such that $g = 1$ on K_1 and $= 0$ on K_2. It follows from the Markov property that $\mathcal{F} \cap C_b(X)$ is a vector lattice. Then, applying the Stone-Weierstrass theorem, we obtain an $h \in \mathcal{F} \cap C_b(X)$ with $\sup\{|h(x) - g(x)| : x \in K_1 \cup K_2\} < 1/4$. It is easy to see that the function $f \equiv 0 \vee (2h - (1/2)) \wedge 1$ enjoys the property described in the first assertion.

To see the second assertion, let μ and ν be finite measures with $\int_X f d\mu = \int_X f d\nu$, $f \in \mathcal{F} \cap C_b(X)$. Take $\varepsilon > 0$ and $g \in C_b(X)$ arbitrarily. Since X is Lusinian, there is a compact set K such that $\mu(X \setminus K) < \varepsilon$ and $\nu(X \setminus K) < \varepsilon$. Applying the Stone-Weierstrass theorem again, we have an $h \in \mathcal{F} \cap C_b(X)$ such that $\sup\{|h(x) - g(x)| : x \in K\} < \varepsilon$. Then we can easily conclude that

$$\left| \int_X g d\mu - \int_X g d\nu \right| \leq \varepsilon(\mu(X) + \nu(X) + 2\|g\|_\infty),$$

where $\|g\|_\infty = \sup\{|g(x)| : x \in X\}$. Letting $\varepsilon \downarrow 0$, we see that $\int_X g d\mu = \int_X g d\nu$ for any $g \in C_b(X)$, which means that $\mu = \nu$. $\square$

For $u \in L^2(X; m)$, its support $supp[u]$ is defined to be the support of the measure $u \cdot m$. We denote by $\mathcal{F}_{cpt}$ the space of $u \in \mathcal{F}$ with compact support. By virtue of Theorem 1.1, we see that $\mathcal{F}_{cpt}$ separates the closed sets in X in the following sense:

Lemma 1.3. *Let F_i, $i = 1, 2$, be disjoint closed sets in X. Then there is a sequence $\{u_n\} \subset \mathcal{F}_{cpt}$ such that $0 \leq u_n \leq u_{n+1} \leq 1$,*

$$\tilde{u}_n = 0 \quad \text{q.e. on } F_1 \qquad \text{and} \qquad \tilde{u}_n \to 1 \quad \text{q.e. on } F_2.$$

Proof. By virtue of Assumption (A.3), we obtain an increasing sequence $\{K_n\}$ of compact sets with $Cap(X \setminus K_n) \downarrow 0$. Without loss of generality, we may assume that $\tilde{e}_{X \setminus K_n} \to 0$ q.e. By Lemma 1.2, there is an $f_n \in \mathcal{F} \cap C_b(X)$ such that $0 \leq f_n \leq 1$, $f_n = 1$ on $F_1 \cap K_n$ and $= 0$ on $F_2 \cap K_n$. Now it suffices to put $u_n = \max\{1 - f_j \vee e_{X \setminus K_j} : 1 \leq j \leq n\}$. $\square$

We end this section with seeing that $\mathcal{F}_{cpt}$ is dense in $\mathcal{F}$.

Lemma 1.4. *For every $u \in \mathcal{F}$, there is a sequence $\{u_n\} \subset \mathcal{F}_{cpt}$ such that $supp[u_n] \subset supp[u]$, $\tilde{u}_n \to \tilde{u}$ q.e., and in $(\mathcal{F}, \mathcal{E}_1)$. If $\|u\|_\infty < \infty$, then $\|u_n\|_\infty \le \|u\|_\infty$.*

Proof. Choose an increasing sequence $\{K_n\}$ of compact sets such that $Cap(X \setminus K_n) \le n^{-2}$. Note that $e_n \equiv \tilde{e}_{X \setminus K_n} \to 0$ q.e. Put $\hat{v}_n = (-n) \vee (u \wedge n)$ and $v_n = (1 - e_n)\hat{v}_n$. It is straightforward to see that $supp[v_n] \subset supp[u]$ and $\tilde{v}_n \to \tilde{u}$ q.e. Moreover, if we put $\| \cdot \|_{\mathcal{E}_1} = \mathcal{E}_1(\cdot, \cdot)^{1/2}$, then we have

$$\begin{aligned}
\|v_n\|_{\mathcal{E}_1} &\le \|\hat{v}_n\|_{\mathcal{E}_1} + \|e_n\|_\infty \|\hat{v}_n\|_{\mathcal{E}_1} + \|\hat{v}_n\|_\infty \|e_n\|_{\mathcal{E}_1} \\
&\le 2\|u\|_{\mathcal{E}_1} + nCap(X \setminus K_n)^{1/2} \\
&\le 2\|u\|_{\mathcal{E}_1} + 1.
\end{aligned}$$

Hence the Cesàro mean $\{u_n\}$ of a subsequence of $\{v_n\}$ converges to u in $\mathcal{F}$.

The above construction also implies the second assertion. □

2. Measures of finite energy integral

A finite positive Borel measure on X is said to be of finite energy integral if there is a constant $C > 0$ such that

$$\int_X |u| d\mu \le C\sqrt{\mathcal{E}_1(u,u)} \qquad \text{for any } u \in \mathcal{F} \cap C_b(X).$$

We denote by $\mathcal{S}_0$ the totality of measures of finite energy integral. Since $1 \in \mathcal{F}$, every $\mu \in \mathcal{S}_0$ is a finite measure and hence inner regular. For $\mu \in \mathcal{S}_0$, a unique $U_\alpha \mu \in \mathcal{F}$ is determined by

$$\mathcal{E}_\alpha(U_\alpha \mu, u) = \int_X u d\mu \qquad u \in \mathcal{F} \cap C_b(X),$$

where $\mathcal{E}_\alpha(u,u) = \mathcal{E}(u,u) + \alpha(u,u)_m$. We call $U_\alpha \mu$ the α-potential of μ. We have

Theorem 2.1. *The following conditions are equivalent to each other for $u \in \mathcal{F}$ and $\alpha > 0$.*
(i) u is an α-potential.
(ii) u is α-excessive : $u \ge 0$, $e^{-\alpha t} T_t u \le u$ m-a.e. for every $t > 0$.
(iii) $u \ge 0$, $\beta G_{\alpha+\beta} u \le u$ m-a.e. for every $\beta > 0$.
(iv) $\mathcal{E}_\alpha(u,v) \ge 0$ for any $v \in \mathcal{F}$ with $v \ge 0$ m-a.e.
(v) $\mathcal{E}_\alpha(u,v) \ge 0$ for any $v \in \mathcal{F} \cap C_b(X)$ with $v \ge 0$.

By Lemma 1.2, if $u \in \mathcal{F}$ satisfies one of the above conditions, then there is a unique $\mu \in \mathcal{S}_0$ with $u = U_\alpha \mu$.

Proof. The equivalence of (ii), (iii) and (iv) can be seen in exactly the same manner as in [11, Proof of Theorem 3.2.1]. The implications (i) ⇒ (v), (iv) ⇒ (v) are trivial. The implication (v) ⇒ (iv) follows from (A.1) and that every normal contraction operates on $\mathcal{E}$. Thus it suffices to show (ii) ⇒ (i).

Let $g_n = n(u - nG_{n+\alpha}u)$ and $\mu_n = g_n \cdot m$. Note that $G_\alpha g_n \to u$ weakly in $\mathcal{F}$ and hence

$$\sup_n \mathcal{E}_\alpha(G_\alpha g_n, G_\alpha g_n) < \infty.$$

In particular, $\mu_n(X) \to \mathcal{E}_\alpha(u, 1)$ and $\sup_n \mu_n(X) < \infty$.

For an arbitrary but fixed $\varepsilon > 0$, take a compact set K with $Cap(X \setminus K) \le \varepsilon$. Then, by Theorem 1.1, we have

$$\begin{aligned} \mu_n(X \setminus K) &\le \int_X e_{X\setminus K} g_n dm \\ &= \mathcal{E}_\alpha(G_\alpha g_n, e_{X\setminus K}) \\ &\le \sqrt{\mathcal{E}_\alpha(G_\alpha g_n, G_\alpha g_n)}\sqrt{\mathcal{E}_\alpha(e_{X\setminus K}, e_{X\setminus K})} \\ &\le \sqrt{1 \vee \alpha}\sqrt{\mathcal{E}_\alpha(G_\alpha g_n, G_\alpha g_n)}\sqrt{Cap(X \setminus K)}. \end{aligned}$$

Hence $\{\mu_n\}$ is tight and a subsequence $\{\mu_{n_j}\}$ converges weakly to a finite Borel measure μ on X. Then, for any $f \in \mathcal{F} \cap C_b(X)$, we have

$$\begin{aligned} \int_X f d\mu &= \lim_{j\to\infty} \int_X f d\mu_{n_j} \\ &= \lim_{j\to\infty} \mathcal{E}_\alpha(G_\alpha g_{n_j}, f) \\ &= \mathcal{E}_\alpha(u, f), \end{aligned}$$

which means that $\mu \in \mathcal{S}_0$ and $u = U_\alpha \mu$. □

Combining the above proof with Lemma 1.2, we obtain

Proposition 2.2. *Let $\mu \in \mathcal{F}_0$ and $g_n = n(U_\alpha\mu - nG_{n+\alpha}U_\alpha\mu)$. Then, $\int_X f g_n dm \to \int_X f d\mu$ for any $f \in C_b(X)$ and $G_\alpha g_n$ converges to $U_\alpha\mu$ weakly in $\mathcal{F}$.*

Now, as in [11, pp.70, 71], we can conclude

Theorem 2.3. *Let $\mu \in \mathcal{S}_0$.*
(i) $\mu(G) \le \sqrt{\mathcal{E}_1(U_1\mu, U_1\mu)}\sqrt{Cap(G)}$ for any open $G \subset X$.
(ii) μ charges no set of zero capacity.
(iii) For every $u \in \mathcal{F}$ and $\alpha > 0$, $\tilde{u} \in L^1(X; \mu)$ and it holds that

$$\int_X \tilde{u} d\mu = \mathcal{E}_\alpha(U_\alpha\mu, u).$$

We call a positive Borel measure μ on X smooth if it satisfies the following conditions:

(S.1) μ charges no set of zero capacity

(S.2) there exists an increasing sequence $\{F_n\}$ of compact sets such that

$$\mu(F_n) < \infty, \qquad n = 1, 2, \ldots \tag{2.1}$$
$$\mu(X \setminus \cup_n F_n) = 0, \tag{2.2}$$
$$\lim_n Cap(K \setminus F_n) = 0 \qquad \text{for any compact } K \subset X. \tag{2.3}$$

We denote by $\mathcal{S}$ the family of all smooth measures. It follows from Assumption (A.3) that $\mu \in \mathcal{S}$ if and only if the conditions (S.1), (2.1), and the following condition (2.4) are satisfied.

$$\lim_n Cap(X \setminus F_n) = 0. \tag{2.4}$$

Lemma 2.4. *Every finite Borel measure charging no set of zero capacity is smooth. In particular, $\mathcal{S}_0 \subset \mathcal{S}$.*

Proof. By Assumption (A.3), there is an increasing sequence $\{K_n\}$ of compact sets such that $Cap(X \setminus K_n) \downarrow 0$. It suffices to set $F_n = K_n$. $\square$

Furthermore, any smooth measure is approximated by measures in $\mathcal{S}_0$:

Theorem 2.5. *$\mu \in \mathcal{S}$ if and only if there exists an increasing sequence $\{F_n\}$ of closed sets satisfying (2.2) and (2.3) and $\mathbf{I}_{F_n} \cdot \mu \in \mathcal{S}_0$.*

Proof. The "Only if" part can be seen in exactly the same manner as in [11, Proof of Theorem 3.2.3].

To see the "if" part, let $\{F_n\}$ be an increasing sequence of closed sets satisfying (2.2) and (2.3) and $\mathbf{I}_{F_n} \cdot \mu \in \mathcal{S}_0$. Let A be a Borel set such that $Cap(A) = 0$. By Theorem 2.3, $\mu(F_n \cap A) = 0$. Combining with (2.2), we have $\mu(A) = 0$. Thus (S.1) is satisfied.

By Assumption (A.3), there is an increasing sequence $\{K_n\}$ of compact sets with $Cap(X \setminus K_n) \downarrow 0$. Put $\widetilde{F}_n = F_n \cap K_n$. Since $Cap(X \setminus \cup_n K_n) = 0$, $\mu(X \setminus \cup_n K_n) = 0$. Hence

$$\mu(X \setminus \cup_n \widetilde{F}_n) = 0.$$

Moreover, the subadditivity of Cap implies that (2.3) holds for $\widetilde{F}_n$. Thus (S.2) is satisfied. $\square$

3. Equilibrium potentials

The function e_G obtained in Theorem 1.1 is called the 1-equilibrium potential of an open set G. In this section, we consider equilibrium potentials for Borel sets.

Lemma 3.1. *For $u \in \mathcal{F}$ and closed $F \subset X$, the following conditions are equivalent.*
(i) $u = U_\alpha \mu$ for some $\mu \in \mathcal{S}_0$ with $supp[\mu] \subset F$.
(ii) $\mathcal{E}_\alpha(u, v) \geq 0$ for any $v \in \mathcal{F}$ with $\tilde{v} \geq 0$ q.e. on F.

Moreover, if F is compact, then each of the above conditions is equivalent to
(iii) $\mathcal{E}_\alpha(u, v) \geq 0$ for any $v \in \mathcal{F} \cap C_b(X)$ with $v \geq 0$ on F.

Proof. The implication (i) $\Rightarrow$ (ii) follows from Theorem 2.3 (iii).

We now assume that (ii) is satisfied. By Theorem 2.1, there is a $\mu \in \mathcal{S}_0$ such that $u = U_\alpha \mu$. Let K be a compact set with $K \cap F = \emptyset$. Due to Lemma 1.3, there is a sequence $\{g_n\}$ of quasi-continuous functions satisfying that

$$0 \leq g_n \leq 1, \quad g_n = 0 \text{ q.e. on } F \quad \text{and } \uparrow 1 \text{ q.e. on } K.$$

Since $\mathcal{E}_\alpha(u, g_n) = 0$, applying Theorem 2.3, we have

$$\begin{aligned} \mu(K) &\leq \liminf_{n \to \infty} \int_X g_n d\mu \\ &= \liminf_{n \to \infty} \mathcal{E}_\alpha(u, g_n) = 0. \end{aligned}$$

Thus the implication (ii) $\Rightarrow$ (i) has been shown.

Now suppose that F is compact. The implication (ii) $\Rightarrow$ (iii) is trivial. Suppose that (iii) is satisfied. Taking advantage of Theorem 2.1, we have a $\mu \in \mathcal{S}_0$ such that $u = U_\alpha \mu$. Let K be a compact set with $K \cap F = \emptyset$. Due to Lemma 1.2, there is a $g \in \mathcal{F} \cap C_b(X)$ satisfying that

$$0 \leq g \leq 1, \quad g = 0 \text{ on } F \quad \text{and } = 1 \text{ on } K.$$

Then,

$$\mu(K) \leq \int_X g d\mu = \mathcal{E}_\alpha(u, g) = 0,$$

and hence the implication (iii) $\Rightarrow$ (i) has been seen. □

Consider now a $B \in \mathcal{B}(X)$ and set

$$\mathcal{L}_B = \{u \in \mathcal{F} : \tilde{u} \geq 1 \text{ q.e. on } B\}.$$

Then $\mathcal{L}_B$ admits a unique element e_B minimizing $\mathcal{E}_1(u, u)$ on $\mathcal{L}_B$. We call e_B the equilibrium potential of B. Moreover, e_B is a unique element of $\mathcal{F}$ satisfying

$$\tilde{e}_B = 1 \qquad \text{q.e. on } B, \tag{3.1}$$

$$\mathcal{E}_1(e_B, v) \geq 0 \qquad \text{for any } v \in \mathcal{F} \text{ with } \tilde{v} \geq 0 \text{ q.e. on B.} \tag{3.2}$$

In particular, if $w \in \mathcal{F}$ and $= 1$ q.e. on B, then $\mathcal{E}_1(e_B, w) = \mathcal{E}_1(e_B, e_B)$. Applying the above lemma, we see that

$$e_B = U_1 \nu_B \qquad \text{for some } \nu_B \in \mathcal{S}_0 \text{ with } supp[\nu_B] \subset \overline{B}. \tag{3.3}$$

ν_B is called the equilibrium measure of B. It has been seen by Fukushima and Kaneko [12] that, for every $B \in \mathcal{B}(X)$,

$$Cap(B) = \mathcal{E}_1(e_B, e_B) = \inf\{\mathcal{E}_1(u, u) : u \in \mathcal{L}_B\}. \tag{3.4}$$

Using the equilibrium potential, we can show

Theorem 3.2. *For a Borel set B, the following conditions are equivalent.*
(i) $Cap(B) = 0$.
(ii) $\mu(B) = 0$ for any $\mu \in \mathcal{S}_0$.
(iii) $\mu(B) = 0$ for any $\mu \in \mathcal{S}_{00} \equiv \{\mu \in \mathcal{S}_0,\ \|U_1\mu\|_\infty < \infty\}$.

Proof. The equivalence of (i) and (ii) can be seen in the same way as [11, p.77].

Suppose that (iii) is fulfilled. Let $\mu \in \mathcal{S}_0$ and $\Gamma_n = \{U_1 \mu \leq n\}$. Choose an increasing sequence $\{K_n\}$ of compact sets such that $Cap(X \setminus K_n) \downarrow 0$. By Theorem 2.3,

$$\mu(X \setminus \cup_n K_n) = 0. \tag{3.5}$$

As in [11, p.77], if we put $\mu_n = (\mu(\Gamma_n \cap K_n)^{-1} \mathbf{I}_{\Gamma_n \cap K_n}) \cdot \mu$, then $\mu_n \in \mathcal{S}_{00}$. Moreover, by (3.5),

$$\mu(B) = \lim \mu(\Gamma \cap K_n) \mu_n(B) = 0$$

which shows the implication (iii) $\Rightarrow$ (ii). □

Consider an α-excessive function $f \in \mathcal{F}$ and an arbitrary set $B \subset X$. Define

$$\mathcal{L}_{f,B} = \{u \in \mathcal{F} : \tilde{u} \geq \tilde{f} \text{ q.e. on } B\}.$$

Then $\mathcal{L}_{f,B}$ admits a unique element f_B minimizing $\mathcal{E}_1(u,u)$ on $\mathcal{L}_{f,B}$. We call f_B the α-reduced function of f on B. Moreover, f_B is a unique element of $\mathcal{F}$ satisfying

$$\tilde{f}_B = \tilde{f} \qquad \text{q.e. on } B \tag{3.6}$$

$$\mathcal{E}_\alpha(f_B, v) \geq 0 \qquad \text{for any } v \in \mathcal{F} \text{ with } \tilde{v} \geq 0 \text{ q.e. on B.} \tag{3.7}$$

Applying Lemma 3.1, we see that

$$f_B = U_\alpha \nu \qquad \text{for some } \nu \in \mathcal{S}_0 \text{ with } \mathit{supp}[\nu] \subset \overline{B}. \tag{3.8}$$

As in [11, pp.78, 79], we have

Lemma 3.3. *Let B be a Borel set. Define*

$$\mathcal{F}_{X\setminus B} = \{u \in \mathcal{F} : \tilde{u} = 0 \textit{ q.e. on } B\}$$

and $\mathcal{H}_\alpha^B$ be its orthogonal complement in $(\mathcal{F}, \mathcal{E}_\alpha)$:

$$\mathcal{F} = \mathcal{F}_{X\setminus B} \oplus \mathcal{H}_\alpha^B.$$

Then, $f = (f - f_B) + f_B$ represents the corresponding orthogonal decomposition.

In connection with the space $\mathcal{H}_\alpha^B$, we finally prove a theorem on the spectral synthesis. An open set G is said to be an α-regular set of $u \in \mathcal{F}$ if

$$\mathcal{E}_\alpha(u, v) = 0 \qquad \text{for } v \in \mathcal{F} \text{ with } \mathit{supp}[v] \subset G.$$

Lemma 3.4. *Let $u \in \mathcal{F}$. If G_1 and G_2 are both α-regular sets of u, then so is $G_1 \cup G_2$.*

Proof. By virtue of Lemma 1.4, it suffices to show

$$\mathcal{E}_\alpha(u, v) = 0 \qquad \text{for any bounded } v \in \mathcal{F}_{cpt} \text{ with } \mathit{supp}[v] \subset G_1 \cup G_2. \tag{3.9}$$

Take a bounded $v \in \mathcal{F}_{cpt}$ with $K \equiv \mathit{supp}[v] \subset G_1 \cup G_2$. Choose open sets G_1' and G_2' such that

$$K \subset G_1' \cup G_2', \quad \text{and} \quad \overline{G_i'} \subset G_i, \ i = 1, 2.$$

We set $K_1 = K \setminus G_2'$ and $K_2 = K \setminus G_1'$. By Lemma 1.2, there is a $\phi \in \mathcal{F} \cap C_b(X)$ such that $0 \leq \phi \leq 1$ and $\phi = 1$ on K_1 and $= 0$ on K_2. Then ϕv, $(1-\phi)v \in \mathcal{F}$, $\mathit{supp}[\phi v] \subset K \cap \overline{G_1'} \subset G_1$ and $\mathit{supp}[(1-\phi)v] \subset G_2$. Hence we have

$$\mathcal{E}_\alpha(u, v) = \mathcal{E}_\alpha(u, \phi v) + \mathcal{E}_\alpha(u, (1-\phi)v) = 0,$$

which completes the proof. $\square$

We define the α-spectrum $\sigma_\alpha(u)$ as the complement of the largest α-regular open set of u.

Lemma 3.5. *Let $\mu \in \mathcal{S}_0$. Then $\sigma_\alpha(U_\alpha \mu) = supp[\mu]$.*

Proof. Let $v \in \mathcal{F}$ satisfy $supp[v] \subset X \setminus supp[\mu]$. Then $\tilde{v} = 0$ q.e. on $supp[\mu]$. By Theorem 2.3, we have

$$\mathcal{E}_\alpha(U_\alpha \mu, v) = 0$$

Thus $X \setminus supp[\mu]$ is an α-regular set of $U_\alpha \mu$ and

$$supp[\mu] \supset \sigma_\alpha(U_\alpha \mu).$$

We now suppose that $supp[\mu] \setminus \sigma_\alpha(U_\alpha \mu) \neq \emptyset$. Choose open sets G_i, $i = 1, 2$, such that $G_1 \cap supp[\mu] \neq \emptyset$, $\overline{G_1} \cap \sigma_\alpha(U_\alpha \mu) = \emptyset$, $G_2 \supset \sigma_\alpha(U_\alpha \mu)$, and $\overline{G_1} \cap \overline{G_2} = \emptyset$. By Lemma 1.3, there exists a sequence $\{\phi_n\} \subset \mathcal{F}$ such that $\phi_n = 0$ q.e. on $\overline{G_2}$ and $\to 1$ q.e. on $\overline{G_1}$. Since

$$supp[\phi_n] \subset X \setminus G_2 \subset X \setminus \sigma_\alpha(U_\alpha \mu),$$

we have

$$\begin{aligned}
\mu(G_1) &\leq \liminf_n \int_X \phi_n d\mu \\
&= \liminf_n \mathcal{E}_\alpha(\phi_n, U_\alpha \mu) \\
&= 0
\end{aligned}$$

which contradicts to that $G_1 \cap supp[\mu] \neq \emptyset$. Thus we obtain the identity

$$supp[\mu] = \sigma_\alpha(U_\alpha \mu).$$

□

Lemma 3.6. *Let $G \subset X$ be open and W_α^G be the closure of $\{u \in \mathcal{F} : \sigma_\alpha(u) \subset G\}$ in $(\mathcal{F}, \mathcal{E}_\alpha)$. Then $W_\alpha^G = \mathcal{H}_\alpha^G$.*

Proof. The inclusion $W_\alpha^G \subset \mathcal{H}_\alpha^G$ is immediate consequence of the definition of $\sigma_\alpha(u)$. The converse inclusion can be seen in exactly the same way as in [11, Proof of Lemma 3.3.4]. □

We are now prepared to repeat the argument used in [11, pp.80, 90] and obtain

Theorem 3.7. *Let F be a closed set and $W_\alpha^F = \{u \in \mathcal{F} : \sigma_\alpha(u) \subset F\}$. Then $W_\alpha^F = \mathcal{H}_\alpha^F$. In particular, each $u \in \mathcal{F}$ can be approximated in $(\mathcal{F}, \mathcal{E}_\alpha)$ by finite linear combinations of α-potentials of measures in $\mathcal{S}_0$ supported by $\sigma_\alpha(u)$.*

4. Beurling-Deny formula

The aim of this section is to establish the following Beurling-Deny formula.

Theorem 4.1. *The Dirichlet form $\mathcal{E}$ can be expressed for $u, v \in \mathcal{F}$ as follows:*

$$\begin{aligned}\mathcal{E}(u,v) &= \mathcal{E}^{(c)}(u,v) \\ &\quad + \int_{X\times X\setminus D} (\tilde{u}(x)-\tilde{u}(y))(\tilde{v}(x)-\tilde{v}(y))J(dxdy) + \int_X \tilde{u}(x)\tilde{v}(x)k(dx).\end{aligned} \tag{4.1}$$

In this expression, $\mathcal{E}^{(c)}$ is a symmetric form satisfying

$$\mathcal{E}^{(c)}(u,v) = 0 \tag{4.2}$$

for $u, v \in \mathcal{F}$ such that v =constant on a neighborhood of supp$[u]$, J is a σ-finite symmetric measure on $X \times X \setminus D$, D being the diagonal set of $X \times X$, satisfying $J(X \times A) = 0$ if $Cap(A) = 0$, and $k \in \mathcal{S}_0$.

Such $\mathcal{E}^{(c)}$, J, and k are determined uniquely by $\mathcal{E}$ and every normal contraction operates on $\mathcal{E}^{(c)}$.

The Proof will be broken into several steps, each being a lemma. In the sequel, we fix an increasing sequence $\{K_n\}$ of compact sets with $Cap(X \setminus K_n) \downarrow 0$ and a decreasing sequence $\{\varepsilon_n\}$ of positive numbers such that $\varepsilon_n \downarrow 0$. Define

$$K^{(n)} = \{(x,y) \in K_n \times K_n : d(x,y) \geq \varepsilon_n\}, \tag{4.3}$$

where d denotes the metric on X.

Lemma 4.2. *Let $t > 0$. There exists a finite symmetric Borel measure σ_t on $X \times X$ such that, for $u, v \in L^2(X; m)$, $(u \otimes v)(x,y) \equiv u(x)v(y) \in L^1(X \times X; \sigma_t)$ and*

$$\frac{1}{t}(T_t u, v)_m = \int_{X\times X} u \otimes v d\sigma_t.$$

Moreover, it holds that

$$\frac{1}{t}(u - T_t u, u)_m = \frac{1}{2}\int_{X\times X} (\tilde{u}(x) - \tilde{u}(y))^2 \sigma_t(dxdy) + \frac{1}{t}(u^2, 1 - T_t 1)_m.$$

Proof. The second assertion is an immediate consequence of the first.

To see the first assertion, set

$$\beta(A,B) = \frac{1}{t}(T_t \mathbf{I}_A, \mathbf{I}_B)_m.$$

By [9, Theorem III.74], there exists a Borel finite measure σ_t on $X \times X$ such that $\beta(A,B) = \sigma_t(A \times B)$. Since the Markov property implies that $\sigma(X \times B) \leq m(B)/t$, we obtain the desired conclusion. □

Lemma 4.3. *Let E be a separable metric space and F be its closed subset. If a sequence $\{\mu_n\}$ of finite measures on E converges weakly to a measure ν on E and if its restriction $\mu|_F$ on F does to a measure ξ on F, then $\nu(A) \geq \xi(A \cap F)$ for any Borel subset of E.*

Proof. It is easily seen that

$$\int_E f d\mu \geq \int_F f|_F d\xi \qquad \text{for } f \in C_b(E),\ f \geq 0.$$

Hence, for open $G \subset E$, we have $\mu(G) \geq \xi(G \cap F)$, which implies the desired inequality. $\square$

Lemma 4.4. *There is a symmetric Borel measure J on $X \times X \setminus D$ such that $J(X \times A) = 0$ if $Cap(A) = 0$, and*

$$\mathcal{E}(u, v) = -2 \int_{X \times X \setminus D} \tilde{u}(x)\tilde{v}(y) J(dxdy) \tag{4.4}$$

for $u, v \in \mathcal{F}$ with $supp[u] \cap supp[v] = \emptyset$.

Proof. Let $\mu_t^{(n)} = \sigma_t|_{K^{(n)}}$. We first show that

$$\sup_t \mu_t^{(n)}(K^{(n)}) < \infty. \tag{4.5}$$

To do this, let $U_n(x)$ be the $\varepsilon_n/6$-neighborhood of $x \in X$. There exist $x_i^{(n)}, y_i^{(n)} \in K_n$, $i = 1, \ldots, N$, such that

$$K^{(n)} \subset \cup_{i=1}^N U_n(x_i^{(n)}) \times U_n(y_i^{(n)}).$$

It is easily seen that $\overline{U_n(x_i^{(n)})} \cap \overline{U_n(y_i^{(n)})} = \emptyset$. By Lemma 1.2, there are $\phi_i^{(n)} \in \mathcal{F} \cap C_b(X)$, $i = 1, \ldots, N$, such that $0 \leq \phi_i^{(n)} \leq 1$, $\phi_i^{(n)} = 1$ on $K_n \cap \overline{U_n(x_i^{(n)})}$ and $= 0$ on $K_n \cap \overline{U_n(y_i^{(n)})}$. Set $f_i^{(n)} = (3\phi_i^{(n)} - 2) \vee 0$ and $g_i^{(n)} = (1 - 3\phi_i^{(n)}) \vee 0$. Since $f_i^{(n)}, g_i^{(n)} \in \mathcal{F} \cap C_b(X)$ and their supports are disjoint for $i = 1, \ldots, N$, it holds

$$\begin{aligned}
\mu_t^{(n)}(K^{(n)} \cap (\overline{U_n(x_i^{(n)})} \times \overline{U_n(y_i^{(n)})}) &\leq \int_{X \times X} f_i^{(n)} \otimes g_i^{(n)} d\sigma_t \\
&= \frac{1}{t}(T_t f_i^{(n)} - f_i^{(n)}, g_i^{(n)})_m \\
&\leq \sqrt{\frac{1}{t}(T_t f_i^{(n)} - f_i^{(n)}, f_i^{(n)})_m}\sqrt{\frac{1}{t}(T_t g_i^{(n)} - g_i^{(n)}, g_i^{(n)})_m} \\
&\leq \sqrt{\mathcal{E}(f_i^{(n)}, f_i^{(n)})}\sqrt{\mathcal{E}(g_i^{(n)}, g_i^{(n)})},
\end{aligned}$$

which implies (4.5).

Since $K^{(n)}$ is compact, for some $t_j \downarrow 0$, $\mu_{t_j}^{(n)}$ converges to a finite symmetric Borel measure $\mu^{(n)}$ on $K^{(n)}$. We next observe

$$\mu^{(n)}(K^{(n)} \cap (X \times A)) = 0 \qquad \text{if } Cap(A) = 0. \tag{4.6}$$

In fact, take a decreasing sequence $\{G_k\}$ of open sets such that $G_k \supset A$ and $Cap(G_k) \downarrow 0$. Then we have

$$\mu^{(n)}(K^{(n)} \cap (X \times G_k)) \leq \liminf_j \mu_{t_j}^{(n)}(K^{(n)} \cap (X \times G_k)). \tag{4.7}$$

On the other hand, we obtain

$$\begin{aligned}\limsup_j \mu_{t_j}^{(n)}(K^{(n)}\cap(X\times G_k)) &\leq \limsup_j \int \sum_i f_i^{(n)}\otimes(g_i^{(n)}e_{G_k})d\sigma_{t_j}\\ &= \limsup_j \sum_i \frac{1}{t_j}(T_{t_j}f_i^{(n)}-f_i^{(n)},g_i^{(n)}e_{G_k})_m\\ &= -\sum_{i=1}^N \mathcal{E}(f_i^{(n)},g_i^{(n)}e_{G_k}).\end{aligned}$$

Since $g_i^{(n)}, e_{G_k}$ are bounded, it is straightforward to see that $g_i^{(n)}e_{G_k}\to 0$ weakly in $\mathcal{F}$ as $k\to\infty$. We therefore have

$$\lim_k \limsup_j \mu_{t_j}^{(n)}(K^{(n)}\cap(X\times G_k))=0 \tag{4.8}$$

Hence, letting k tend to infinity in (4.7), we obtain (4.6).

Let $u,v\in\mathcal{F}$ be bounded. Then we may assume that their quasi-continuous versions $\tilde u,\tilde v$ are also bounded. Choose a decreasing sequence $\{G_k\}$ of open sets such that $Cap(G_k)\downarrow 0$ and $\tilde u,\tilde v$ are both continuous on each $X\setminus G_k$. For each k, by Uryson's theorem, there are $u_k,v_k\in C_b(X)$ such that $u_k=\tilde u$ and $v_k=\tilde v$ on $X\setminus G_k$ and $\|u_k\|_\infty\leq\|\tilde u\|_\infty$ and $\|v_k\|_\infty\leq\|\tilde v\|_\infty$. Then, we have

$$\begin{aligned}&|\int_{K^{(n)}}\tilde u(x)\tilde v(y)\mu_{t_j}^{(n)}(dxdy)-\int_{K^{(n)}}\tilde u(x)\tilde v(y)\mu^{(n)}(dxdy)|\\ &\quad\leq\int_{K^{(n)}}|\tilde u(x)\tilde v(y)-u_k(x)v_k(y)|\mu_{t_j}^{(n)}(dxdy)\\ &\qquad+\int_{K^{(n)}}|\tilde u(x)\tilde v(y)-u_k(x)v_k(y)|\mu^{(n)}(dxdy)\\ &\qquad+|\int_{K^{(n)}}u_k(x)v_k(y)\mu_{t_j}^{(n)}(dxdy)-\int_{K^{(n)}}u_k(x)v_k(y)\mu^{(n)}(dxdy)|\\ &\quad\leq 4\|\tilde u\|_\infty\|\tilde v\|_\infty(\mu_{t_j}^{(n)}(K^{(n)}\cap(X\times G_k))+\mu^{(n)}(K^{(n)}\cap(X\times G_k)))\\ &\qquad+|\int_{K^{(n)}}u_k(x)v_k(y)\mu_{t_j}^{(n)}(dxdy)-\int_{K^{(n)}}u_k(x)v_k(y)\mu^{(n)}(dxdy)|.\end{aligned}$$

Thus it follows from (4.7) and (4.8) that

$$\lim_j\int_{K^{(n)}}\tilde u(x)\tilde v(y)\mu_{t_j}^{(n)}(dxdy)=\int_{K^{(n)}}\tilde u(x)\tilde v(y)\mu^{(n)}(dxdy). \tag{4.9}$$

This yields that

$$\begin{aligned}\int_{K^{(n)}}\tilde u(x)\tilde v(y)\mu^{(n)}(dxdy) &= \lim_j\int_{K^{(n)}}\tilde u(x)\tilde v(y)\mu_{t_j}^{(n)}(dxdy)\\ &= \lim_j\frac{1}{t_j}(T_{t_j}u-u,v)_m\\ &= -\mathcal{E}(u,v)\end{aligned} \tag{4.10}$$

for $u,v\in\mathcal{F}$ such that $supp[u],supp[v]\subset K_n$ and $d(supp[u],supp[v])\geq\varepsilon_n$.

We now finish the construction of J. By the diagonal argument, for some $t_j\downarrow 0$, each $(1/2)\sigma_{t_j}|_{K^{(n)}}$ converges weakly to a finite symmetric Borel measure J_n on $K^{(n)}$ as $j\to\infty$.

(We will use 'J_n' for '$\frac{1}{2}\mu^{(n)}$' in the above observation.) Setting $J_n((X\times X\setminus D)\setminus K^{(n)})=0$, we may regard J_n as a measure on $X\times X\setminus D$. By Lemma 4.3, it holds

$$J_{n+1}(A)\geq J_n(A) \qquad \text{for } A\in\mathcal{B}(X\times X\setminus D).$$

We define a Borel measure on $X\times X\setminus D$ by

$$J(A)=\lim_n J_n(A).$$

It is obvious that J is symmetric and $J(X\times B)=0$ if $Cap(B)=0$.

We finally show that (4.4) holds. To do this, take $u,v\in\mathcal{F}$ with $supp[u]\cap supp[v]=\emptyset$. By Lemma 1.4, we may assume that $u,v\in\mathcal{F}_{cpt}$ and are bounded and nonnegative. Choose n_0 such that $d(supp[u],supp[v])\geq\varepsilon_{n_0}$. Let $u_n=u(1-e_{X\setminus K_n})$ and $v_n=v(1-e_{X\setminus K_n})$. Then, $\tilde{u}_n\to\tilde{u}$, $\tilde{v}_n\to\tilde{v}$ q.e. and weakly in $\mathcal{F}$, and $supp[u_n\otimes v_m]\subset K^{(n)}$ for $n,m>n_0$. Hence, by (4.10),

$$\int_{X\times X\setminus D}\tilde{u}_n\otimes\tilde{v}_m dJ=\lim_k\int_{X\times X\setminus D}\tilde{u_n}\otimes\tilde{v_m}dJ_k=-\frac{1}{2}\mathcal{E}(u_n,v_m).$$

Letting $n,m\to\infty$, we obtain (4.4). □

Lemma 4.5. *There exists a $k\in\mathcal{S}_0$ such that $\frac{1}{t}(1-T_t1)\cdot m\to k$ weakly on X and*

$$\lim_{t\downarrow 0}\frac{1}{t}(u^2,1-T_t1)_m=\int_X\tilde{u}^2dk \qquad \text{for } u\in\mathcal{F}. \tag{4.11}$$

Proof. We set

$$k_t=\frac{1}{t}(1-T_t1)\cdot m.$$

By Lemma 4.2,

$$\int_X u^2dk_t\leq\frac{1}{t}(u-T_tu,u)_m\uparrow\mathcal{E}(u,u) \qquad \text{as } t\downarrow 0 \text{ for } u\in\mathcal{F}. \tag{4.12}$$

It follows from (4.12) with 1 and $e_{X\setminus K_n}$ substituted for u that $\sup_t k_t(X)<\infty$ and $\{k_t\}$ is tight. If k_0 is the limit of a converging subsequence $\{k_{t_j}\}$, then

$$\int_X f dk_0=\lim_j\frac{1}{t_j}(f,1-T_{t_j}1)_m=\mathcal{E}(f,1)$$

for every $f\in\mathcal{F}\cap C_b(X)$. Thus, by Lemma 1.2, there exists a unique finite Borel measure k to which k_t converges weakly as $t\downarrow 0$. By (4.12), it holds

$$\int_X u^2dk\leq\mathcal{E}(u,u) \qquad \text{for } u\in\mathcal{F}\cap C_b(X). \tag{4.13}$$

Hence $k\in\mathcal{S}_0$.

Let $u\in\mathcal{F}$. There exists a sequence $\{u_{nm}\}\subset\mathcal{F}\cap C_b(X)$ such that $u_{nm}\to\tilde{u}_n$ q.e. and in $\mathcal{F}$ as $m\to\infty$, where $u_n=(-n)\vee(u\wedge n)$. By Theorem 2.3 and (4.13), we obtain $\int_X\tilde{u}_n^2dk\leq\mathcal{E}(u_n,u_n)$. Letting $n\to\infty$,

$$\int_X\tilde{u}^2dk\leq\mathcal{E}(u,u) \qquad u\in\mathcal{F}. \tag{4.14}$$

It is straightforward to see that, for every $v \in \mathcal{F} \cap C_b(X)$,

$$\begin{aligned}&\left|\|u\|_{L^2(X;k_t)} - \|\tilde{u}\|_{L^2(X;k)}\right| \\ &\qquad \leq \|u-v\|_{L^2(X;k_t)} + \|u-v\|_{L^2(X;k)} + \left|\|v\|_{L^2(X;k_t)} - \|v\|_{L^2(X;k)}\right|\end{aligned}$$

Combining this with (4.12) and (4.14), we obtain (4.11). □

Lemma 4.6. *There exists a symmetric form $\mathcal{E}^{(c)}$ as stated in Theorem 4.1.*

Proof. Let $u \in \mathcal{F}$ be bounded. By Lemmas 4.2, 4.5 and (4.9), we have

$$\begin{aligned}\mathcal{E}(u,u) = \lim_j \frac{1}{2}\int_{X\times X\setminus K^{(n)}} (\tilde{u}(x)-\tilde{u}(y))^2 \sigma_{t_j}(dxdy) \\ + \int_{X\times X\setminus D} (\tilde{u}(x)-\tilde{u}(y))^2 J_n(dxdy) + \int_X \tilde{u}^2(x)k(dx).\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}\mathcal{E}(u,u) = \lim_n \lim_j \frac{1}{2}\int_{X\times X\setminus K^{(n)}} (\tilde{u}(x)-\tilde{u}(y))^2 \sigma_{t_j}(dxdy) \\ + \int_{X\times X\setminus D} (\tilde{u}(x)-\tilde{u}(y))^2 J(dxdy) + \int_X \tilde{u}^2(x)k(dx).\end{aligned}$$

By the monotone convergence theorem, for every $u \in \mathcal{F}$, it holds that

$$\begin{aligned}\mathcal{E}(u,u) = \lim_M \lim_n \lim_j \frac{1}{2}\int_{X\times X\setminus K^{(n)}} (\tilde{u}_M(x)-\tilde{u}_M(y))^2 \sigma_{t_j}(dxdy) \\ + \int_{X\times X\setminus D} (\tilde{u}(x)-\tilde{u}(y))^2 J(dxdy) + \int_X \tilde{u}^2(x)k(dx),\end{aligned}$$

where $u_M = (-M) \vee (u \wedge M)$. Thus, by setting

$$\mathcal{E}^{(c)}(u,v) = \lim_M \lim_n \lim_j \frac{1}{2}\int_{X\times X\setminus K^{(n)}} (\tilde{u}_M(x)-\tilde{u}_M(y))(\tilde{v}_M(x)-\tilde{v}_M(y))\sigma_{t_j}(dxdy),$$

we obtain a symmetric form $\mathcal{E}^{(c)}$ satisfying (4.1) and (4.2). Moreover, every normal contraction operates on this $\mathcal{E}^{(c)}$. □

Lemma 4.7. *J is σ-finite.*

Proof. As was seen in the proof of Lemma 4.4, for each n, there are $f_i^{(n)}, g_i^{(n)} \in \mathcal{F} \cap C_b(X)$ and $N = N(n) \in \mathbf{N}$ such that $\mathbf{I}_{K^{(n)}} \leq \sum_{i=1}^N f_i^{(n)} \otimes g_i^{(n)}$. Hence

$$J(K^{(n)}) \leq \sum_{i=1}^N \int_X f_i^{(n)} \otimes g_i^{(n)} dJ = -2\sum_{i=1}^N \mathcal{E}(f_i^{(n)}, g_i^{(n)}) < \infty.$$

Obviously $J(X \times X \setminus \cup_n K^{(n)}) = 0$. Thus, J is σ-finite. □

Lemma 4.8. *The decomposition (4.1) is unique.*

Proof. Since $\int_X udk = \mathcal{E}(u, 1)$ for $u \in \mathcal{F} \cap C_b(X)$, the uniqueness of k follows from Lemma 1.2.

Note that

$$\int_{X \times X \setminus D} \tilde{u} \otimes \tilde{v} dJ = -2\mathcal{E}(u, v) \tag{4.15}$$

for $u, v \in \mathcal{F}$ with $supp[u] \cap supp[v] = \emptyset$. Let K be a compact set in X and $\delta > 0$. If we set $K(\delta) = \{(x, y) \in K \times K : d(x, y) \geq \delta\}$, then by using (4.15), Lemma 1.2, and the argument similar to that in the proof of Lemma 4.4, we can conclude the uniqueness of J on $K(\delta)$. Since every compact set in $X \times X \setminus D$ is covered by such $K(\delta)$ and X is Lusinian, we have the uniqueness of J. □

In the case of locally compact state spaces, two approaches to the Beurling-Deny formula are known ; one is analytic and the other is probabilistic. The above approach that we employed for the Dirichlet form on a general metric space (which may not be locally compact) is the analytic one. As will be seen in Section 7, the stochastic calculus related to the Dirichlet form can be developed in the present situation. Then, by repeating the argument in [14], we can establish the Beurling-Deny formula in the probabilistic way.

5. Hunt processes

We have been studying Dirichlet forms from the analytic point of view. We now proceed to the probabilistic investigation of them.

We continue to assume the hypotheses (A.1)–(A.3). Then, combining [11, Chapter6] and [2], we can conclude

Theorem 5.1. *There exists a Hunt process* $\mathbf{M} = (\Omega, \mathcal{M}, X_t, P_x)$ *on* X *associated with* $(\mathcal{E}, \mathcal{F})$*: for any Borel measurable, bounded* $u : X \to \mathbf{R}$ *and* $t > 0$,

$$(T_t u)(x) = E_x[u(X_t)] \qquad \text{for } m\text{-a.e. } x \in X, \tag{5.1}$$

where E_x *stands for the expectation with respect to* P_x.

Also see [17]. Thus we have a symmetric Hunt process $\mathbf{M}$. Its transition function is denoted by $\{p_t, t > 0\}$ and the resolvent $\{R_\alpha, \alpha > 0\}$ of $\mathbf{M}$ is defined by

$$R_\alpha(x, E) = \int_0^\infty e^{-\alpha t} p_t(x, E) dt.$$

We now recall several notions of smallness of sets related to $\mathbf{M}$. A point x is said to be a regular point of a nearly Borel set B if $P_x(\sigma_B > 0) = 0$, where

$$\sigma_B = \inf\{t > 0 : X_t \in B\}. \tag{5.2}$$

The totality of the *regular points* of B is denoted by B^r. A set A is said to be *finely open* if the set $X \setminus A$ is thin at each $x \in A$, i.e., there is a nearly Borel set $B = B(x)$ such that $x \notin B^r$ and $B \supset X \setminus A$. We say a set A is *thin* if it is contained in a nearly Borel set B with $B^r = \emptyset$. A is said to be *semi-polar* if it is contained in a countable union of thin sets. If $A \subset B$ for a nearly Borel set B such that $P_x(\sigma_B < \infty) = 0$ for any $x \in X$, then A is called *polar*. A set N is called *exceptional* if there is a Borel set $\widetilde{N} \supset N$ such that $P_m(\sigma_{\widetilde{N}} < \infty) = 0$, where $P_m(\cdot) = \int_X P_x(\cdot) m(dx)$. We say that a set N is *properly exceptional* if it is Borel and $X \setminus N$ is $\mathbf{M}$-invariant: $P_x(X_t, X_{t-} \in (X \setminus N)_\Delta,\ t \geq 0) = 1$ for any $x \in X \setminus N$, where Δ is the death point of $\mathbf{M}$ that is joined as an isolated point.

Taking advantage of recent general results in the potential theory on Markov processes (right processes), we can recover the assertions in [11, §4.2] on the relationship among the above notions of smallness. All results can be found in [15], however, for the sake of completeness, we summarily state as a theorem:

Theorem 5.2. *(i) A nearly Borel, finely open, m-negligible set is exceptional.*
(ii) A set is exceptional if and only if it is contained in a properly exceptional set.
(iii) Any semi-polar set is exceptional.
(vi) Let $\{u_n\}$ be a decreasing sequence of α-excessive function with respect to p_t:$u_n \geq 0$ and $e^{-\alpha t} p_t u_n \leq u_n$ on X. If $\lim_n u_n = 0$ m-a.e., then $u = 0$ except for an exceptional set.

We now identify the above notions with those analytic ones in the proceeding sections.

Theorem 5.3. *A set N is exceptional if and only if $Cap(N) = 0$.*

Proof. It suffices to notice that we can choose a decreasing sequence $\{A_n\}$ of open sets for relatively compact open sets in [11, Proof of Theorem 4.3.1], because $Cap(X) < \infty$. □

We now investigate the quasi-continuity. As usual, every Borel measurable function $f : X \to \mathbf{R}$ is extended to $X_\Delta = X \cup \{\Delta\}$ by setting $f(\Delta) = 0$.

Theorem 5.4. *(i) If u is quasi-continuous, then there is a properly exceptional set N such that u is Borel measurable on $X \setminus N$ and*

$$P_x(u(X_t) \text{ is right continuous and } \lim_{s \uparrow t} u(X_s) = u(X_{t-}) \text{ for any } t \geq 0) = 1,$$

for any $x \in X \setminus N$.
(ii) Let $u \in \mathcal{F}$. Suppose that there is a nearly Borel exceptional set N such that $X \setminus N$ is finely open and u is nearly Borel and finely continuous on $X \setminus N$. Then, u is quasi-continuous.

Proof. To show (i), we follow [11, Proof of Theorem 4.3.2].

Let $\{A_n\}$ be a decreasing sequence of open sets such that $Cap(A_n) \downarrow 0$ and u is continuous on each $X \setminus A_n$. By virtue of Assumption (A.3), we may assume that each $X \setminus A_n$ is compact.

Using Theorem 5.2 (iv), as in [11, Proof of Theorem 4.3.2], we can show that there exists a properly exceptional set N such that

$$P_x(\lim_n \sigma_{A_n} = \infty) = 1 \qquad x \in X \setminus N.$$

Then the assertion follows from the continuity of u on each $X \setminus A_n$ and the compactness of $X \setminus A_n$.

The assertion (ii) can be seen in the same way as in [11, Proof of Theorem 4.3.2]. □

By this observation, we can make clear the relationship between $\{T_t\}$ (resp. $\{G_\alpha\}$) and $\{p_t\}$ (resp. $\{R_\alpha\}$). Indeed, repeating the argument in [11, Proof of Theorem 4.3.3] with $C_b(X)$ for $C_0(X)$, we obtain

Theorem 5.5. *For any nonnegative universally measurable function $u \in L^2(X;m)$,*
(i) $p_t u$ is a quasi-continuous version of $T_t u$, $t > 0$,
(ii) $R_\alpha u$ is a quasi-continuous version of $G_\alpha u$, $\alpha > 0$.

Furthermore, the argument in [11, pp.106-110] also works in our situation and bears:

Theorem 5.6. *Let $B \in \mathcal{B}(X)$. Define*

$$p_B^1(x) = E_x[e^{-\sigma_B}], \quad \dot{p}_B^1(x) = E_x[e^{-\dot{\sigma}_B}], \quad \text{and} \quad H_\alpha^B u(x) = E_x[e^{-\alpha\sigma_B} u(X_{\sigma_B})],$$

where $\dot{\sigma}_B = \inf\{t \geq 0 : X_t \in B\}$. Then
(i) p_B^1 and $\dot{p}_B^1$ are both quasi-continuous version of e_B,
(ii) for $u \in \mathcal{F}$, $H_\alpha^B \tilde{u}$ is a quasi-continuous version of $\mathcal{P}_{\mathcal{H}_\alpha^B} u$, where $\mathcal{P}_{\mathcal{H}_\alpha^B}$ the orthogonal projection of $\mathcal{F}$ onto $\mathcal{H}_\alpha^B$. (For the definition of $\mathcal{H}_\alpha^B$, see Section 3.)

6. Continuity, killing, and jumps of sample paths

As in the previous sections, we consider the Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfying (A.1)–(A.3) and the associated Hunt process $\mathbf{M}$. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is said to possess the local property if $\mathcal{E}(u, v) = 0$ for $u, v \in \mathcal{F}$ such that $supp[u] \cap supp[v] = \emptyset$. We have

Theorem 6.1. *The following conditions are equivalent to each other.*
(i) $(\mathcal{E}, \mathcal{F})$ possesses the local property.
(ii) For any open set $G \subset X$, the hitting distribution

$$H_\alpha^{X\setminus G}(x, dy) = E_x[e^{-\alpha\sigma_{X\setminus G}}; X_{\sigma_{X\setminus G}} \in dy]$$

is concentrated on the boundary ∂G, q.e. $x \in G$.
(iii) There exists a properly exceptional set N such that

$$P_x(X_t \text{ is continuous in } t \in [0, \zeta)) = 1 \qquad x \in X \setminus N, \tag{6.1}$$

where ζ is the life time of $\mathbf{M}$.

Proof. We first show the implication (i) ⇒ (ii). Let $G \subset X$ be open. We set

$$G_n = \{x \in X : d(x, G) < 1/n\} \qquad \text{and} \qquad F_n = X \setminus G_n$$

By Lemma 1.3, for each n, there exists a sequence $\{u_{nk}\} \subset \mathcal{F}_{cpt}$ such that $0 \leq \tilde{u}_{nk} \leq 1$ q.e., $\tilde{u}_{nk} = 0$ q.e. on G_{2n}, and $\tilde{u}_{nk} \to 1$ q.e. on F_n as $k \to \infty$.

Let $\mathcal{P}$ be the orthogonal projection of $\mathcal{F}$ onto $\mathcal{F}_G \equiv \{u \in \mathcal{F} : \tilde{u} = 0 \text{ q.e. on } X \setminus G\}$, where $\mathcal{F}$ is thought of as the Hilbert space equipped with the inner product $\mathcal{E}_\alpha$. Since $supp[u_{nk}] \subset X \setminus G_{2n}$, it follows from the local property that

$$\mathcal{E}_\alpha(\mathcal{P}u_{nk}, v) = \mathcal{E}_\alpha(u_{nk}, \mathcal{P}v) = \mathcal{E}_\alpha(u_{nk}, v) = 0 \qquad v \in \mathcal{F}_G.$$

Hence $\mathcal{P}u_{nk} = 0$ q.e. on X. Applying Theorem 5.6 (ii), we have

$$H_\alpha^{X\setminus G}\tilde{u}_{nk}(x) = 0 \qquad \text{for q.e. } x \in G.$$

By Theorems 5.2 and 5.3, there is a properly exceptional set N such that $\tilde{u}_{nk} = 0$ on $G_{2n} \setminus N$, $\tilde{u}_{nk} \to 1$ on $F_n \setminus N$, and $H_\alpha^{X\setminus G}\tilde{u}_{nk} = 0$ on $G \setminus N$. Then we have

$$H_\alpha^{X\setminus G}\mathbf{I}_{F_n}(x) \leq \lim_k H_\alpha^{X\setminus G}u_{nk}(x) = 0 \qquad x \in G \setminus N.$$

Now letting $n \to \infty$, we obtain

$$H_\alpha^{X\setminus G}\mathbf{I}_{X\setminus \overline{G}}(x) = 0 \qquad x \in G \setminus N.$$

Thus (ii) follows from (i).

We next assume that (ii) holds. Let $u, v \in \mathcal{F}$ satisfy $supp[u] \cap supp[v] = \emptyset$. Without loss of generality, we may assume that u is nonnegative. We set $G = \{x \in X : d(x, supp[u]) > d(x, supp[v])\}$. Obviously, G is open, contains $supp[v]$, and $supp[u] \cap \overline{G} = \emptyset$. Since $\mathcal{P}v = v$,

$$\mathcal{E}(u, v) = \mathcal{E}_\alpha(u, v) = \mathcal{E}_\alpha(\mathcal{P}u, v). \tag{6.2}$$

On the other hand, it follows from Theorem 5.6 that

$$\mathcal{P}u(x) = \tilde{u}(x) - \int_{\partial G} \tilde{u}(y) H_\alpha^{X\setminus G}(x, dy) = 0 \qquad \text{q.e. } x \in G.$$

Hence $\mathcal{P}u = 0$ q.e. on X. Combining with (6.2), we have $\mathcal{E}(u, v) = 0$. Thus (i) follows.

The implication (iii) $\Rightarrow$ (ii) is trivial and the converse implication (ii) $\Rightarrow$ (iii) can be seen in the same way as in [11, p.114]. $\square$

We now proceed to the probabilistic interpretation of the measures k and J obtained in Theorem 4.1. In the following, we occasionally denote the integral of a function v with respect to a positive measure μ by $<\mu, v>$ or $<v, \mu>$. Moreover, $E_{h\cdot m}$ stands for the integration with respect to the measure $\int_X h(x)P_x(\cdot)m(dx)$. As [11, Lemma 4.5.2], we have

Proposition 6.2. *(i) For any nonnegative Borel measurable $f, h : X \to \mathbf{R}$, and $t > 0$,*

$$E_{h\cdot m}[f(X_{\zeta-}); \zeta \leq t] = \int_0^t <fk, p_s h> ds. \tag{6.3}$$

(ii) For $\alpha > 0$ and nonnegative $f \in C_b(X)$, $E_x[e^{-\alpha\zeta} f(X_{\zeta-})]$ is a quasi-continuous version of the potential $U_\alpha(f \cdot k)$.

We now investigate J. Let G be an arbitrary but fixed open set in X. Define the kernel $R_\alpha^G(x, E)$ by

$$R_\alpha^G(x, E) = E_x[\int_0^{\tau_G} \mathbf{I}_E(X_t) dt],$$

where $\tau_G = \sigma_{X\setminus G}$. It is known [15] that $\{R_\alpha^G, \alpha > 0\}$ is an m-symmetric kernel and Dynkin's formula holds;

$$R_\alpha f = R_\alpha^G f + H_\alpha^G R_\alpha f \qquad \text{for nonnegative Borel } f,$$

where $\{R_\alpha\}$ is the resolvent kernel of $\mathbf{M}$. Combining this with Theorems 5.5 and 5.6, we have

$$\mathcal{E}_\alpha(R_\alpha^G f, v) = (f, v)_m \qquad \text{for any } v \in \mathcal{F}_G. \tag{6.4}$$

Let $\mathbf{M}_G$ be the part of $\mathbf{M}$ on G, i.e., $\mathbf{M}_G = (X_t^G, P_x)_{x\in G}$, where $X_t^G = X_t$ for $t < \tau_G$ and $= \Delta$ for $t \geq \tau_G$. Obviously $\{R_\alpha^G\}$ is the resolvent kernel of $\mathbf{M}_G$. Hence it follows from (6.4) that

Lemma 6.3. *Define the part $(\mathcal{E}_G, \mathcal{F}_G)$ of $(\mathcal{E}, \mathcal{F})$ on G by*

$$\mathcal{E}_G(u, v) = \mathcal{E}(u, v) \qquad u, v \in \mathcal{F}_G.$$

Then $\mathbf{M}_G$ is associated with the Dirichlet form $(\mathcal{E}_G, \mathcal{F}_G)$.

Unfortunately the Dirichlet form $\mathcal{E}_G$ does not satisfy Assumptions (A.1)–(A.3). However, this form plays a key role in the investigation of J.

Take a $v \in \mathcal{F}$ such that $supp[v] \cap \overline{G} = \emptyset$ and define

$$J_v(dx) = 2I_G(x) \int \tilde{v}(y) J(dxdy).$$

For every $u \in \mathcal{F}_G$, it holds that

$$\int_G |\tilde{u}| dJ_v = -\mathcal{E}(|u|, v) \leq \mathcal{E}(v, v)^{1/2} \mathcal{E}(u, u)^{1/2}.$$

Since $\mathcal{E}_G = \mathcal{E}$ on $\mathcal{F}_G$, there exists a $U_\alpha^G J_v \in \mathcal{F}_G$ such that

$$\int_G \tilde{u} dJ_v = \mathcal{E}_{G,\alpha}(U_\alpha^G J_v, u) = \mathcal{E}_\alpha(U_\alpha^G J_v, u) \qquad u \in \mathcal{F}_G.$$

By Theorems 4.1 and 5.6, we obtain

$$\mathcal{E}_\alpha(v - H_\alpha^{X\setminus G} v, u) = -\mathcal{E}_\alpha(U_\alpha^G J_v, u) \qquad u \in \mathcal{F}_G,$$

and hence $v - H_\alpha^{X\setminus G} v = -U_\alpha^G J_v$. In particular, for any nonnegative Borel h vanishing outside of G,

$$E_{h\cdot m}[e^{-\alpha\tau_G} f(X_{\tau_G})] = 2 \int R_\alpha^G h \otimes v dJ. \tag{6.5}$$

This formula is strengthened as follows.

Proposition 6.4.
(i) For any bounded Borel measurable functions $f, g, h \geq 0$ such that $supp[f], supp[h] \subset G$ and $supp[g] \cap \overline{G} = \emptyset$,

$$E_{h\cdot m}[f(X_{\tau_G-})g(X_{\tau_G}); \tau_G \leq t] = 2 \int_0^t \left[\int p_s^G h(x) f(x) g(y) J(dxdy) \right] ds,$$

where $\{p_t^G\}$ is the transition function of $\mathbf{M}_G$.
(ii) $E_x[e^{-\alpha\tau_G} f(X_{\tau_G-})g(X_{\tau_G})]$ on G is a quasi-continuous version of $U_\alpha^G(fJ_g)$ for $\alpha > 0$, $f, g \in \mathcal{F}$ with $supp[f] \subset G$ and $supp[g] \cap \overline{G} = \emptyset$.

Proof. (ii) is a consequence of (i) and Theorem 5.4.

To see (i), first suppose that $f, g \in \mathcal{F}$ and are bounded, and $h = R_\alpha^G h'$ for some bounded $h' \in \mathcal{F}_G$. Then, as in [11, p.119], we obtain

$$\begin{aligned} E_{h\cdot m}[f(X_{\tau_G -})g(X_{\tau_G}); \tau_G \le t] &= \lim_n \frac{n}{t} \int_0^t (p_s^G h, f p_{t/n} g)_m ds \\ &= \int_0^t \lim_n (f p_s^G h, p_{t/n} g - g)_m ds \\ &= -\int_0^t \mathcal{E}(f p_s^G h, g) ds \\ &= 2 \int_0^t \left[\int (f p_s^G h) \otimes g dJ \right] ds. \end{aligned}$$

The assertion for general f, g, h, follows by applying the monotone class theorem. □

7. Stochastic calculus

Let X, m, $(\mathcal{E}, \mathcal{F})$, and $\mathbf{M} = (\Omega, \mathcal{M}, X_t, P_x)$ be as before. A real valued function $A_t(\omega)$, $t > 0$, $\omega \in \Omega$, is called an additive functional (abbreviated to AF) if it is a perfect additive functional in the ordinary sense but with respect to the restricted Hunt process $\mathbf{M}_{X \setminus N}$, N being a properly exceptional set which depends on A in general. For details, see [11, §5.1]. Two AF's A^1 and A^2 are said to be equivalent if for each $t > 0$ $P_x(A_t^1 = A_t^2) = 1$ q.e. $x \in X$. A positive continuous AF (PCAF in abbreviation) means a nonnegative continuous AF and the totality of all PCAF's is denoted by $\mathcal{A}_c^+$.

As in [11, §5.1], we can establish

Theorem 7.1. *The family of all equivalent classes of $\mathcal{A}_c^+$ and $\mathcal{S}$ are in one to one correspondence specified by the following relation:*

$$\lim_{t \downarrow 0} \frac{1}{t} E_{h\cdot m}[(f \cdot A)_t] = < f \cdot \mu, h >, \qquad A \in \mathcal{A}_c^+,\ \mu \in \mathcal{S},$$

for any γ-excessive $h (\gamma > 0)$ and Borel measurable $f \ge 0$, where $(f \cdot A)_t = \int_0^t f(X_s) dA_s$.

Moreover, if $\mu \in \mathcal{S}_0$, then $U_A^1(f \cdot \mu) = \int_0^\infty e^{-t} f(X_t) dA_t$ is a quasi-continuous version of the 1-potential $U_1(f \cdot \mu)$.

Proof. In the argument in [11, §5.1] in order to see the above relationship, the locally compactness of the state space is used only in the proof of Lemma 5.1.6. So what we have to do is to see that the assertion of [11, Lemma 5.1.6] holds in our situation, i.e. to show the equivalence of the following three conditions for an increasing sequence $\{F_n\}$ of closed sets:

(i) $Cap(K \setminus F_n) \downarrow 0$ for every compact $K \subset X$,

(ii) $Cap(X \setminus F_n) \downarrow 0$,

(iii) $P_x(\lim_n \sigma_{X \setminus F_n} < \zeta)$=0 q.e. $x \in X$.

The implication (i) $\Rightarrow$ (ii) is a consequence of Assumption (A.3) and the converse implication (ii) $\Rightarrow$ (i) is trivial.

To see the equivalence of (ii) and (iii), recall that $p_n(x) \equiv E_x[e^{-\sigma_{X\setminus F_n}}]$ is a quasi-continuous version of $e_{X\setminus F_n}$. Hence $p_n \to 0$ q.e. if $Cap(X \setminus F_n) \downarrow 0$, which means the implication (ii) $\Rightarrow$ (iii) holds. By the bounded convergence theorem, $p_n \to 0$ q.e. if (iii) holds. On the other hand, since $\mathcal{E}_1(p_n - p_m, p_n - p_m) = |Cap(X \setminus F_n) - Cap(X \setminus F_m)|$, $\{p_n\}$ is a Cauchy sequence in $\mathcal{F}$. Thus the implication (iii) $\Rightarrow$ (ii) is verified. $\square$

For AF's A_t, B_t, we set

$$e(A, B) = \lim_{t\downarrow 0} \frac{1}{2t} E_m[A_t B_t], \qquad e(A) = e(A, A),$$

and, for $u \in \mathcal{F}$, we define an AF $A^{[u]}$ by

$$A_t^{[u]} = \tilde{u}(X_t) - \tilde{u}(X_0).$$

Then, by Lemma 4.5, we have

$$e(A^{[u]}) = \mathcal{E}(u, u) - \frac{1}{2} \int_X \tilde{u}^2 dk.$$

We now consider the space $\mathcal{M}$ of AF's M with $E_x[M_t] = 0$ and $E_x[M_t^2] < \infty$ q.e., $t > 0$, $\overset{\circ}{\mathcal{M}} \equiv \{M \in \mathcal{M} : e(M) < \infty\}$, and the space $\mathcal{N}_c$ of continuous AF's N such that $e(N) = 0$ and $E_x[|N_t|] < \infty$ q.e., $t > 0$. Every $M \in \overset{\circ}{\mathcal{M}}$ determines a unique $<M> \in \mathcal{A}_c^+$ such that $E_x[<M>_t] = E_x[M_t^2]$ q.e., $t > 0$. As in [11, §5.2, 5.3], we obtain the following assertions.

(AF.i) $\overset{\circ}{\mathcal{M}}$ is a real Hilbert space with inner product $e(\cdot, \cdot)$.

(AF.ii) For any Cauchy sequence $\{M^{(n)}\} \subset \overset{\circ}{\mathcal{M}}$, there is a unique $M \in \overset{\circ}{\mathcal{M}}$ and a subsequence $M^{(n_k)}$ such that $e(M^{(n)} - M) \to 0$ and for q.e. $x \in X$, $P_x(\lim_k M_t^{(n_k)} = M_t$ uniformly on any finite interval of $t) = 1$.

(AF.iii) For each $u \in \mathcal{F}$, there is a unique $(M^{[u]}, N^{[u]}) \in \overset{\circ}{\mathcal{M}} \times \mathcal{N}_c$ such that $A^{[u]} = M^{[u]} + N^{[u]}$, and it holds

$$e(M^{[u]}) = \mathcal{E}(u, u) - \frac{1}{2} \int_X \tilde{u}^2 dk.$$

(AF.iv) Let $u \in \mathcal{F}_b (\equiv \{w \in \mathcal{F} : w \text{ is bounded}\})$. If we denote by $\mu_{<u>}$ the smooth measure associated with $M^{[u]}$, then it holds that

$$\int_X \tilde{f} d\mu_{<u>} = 2\mathcal{E}(uf, u) - \mathcal{E}(u^2, f)$$

for any $f \in \mathcal{F}_b$.

(AF.v) For AF A and $u \in \mathcal{F}$, the following three conditions are equivalent:

(a) $A = N^{[u]}$,

(b) $A \in \mathcal{N}_c$ and $E_x[A_t] = p_t\tilde{u}(x) - \tilde{u}(x)$ q.e. $x \in X$, $t > 0$,

(c) $A \in \mathcal{N}_c$, $\lim_{t\downarrow 0} E_x[A_t] = 0$ q.e., and $\lim_{t\downarrow 0} E_{v\cdot m}[A_t] = -\mathcal{E}(u, v)$ for $v \in \mathcal{F}$.

(AF.vi) The following two conditions are equivalent to each other for $u \in \mathcal{F}$:

(i) $N^{[u]}$ is a continuous AF of bounded variation,

(ii) there exist smooth measures ν^1 and ν^2 such that $\mathcal{E}(u,v) =< (\nu^1 - \nu^2)|_{F_k}, \tilde{v} >$ for any $v \in \mathcal{F}_{F_k}$, $k = 1, 2, \ldots$, where $\{F_k\}$ is an increasing sequence of closed set with $Cap(X \setminus F_k) \downarrow 0$ and $\mathbf{I}_{F_k} \cdot \nu^i \in \mathcal{S}_0$, $i = 1, 2$, $k = 1, 2, \ldots$.

In the remainder of this section, we see that the stochastic calculus related to the Dirichlet form discussed in [11, §5.4] remains valid in our situation. In the sequel, $\mathcal{E}^{(c)}$, J, and k denote the ones appearing in the Beurling-Deny formula (Theorem 4.1).

For $u, v \in \mathcal{F}$, we define

$$\mu_{<u,v>} = \frac{1}{2}(\mu_{<u+v>} - \mu_{<u>} - \mu_{<u>}).$$

It is easily seen that, for $u, v, f \in \mathcal{F}_b$,

$$\begin{aligned} \int_X \tilde{f} d\mu_{<u,v>} &= \mathcal{E}(uf, v) - \mathcal{E}(vf, u) - \mathcal{E}(uv, f) \\ &= \lim_{t\downarrow 0} \frac{1}{t} \int_{X\times X\setminus D} (\tilde{u}(y) - \tilde{u}(x))(\tilde{v}(y) - \tilde{v}(x))\tilde{f}(x) p_t(x, dy) m(dx) + <\tilde{u}\tilde{v}\tilde{f}, k> . \end{aligned} \tag{7.1}$$

Then we have

Lemma 7.2. *For $u, v, f \in \mathcal{F}_b$, it holds that*

$$\begin{aligned} &\int_X \tilde{f} d\mu_{<u^2,v>} - 2\int_X \tilde{f}\tilde{u} d\mu_{<u,v>} \\ &\quad = 2\int_{X\times X\setminus D} (\tilde{u}(y) - \tilde{u}(x))^2 (\tilde{v}(y) - \tilde{v}(x))\tilde{f}(x) J(dxdy) - <\tilde{u}^2\tilde{v}\tilde{f}, k> . \end{aligned} \tag{7.2}$$

Proof. By (7.1), we have

$$\text{LHS of (7.2)} = \lim_{t\downarrow 0} \frac{1}{t} \int_{X\times X} (\tilde{u}(y) - \tilde{u}(x))^2 (\tilde{v}(y) - \tilde{v}(x))\tilde{f}(x) p_t(x, dy) m(dx) - <\tilde{u}^2\tilde{v}\tilde{f}, k> .$$

Choose an increasing sequence $\{K_n\}$ of compact sets such that $Cap(X \setminus K_n) \downarrow 0$ and $\tilde{v}$ is continuous on each K_n. Then there is a decreasing sequence $\{\varepsilon_n\}$ of positive numbers such that $\varepsilon_n \downarrow 0$ and

$$|\tilde{v}(y) - \tilde{v}(x)| \leq \frac{1}{n} \qquad \text{if } x, y \in K_n \text{ and } d(x, y) < \varepsilon_n.$$

Now we set

$$\begin{aligned} &K^{(n)} = \{(x, y) \in K_n \times K_n : d(x, y) \geq \varepsilon_n\} \\ &G_n = X \setminus K_n. \end{aligned}$$

As was seen in the proof of Lemma 4.4, for some decreasing sequence $\{t_j\}$ of positive numbers with $t_j \downarrow 0$, each $(1/2t_j) p_{t_j}(x, dy) m(dx)|_{K^{(n)}}$ converges weakly to a Borel measure

J_n on $K^{(n)}$ and $\lim_n J_n = J$. We then decompose as

$$\begin{aligned}
&\frac{1}{t}\int_{X\times X}(\tilde{u}(y)-\tilde{u}(x))^2(\tilde{v}(y)-\tilde{v}(x))\tilde{f}(x)p_t(x,dy)m(dx) = \mathrm{I}_n(t)+\mathrm{II}_n(t)+\mathrm{III}_n(t),\\
&\mathrm{I}_n(t) = \frac{1}{t}\int_{K^{(n)}}(\tilde{u}(y)-\tilde{u}(x))^2(\tilde{v}(y)-\tilde{v}(x))\tilde{f}(x)p_t(x,dy)m(dx),\\
&\mathrm{II}_n(t) = \frac{1}{t}\int_{K_n\times K_n\setminus K^{(n)}}(\tilde{u}(y)-\tilde{u}(x))^2(\tilde{v}(y)-\tilde{v}(x))\tilde{f}(x)p_t(x,dy)m(dx),\\
&\mathrm{III}_n(t) = \frac{1}{t}\int_{X\times X\setminus K_n\times K_n}(\tilde{u}(y)-\tilde{u}(x))^2(\tilde{v}(y)-\tilde{v}(x))\tilde{f}(x)p_t(x,dy)m(dx).
\end{aligned}$$

As in the proof of Lemma 4.4, we obtain

$$\lim_{n\to\infty}\lim_{j\to\infty}\mathrm{I}_n(t_j) = 2\int_{X\times X\setminus D}(\tilde{u}(y)-\tilde{u}(x))^2(\tilde{v}(y)-\tilde{v}(x))\tilde{f}(x)J(dxdy).$$

Let $\nu_t(dxdy) = (1/t)(\tilde{u}(y)-\tilde{u}(x))^2p_t(x,dy)m(dx)$. It then follows that

$$\begin{aligned}
|\mathrm{II}_n(t)| &\le \frac{1}{n}\|f\|_\infty\nu_t(X\times X)\\
|\mathrm{III}_n(t)| &\le 2\|v\|_\infty\|f\|_\infty\nu_t(X\times G_n).
\end{aligned}$$

By Lemma 4.2, $\nu_t(X\times X)\le 2\mathcal{E}(u,u)$ and hence

$$\lim_{n\to\infty}\lim_{t\downarrow 0}\mathrm{II}_n(t) = 0.$$

Moreover we have

$$\begin{aligned}
\nu_t(X\times G_n) &\le \frac{1}{t}\int_{X\times X}(\tilde{u}(y)-\tilde{u}(x))^2\tilde{e}_{G_n}(y)p_t(x,dy)m(dx)\\
&= \frac{1}{t}\{(u^2e_{G_n},p_t1)_m - 2(p_tu,ue_{G_n})_m + (u^2,p_te_{G_n})_m\}\\
&= \frac{1}{t}\{-(u^2e_{G_n},1-p_t1)_m + 2(u-p_tu,ue_{G_n})_m - (u^2,e_{G_n}-p_te_{G_n})_m\}\\
&\to -<\tilde{u}^2\tilde{e}_{G_n},k> +2\mathcal{E}(u,ue_{G_n}) - \mathcal{E}(u^2,e_{G_n}) \qquad \text{as } t\downarrow 0.
\end{aligned}$$

Since $\tilde{e}_{G_n},\tilde{u}\tilde{e}_{G_n},\tilde{u}^2\tilde{e}_{G_n}\to 0$ q.e. and weakly in $\mathcal{F}$, it holds that

$$\lim_{n\to\infty}\lim_{t\downarrow 0}\mathrm{III}_n(t) = 0.$$

The proof is completed. □

Let $\overset{c}{M}{}^{[u]}$ be the continuous part of $M^{[u]}$, and $\overset{c}{\mu}_{<u>}$ be the smooth measure associated with $<\overset{c}{M}{}^{[u]}>$. If we set

$$\overset{c}{\mu}_{<u,v>} = \frac{1}{2}\{\overset{c}{\mu}_{<u+v>} - \overset{c}{\mu}_{<u>} - \overset{c}{\mu}_{<v>}\},$$

then, as in [14], we can conclude from Lemma 7.2 that the following derivation property of $\overset{c}{\mu}_{<u>}$ holds.

Theorem 7.3. *For $u, v, w \in \mathcal{F}_b$, it holds that*

$$\overset{c}{\mu}_{<uv,w>} = \tilde{u} \cdot \overset{c}{\mu}_{<v,w>} + \tilde{v} \cdot \overset{c}{\mu}_{<u,w>} .$$

Now repeating the argument in [11, §5.4], we can show the following assertions on the stochastic calculus.

(sc.i) For $u_1, \ldots, u_n \in \mathcal{F}_b$ and $\Phi \in C^1(\mathbf{R}^n)$, it holds that

$$\overset{c}{\mu}_{<\Phi(\mathbf{u}),v>} = \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}(\mathbf{u}) \cdot \overset{c}{\mu}_{<u_i,v>} \text{ for any } v \in \mathcal{F}_b, \tag{7.3}$$

where $\mathbf{u} = (u_1, \ldots, u_n)$.

(sc.ii) For $u_1, \ldots, u_n \in \mathcal{F}$ and $\Phi \in C_b^1(\mathbf{R}^n)$($\equiv$ the space of all continuously differentiable bounded functions with bounded derivatives), the identity (7.3) remains valid.

(sc.iii) For each $M \in \mathcal{M}_1 \equiv \{M \in \mathcal{M} : \mu_{<M>}(X) < \infty\}$ and $f \in L^2(X; \mu_{<M>})$, there exists a unique $f \cdot M \in \overset{\circ}{\mathcal{M}}$ such that

$$e(f \cdot M, L) = \frac{1}{2} \int_X f(x) \mu_{<M,L>}(dx) \qquad L \in \overset{\circ}{\mathcal{M}}, \tag{7.4}$$

where $\mu_{<M,L>} = (1/2)\{\mu_{<M+L>} - \mu_{<M>} - \mu_{<L>}\}$. Moreover, the mapping $f \mapsto f \cdot M$ is an isometry of $L^2(X; \mu_{<M>})$ to $(\overset{\circ}{\mathcal{M}}, 2e)$.

(sc.iv) For $u_1, \ldots, u_n \in \mathcal{F}_b$ and $\Phi \in C^1(\mathbf{R}^n)$, $\Phi(\mathbf{u}) \in \mathcal{F}_b$ and satisfies that

$$\overset{c}{M}{}^{[\Phi(\mathbf{u})]} = \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}(\mathbf{u}) \cdot \overset{c}{M}{}^{[u_i]}. \tag{7.5}$$

(sc.v) For $u_1, \ldots, u_n \in \mathcal{F}$ and $\Phi \in C_b^1(\mathbf{R}^n)$, the identity (7.3) remains valid.

(sc.vi) If either (i) $u_1, \ldots, u_n \in \mathcal{F}_b$ and $\Phi \in C^1(\mathbf{R}^n)$ or (ii) $u_1, \ldots, u_n \in \mathcal{F}$ and $\Phi \in C_b^1(\mathbf{R}^n)$, then it holds that

$$\mathcal{E}^{(c)}(\Phi(\mathbf{u}), \Phi(\mathbf{u})) = \frac{1}{2} \sum_{i,j=1}^{n} \int_X \frac{\partial \Phi}{\partial x_i}(\mathbf{u}) \frac{\partial \Phi}{\partial x_j}(\mathbf{u}) d\overset{c}{\mu}_{<u_i,u_j>} . \tag{7.6}$$

8. Closable parts of pre-Dirichlet forms

Throughout this section, in addition to Assumptions (A.1)-(A.3), we assume the existence of a subalgebra $\mathcal{C} \subset \mathcal{F} \cap C_b(X)$ such that

(C.1) $1 \in \mathcal{C}$, $\mathcal{C}$ is dense in $\mathcal{F}$, and a countable subset of $\mathcal{C}$ separates the points of X,

(C.2) for each $\varepsilon > 0$, there is a function $\beta_\varepsilon : \mathbf{R} \to [-\varepsilon, 1+\varepsilon]$ such that $0 \le \beta_\varepsilon(t) - \beta_\varepsilon(s) \le t - s$, $t \ge s$, $\beta_\varepsilon(t) = t$, $0 \le t \le 1$, and $\beta_\varepsilon(u) \in \mathcal{C}$ whenever $u \in \mathcal{C}$.

(C.3) for some $\delta > 0$ and $a > 0$, $\beta_\delta(t) = \beta_\delta(-a)$, $t \le -a$, and $= \beta_\delta(a)$, $t \ge a$.

A pre-Dirichlet form $(\mathcal{A}, \mathcal{C})$ is by definition a nonnegative definite symmetric bilinear form on $\mathcal{C} \times \mathcal{C}$ with $\mathcal{A}(\beta_\varepsilon(u), \beta_\varepsilon(u)) \le \mathcal{A}(u, u)$ for $u \in \mathcal{C}$, $\varepsilon > 0$. Let $\mu \in \mathcal{FM}$($\equiv$the space of finite Borel measures on X). A pre-Dirichlet form is said to be closable on

$L^2(X;\mu)$ if $\mathcal{A}(u_n,u_n)\to 0$ whenever $\{u_n\}$ is $\mathcal{A}$-Cauchy and $u_n\to 0$ in $L^2(X;\mu)$. $(\mathcal{E},\mathcal{C})$, the restriction of $\mathcal{E}$ on $\mathcal{C}\times\mathcal{C}$, is a pre-Dirichlet form closable on $L^2(X;m)$. As was seen in [13, §6], for each $\mu\in\mathcal{FM}$, there exists a unique pre-Dirichlet form closable on $L^2(X;\mu)$, say $(\mathcal{E}^{cls(\mu)},\mathcal{C})$, which is maximal in the sense that if a pre-Dirichlet form $(\mathcal{A},\mathcal{C})$ closable on $L^2(X;\mu)$ is dominated by $(\mathcal{E},\mathcal{C})$, i.e. $\mathcal{A}(u,u)\le\mathcal{E}(u,u)$, $u\in\mathcal{C}$, then $(\mathcal{A},\mathcal{C})$ is dominated by $(\mathcal{E}^{cls(\mu)},\mathcal{C})$. We call $(\mathcal{E}^{cls(\mu)},\mathcal{C})$ the closable part of $(\mathcal{E},\mathcal{C})$ on $L^2(X;\mu)$. In this section, we aim at establishing a characterization of $\mathcal{E}^{cls(\mu)}$ analogous to [13] and showing the tightness of the capacity associated with $\mathcal{E}^{cls(\mu)}$.

We first recall the extended Dirichlet space $(\mathcal{E},\mathcal{F}_e)$. We say $u\in\mathcal{F}_e$ if there is an $\mathcal{E}$-Cauchy sequence $\{u_n\}\subset\mathcal{F}$ with $u_n\to u$ m-a.e. $\mathcal{E}$ on $\mathcal{F}$ is extended to $\mathcal{F}_e$, again denoted by $\mathcal{E}$, as $\mathcal{E}(u,u)=\lim_n\mathcal{E}(u_n,u_n)$. We have

Lemma 8.1. *(i) Let $u\in\mathcal{F}_e$. Denoting R_β the resolvent of the Hunt process $\mathbf{M}$ associated with $\mathcal{E}$, define $\sigma_\beta(dxdy)=\beta R_\beta(x,dy)m(dx)$, $s_\beta(x)=\beta R(x,X)$, and*

$$\mathcal{E}^\beta(u,u)=\frac{\beta}{2}\int_{X\times X}(u(x)-u(y))^2\sigma_\beta(dxdy)+\beta\int_X u(x)^2(1-s_\beta(x))m(dx).$$

Then $\mathcal{E}(u,u)=\lim_{\beta\uparrow\infty}\mathcal{E}^\beta(u,u)$.
(ii) Every normal contraction operates on $(\mathcal{E},\mathcal{F}_e)$.
(iii) $\mathcal{F}=\mathcal{F}_e\cap L^2(X;m)$.
(iv) Every $u\in\mathcal{F}_e$ admits a quasi-continuous m-modification.

Proof. The first three assertions can be seen in the same way as in [11, Lemma 1.5.4]. To see (iv), it suffices to note that, for every $u\in\mathcal{F}_e$ and $n\ge 1$, $(-n)\vee(u\wedge n)\in\mathcal{F}$ and admits a quasi-continuous m-version. □

In what follows, $u\in\mathcal{F}_e$ is always assumed to be quasi-continuous. For a finely closed Borel set F and $u\in\mathcal{F}_e$, define

$$\mathcal{P}^F u(x)=E_x[u(X_{\sigma_F})].$$

Oshima [18] has shown that

Lemma 8.2. *For $u\in\mathcal{F}_e$,*
(i) $\mathcal{P}^F u\in\mathcal{F}_e$, is quasi-continuous, and enjoys $\mathcal{P}^F u=\mathcal{P}^F(\mathcal{P}^F u)$ q.e.,
(ii) $\mathcal{E}(\mathcal{P}^F u,v)=0$ for $v\in\mathcal{F}_e$, bounded and $=0$ q.e. on F,
(iii) $\mathcal{E}(\mathcal{P}^F u,\mathcal{P}^F u)\le\mathcal{E}(u,u)$.

We now proceed to the review on time changed processes. Let $\nu\in\mathcal{S}\cap\mathcal{FM}$ $(\nu\ne 0)$ and A^ν be the associated PCAF. We set

$$F[\nu]=\{x\in X\setminus N: P_x[A^\nu_t>0,\ t>0]=1\},\tag{8.1}$$

where N is an exceptional set of A^ν. Using [19, (64.2)] instead of [8, V(3.9)], as in [11, Lemma 5.5.1], we can conclude that

$$F[\nu]\subset S[\nu]\equiv supp[\nu]\qquad\text{and}\qquad \nu(S[\nu]\setminus F[\nu])=0.\tag{8.2}$$

Setting $\tau_t^\nu = \inf\{s > 0 : A_s^\nu > t\}$, the time changed process $\mathbf{M}^{tch(\nu)}$ by A^ν is given by

$$\mathbf{M}^{tch(\nu)} = (X_{\tau_t^\nu}, P_x) \tag{8.3}$$

and is a ν-symmetric standard process on $F[\nu]$. We denote by $(\mathcal{E}^{tch(\nu)}, \mathcal{F}^{tch(\nu)})$ the Dirichlet form on $L^2(F[\nu]; \nu)$ associated with $\mathbf{M}^{tch(\nu)}$. It has been seen by Fitzsimmons [10] that

$$\begin{aligned}
&\mathcal{F}^{tch(\nu)} = \{u \in L^2(F[\nu]; \nu) : u = v|_{F[\nu]} \ \nu\text{-a.e. for some } v \in \mathcal{F}_e \text{ with } \mathcal{P}^{F[\nu]}v = v \text{ q.e.}\}\\
&\mathcal{E}^{tch(\nu)}(v|_{F[\nu]}, v|_{F[\nu]}) = \mathcal{E}(\mathcal{P}^{F[\nu]}v, \mathcal{P}^{F[\nu]}v) \quad \text{for } v \in \mathcal{F}_e \text{ with } \mathcal{P}^{F[\nu]}v = v \text{ q.e.}
\end{aligned}$$

By (8.2), we may think of $(\mathcal{E}^{tch(\nu)}, \mathcal{F}^{tch(\nu)})$ as a Dirichlet form on $L^2(S[\nu]; \nu)$. In this case, we write $(\mathcal{E}^\nu, \mathcal{F}^\nu)$. Now we can show the regularity of $\mathcal{F}^\nu$.

Lemma 8.3. *Let $\mathcal{C}^\nu = \{u|_{S[\nu]} : u \in \mathcal{C}\}$. Then $\mathcal{C}^\nu \subset \mathcal{F}^\nu$, and is dense in $\mathcal{F}^\nu$.*

Proof. Let $v \in \mathcal{C}$. Since $F[\nu] \setminus F[\nu]^r$ is semipolar (cf. [19, 8]), by Theorems 5.2 and 5.3, we have

$$v = \mathcal{P}^{F[\nu]}v \qquad \text{q.e. and } \nu\text{-a.e. on } F[\nu]. \tag{8.4}$$

Combining this with Lemma 8.2, we have $v|_{S[\nu]} \in \mathcal{F}^\nu$.

Next let $u \in \mathcal{F}^\nu$. We may assume that u is bounded. Choose $v_0 \in \mathcal{F}_e \cap L^2(X; \nu)$ with $\mathcal{P}^{F[\nu]}v_0 = v_0$ q.e. and $v_0|_{F[\nu]} = u$. Define $v = (-\|u\|_\infty) \vee (v_0 \wedge \|u\|_\infty)$. Then, by Lemma 8.1, $v \in \mathcal{F}$ and moreover we have

$$u = \mathcal{P}^{F[\nu]}v|_{F[\nu]} = v \quad \nu\text{-a.e.} \tag{8.5}$$

Choose $u_n^0 \in \mathcal{C}$ such that $u_n^0 \to v$ q.e. and in $\mathcal{F}$, and put $u_n = \|u\|_\infty \beta_1(u_n^0/\|u\|_\infty)$. Then $\{u_n\}$ is uniformly bounded, and $u_n \to v$ q.e. and in $\mathcal{F}$. Hence, by Lemma 8.2,

$$\begin{aligned}
&\mathcal{E}_1^\nu(u_n|_{S[\nu]} - u, u_n|_{S[\nu]} - u)\\
&\quad = \mathcal{E}(\mathcal{P}^{F[\nu]}u_n - \mathcal{P}^{F[\nu]}v, \mathcal{P}^{F[\nu]}u_n - \mathcal{P}^{F[\nu]}v) + (u_n - u, u_n - u)_\nu\\
&\quad \le \mathcal{E}(u_n - v, u_n - v) + (u_n - u, u_n - u)_\nu \to 0,
\end{aligned}$$

which implies the denseness of $\mathcal{C}^\nu$ in $\mathcal{F}^\nu$. □

Every $\mu \in \mathcal{FM}$ is decomposed as

$$\mu = \mu_0 + \mu_1 \qquad \mu_0 \in \mathcal{S} \text{ and } \mu_1 = \mathbf{I}_N \cdot \mu \text{ for some exceptional set } N. \tag{8.6}$$

See [13, Lemma 2.1]. If $\mu_0 \neq 0$, we then define a pre-Dirichlet form closable on $L^2(X; \mu)$ by

$$\mathcal{E}^\mu(u, u) = \mathcal{E}^{\mu_0}(u|_{S[\mu_0]}, u|_{S[\mu_0]}), \qquad u \in \mathcal{C},$$

and if $\mu_0 = 0$ then $\mathcal{E}^\mu \equiv 0$. We are now ready to state the main result of this section.

Theorem 8.4. *Let $\mu \in \mathcal{FM}$ satisfy $S[\mu] = X$. Then*
(i) $(\mathcal{E}^\mu, \mathcal{C}) = (\mathcal{E}^{cls(\mu)}, \mathcal{C})$.
(ii) The capacity associated with the closure of $(\mathcal{E}^{cls(\mu)}, \mathcal{C})$ on $L^2(X; \mu)$ is tight.
(iii) If $Cap(X \setminus F[\mu_0]) = 0$, then $(\mathcal{E}, \mathcal{C})$ is closable on $L^2(X; \mu)$ and the closure is realized by a Hunt process $\mathbf{M}^\mu = (X_t^\mu, P_x^\mu)$ such that

(a) *"the law of $X^\mu_\bullet$ under P^μ_x" = "the law of $X_{\tau^{\mu_0}_\bullet}$ under P_x"* *for $x \in X \setminus N$,*

(b) $P^\mu_x[X^\mu_t = x$, *for* $t \geq 0] = 1$ *for $x \in N$,*

where $\mathbf{M}^{\mu_0} = (X_{\tau^{\mu_0}_t}, P_x)$ is the time changed process by A^{μ_0}, and N is an exceptional Borel set such that $\mu_1 = \mathbf{I}_N \cdot \mu$ and $X \setminus N \subset F[\mu_0]$.

For the proof, we first mention that under Assumptions (C.1)-(C.3) compact sets are separated in the stronger sense than Lemma 1.2.

Lemma 8.5. *For any disjoint compact sets K_1 and K_2, there exist a decreasing sequence $\{G_n\}$ of open sets and a sequence $\{f_n\} \subset \mathcal{C}$ such that $\cap_n G_n = K_1$, $G_1 \cap K_2 = \emptyset$, $0 \leq f_n \leq 1$, and $f_n = 1$ on K_1 and $= 0$ on $X \setminus G_n$.*

Proof. Due to (C.1), we may assume that there exists a $D \subset \mathcal{C}$ such that D is a $\mathbf{Q}$-algebra and separates the points of X. Let $\overline{D}$ be its closure in $C_b(X)$. By Gelfand's representation theorem, there is a compact metric space $\overline{X}$ such that (i) X is imbedded in $\overline{X}$ densely and continuously and (ii) the restriction to X gives rise to an isomorphism from $C(\overline{X})$ to $\overline{D}$.

Note that K_1 and K_2 are also compact in $\overline{X}$. Hence there are a sequence $\{g_n\} \subset C(\overline{X})$ and a decreasing sequence $\{G'_n\}$ of open sets in $\overline{X}$ such that $\cap_n G'_n = K_1$, $G'_1 \cap K_2 = \emptyset$, $0 \leq g_n \leq 1$, and $g_n = 1$ on K_1 and $= 0$ on $\overline{X} \setminus G'_n$. Let $G_n = X \cap G'_n$. Choose $f'_n \in D$ with $\|\,(g_n|_X) - f'_n\|_\infty < 1/4$, and set $f_n = \{\beta_\delta\,(4a(f'_n - (1/2))) - \beta_\delta(-a)\}/\{\beta_\delta(a) - \beta_\delta(-a)\}$. It is easy to see that $\{f_n\}$ and $\{G_n\}$ satisfy the desired properties. □

By this lemma, we obtain the following representation of Cap.

Lemma 8.6. *For every compact $K \subset X$, it holds that*

$$Cap(K) = \inf\{\mathcal{E}_1(u,u) : u \in \mathcal{C} \text{ and } \geq 1 \text{ on } K\}.$$

Proof. Choose $\{u_n\} \subset \mathcal{C}$ such that $u_n \geq 1$ on K and

$$\mathcal{E}_1(u_n,u_n) \downarrow I(K) \equiv \inf\{\mathcal{E}_1(u,u) : u \in \mathcal{C} \text{ and } \geq 1 \text{ on } K\}.$$

Set $u'_n = \beta_{1/n}(u_n)$. Then, $\mathcal{E}_1(u'_n,u'_n) \downarrow I(K)$ and $u'_n \to u$ in $\mathcal{F}$ for some $u \in \mathcal{F}$. By Theorem 1.1, we have

$$\tilde{u} = 1 \qquad \text{q.e. on } K. \tag{8.7}$$

If $v \in \mathcal{C}$ and ≥ 0 on K, then

$$\mathcal{E}_1(u + \frac{1}{n}v, u + \frac{1}{n}v) = \lim_k \mathcal{E}_1(u'_k + \frac{1}{n}v, u'_k + \frac{1}{n}v) \geq I(K) = \mathcal{E}_1(u,u).$$

Hence we have

$$\mathcal{E}_1(u,v) \geq 0 \qquad \text{for } v \in \mathcal{C} \text{ with } v \geq 0 \text{ on } K. \tag{8.8}$$

Now let $w \in \mathcal{F} \cap C_b(X)$ satisfy $w \geq 0$ on X. Choose $\{w_n\} \subset \mathcal{C}$ converging to w in $\mathcal{F}$ and define $w'_n = \|w\|_\infty\{(1/n) + \beta_{1/n}(w_n/\|w\|_\infty)\}$. Then, it is easily seen that $\sup_n \mathcal{E}_1(w'_n,w'_n) < \infty$. Hence the Cesàro mean w''_n of a subsequence of $\{w'_n\}$ converges to w in $\mathcal{F}$. By virtue of (8.8), we have

$$\mathcal{E}_1(u,w) = \lim_n \mathcal{E}_1(u,w''_n) \geq 0.$$

Thus

$$u = U_1\mu \qquad \text{for some } \mu \in \mathcal{S}_0. \tag{8.9}$$

Take a compact set F with $F \cap K = \emptyset$. By Lemma 8.5, we obtain a nonnegative $f \in \mathcal{C}$ such that $f = 1$ on F and $= 0$ on K. We then have

$$\mu(F) \leq \int_X f d\mu = \mathcal{E}_1(u, f) = 0.$$

Thus $supp[\mu] \subset K$. By Lemma 3.1 and (8.7), we have $u = e_K$. □

These lemmas implies that the following assertion on traps, obtained in [13], also holds in our situation.

Lemma 8.7. *Let $m' \in \mathcal{FM}$ and $(\mathcal{E}', \mathcal{F}')$ be another Dirichlet form on $L^2(X; m')$ with $\mathcal{C} \subset \mathcal{F}'$. Assume that $\mathcal{E}'(u,u) \leq \mathcal{E}(u,u)$, $u \in \mathcal{C}$. Then for any exceptional Borel set N,*

$$G'_\alpha(\mathbf{I}_N u) = \frac{1}{\alpha}\mathbf{I}_N u, \ m'\text{-a.e. } \ \text{for } u \in L^2(X; m'), \tag{8.10}$$

where $\{G'_\alpha\}$ is the resolvent of $\mathcal{E}'$.

Proof. Since X is Lusinian, it suffices to show (8.10) in the case where $N = K$, $u = \mathbf{I}_K$ for some exceptional compact set K. By Lemmas 8.5 and 8.6, there exist sequences $\{g_n\}$, $\{w_k\} \subset \mathcal{C}$ and a decreasing sequence $\{G_k\}$ of open sets such that (i) $g_n \geq 1$ on K and $\mathcal{E}_1(g_n, g_n) \to 0$ as $n \to \infty$, (ii) $0 \leq w_k \leq 1$, $w_k = 1$ on K and $= 0$ outside of G_k, and (iii) $\cap_k G_k = K$. Repeating the argument in [13, Proof of Lemma 4.1] with g_n, w, and G there replaced by the above g_n, w_k and G_k, respectively, and letting $k \to \infty$, we obtain the desired conclusion. We omit the details. □

Proof of Theorem 8.4. The third assertion follows from the first and Lemma 8.7. For the detailed argument, see [13].

To see the first assertion, let $(\mathcal{A}, \mathcal{C})$ be a pre-Dirichlet form closable on $L^2(X; \mu)$, which is dominated by $(\mathcal{E}, \mathcal{C})$. It follows from Lemma 8.7 that $(\mathcal{A}, \mathcal{C})$ is also closable on $L^2(X; \mu_0)$. See [13, Lemma 4.2].

Take $u \in \mathcal{C}$. By Lemma 8.2, then $\mathcal{P}^{F[\mu_0]}u \in \mathcal{F}$, is bounded and

$$\mathcal{E}^\mu(u,u) = \mathcal{E}(\mathcal{P}^{F[\mu_0]}u, \mathcal{P}^{F[\mu_0]}u).$$

Choose a uniformly bounded sequence $\{u_n\} \subset \mathcal{C}$ such that $u_n \to \mathcal{P}^{F[\mu_0]}u$ q.e. and in $\mathcal{F}$. Since $\mu_0 \in \mathcal{S}$, by (8.4), $u_n \to u$ in $L^2(X; \mu_0)$. On the other hand, $\mathcal{A}$ being dominated by $\mathcal{E}$, $\{u_n\}$ is an $\mathcal{A}$-Cauchy sequence. Thus the closability of $\mathcal{A}$ on $L^2(X; \mu_0)$ implies that

$$\mathcal{A}(u,u) = \lim_n \mathcal{A}(u_n, u_n) \leq \lim_n \mathcal{E}(u_n, u_n) = \mathcal{E}^\mu(u,u).$$

Thus the proof of the first assertion is complete.

To see the second assertion, we denote by $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ the closure of $(\mathcal{E}^\mu, \mathcal{C})(= (\mathcal{E}^{cls(\mu)}, \mathcal{C}))$ on $L^2(X; \mu)$, and by $Cap^\mu(\cdot)$ the associated capacity. Since X is Lusinian, by virtue of (A.3), we can choose an increasing sequence $\{K_n\}$ of compact sets in X such that

$$Cap(X \setminus K_n) \downarrow 0 \qquad \text{and} \qquad \mu(N \setminus K_n) \downarrow 0. \tag{8.11}$$

Let e_n be the 1-equilibrium potential of $X \setminus K_n$. It then follows from Lemma 8.7 that

$$e_n \cdot \mathbf{I}_{X\setminus N} \in \mathcal{F}^\mu \qquad \text{and} \qquad \mathcal{E}^\mu(e_n \cdot \mathbf{I}_{X\setminus N}, e_n \cdot \mathbf{I}_{X\setminus N}) \leq \mathcal{E}(e_n, e_n). \tag{8.12}$$

Indeed, choose a uniformly bounded sequence $\{u_{nj}\}$ of elements of $\mathcal{C}$ satisfying that $u_{nj} \to e_n$ as $j \to \infty$ in $\mathcal{F}$ and q.e. Then Lemma 8.7 implies that

$$\begin{aligned} &u_{nj} \cdot \mathbf{I}_{X\setminus N} \in \mathcal{F}^\mu, \\ &\mathcal{E}^\mu(u_{nj} \cdot \mathbf{I}_{X\setminus N}, u_{nj} \cdot \mathbf{I}_{X\setminus N}) = \mathcal{E}^\mu(u_{nj}, u_{nj}) \leq \mathcal{E}(u_{nj}, u_{nj}). \end{aligned}$$

(cf. [13, Proof of Lemma 4.2]). Note that $\mu_0 = \mu|_{X\setminus N}$. Then, since μ_0 is an element of $\mathcal{S}$, $u_{nj} \cdot \mathbf{I}_{X\setminus N} \to e_n \cdot \mathbf{I}_{X\setminus N}$ μ-a.e. Hence we obtain (8.12).

Applying Lemma 8.7 again, we see that $\mathbf{I}_A \in \mathcal{F}^\mu$ and $\mathcal{E}^\mu(\mathbf{I}_A, \mathbf{I}_A) = 0$ for any Borel set $A \subset N$. Let

$$v_n \equiv e_n \cdot \mathbf{I}_{X\setminus N} + \mathbf{I}_{N\setminus K_n}.$$

It follows from (8.12) that $v_n \to 0$ in $\mathcal{F}^\mu$. Since $e_n = 1$ q.e. on $X \setminus K_n$, $v_n \geq 1$ μ-a.e. on $X \setminus K_n$. Thus we obtain

$$Cap^\mu(X \setminus K_n) \leq \mathcal{E}_1^\mu(v_n, v_n) \downarrow 0,$$

which completes the proof of the second assertion. □

After completing the paper, we noticed [3] and [4]. In [3], Albeverio-Ma considered the local property, which we discussed in Section 6. In [4], Albeverio-Ma-Röckner obtained the Beurling-Deny formula for quasi-regular Dirichlet forms. Their proof is completely different from ours in Section 4.

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