# DIFFERENTIAL CALCULUS
# ON A BASED LOOP GROUP

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Dedicated to Professor Shinzo Watanabe on his 60th birthday

We discuss differential calculus on a based loop group. The calculus on a submanifold of the Wiener space was well-developed. In the paper we transfer it to a based loop group through an Itô map.

## 1 Introduction

In this paper, we develop a differential calculus on the path space and the based loop space of a Lie group. Loop groups received much attention by many researchers recently ([15] [8] [24] [25] [17] [19] ... ). On the other hand, the calculus on a submanifold of the Wiener space was also developed by many authors, e.g., Airault-Malliavin [7], Airault [6], Airault-Van Biesen [9], Van Biesen [31]. In the case of loop groups, there exist natural isomorphisms between submanifolds of the Wiener space and loop spaces. We can connect the calculus on the submanifold of the Wiener space to that on the based loop space. The isomorphism is given by the Itô map. We calculate the Lie bracket and the Ricci curvature on the based loop group.

The organization of this paper is as follows. In the section 2, we develop the calculus on the path space of a Lie group. We show that the divergence of an adapted vector field is the stochastic integral. In the section 3, we discuss the based loop group. The explicit forms of the second fundamental form and the Ricci curvature are given. We will give a sufficient condition for the spectral gap on the based loop group.

## 2 Path space of a group

In order to fix notation, we introduce necessary notions. Let $G$ be a connected compact $d$-dimensional Lie group and we denote the unit element by $e$. We denote the set of all left invariant vector fields by $\mathfrak{g}$ that is called the Lie algebra of $G$. $\mathfrak{g}$ is sometimes identified with $T(G)_e$. We fix an $\mathrm{Ad}(G)$-invariant inner product $(\cdot|\cdot)_{\mathfrak{g}}$ in $\mathfrak{g}$. We fix a constant $T > 0$ and denote the $G$-valued path space on $[0,T]$ by

$$PG := \{\gamma\colon [0,T]\to G\,;\gamma \text{ is continuous and } \gamma_0 = e\}. \tag{1}$$

$PG$ has a natural group structure as follows: for $\gamma, \xi \in PG$, define $\gamma\xi \in PG$ by

$$(\gamma\xi)_t = \gamma_t \xi_t.$$

Moreover, we denote the based loop space of $G$ as follows

$$\Omega G := \{\gamma\colon [0,T]\to G\,;\gamma \text{ is continuous and } \gamma_0 = \gamma_T = e\}.$$

It is easy to see that $\Omega G$ is a normal subgroup of $PG$ and $PG/\Omega G \cong G$.

We take an orthonormal basis $\{X_1, X_2, \dots, X_d\}$ of $\mathfrak{g}$. We consider the following stochastic differential equation on $G$:

$$\begin{cases} d\gamma_t = \displaystyle\sum_{i=1}^{d} X_i(\gamma_t)\circ dB_t^i, \\ \gamma_0 = e \end{cases} \tag{2}$$

where $(B_t^1, \dots, B_t^d)_{t\in[0,T]}$ is a $d$-dimensional Brownian motion, and $\circ$ stands for the Stratonovich symmetric stochastic integral. In the sequel, following the custom, we omit the summation sign for repeated indices appearing once at the top and once at the bottom. Setting $B_t = B_t^i X_i$, $(B_t)$ is a Brownian motion on $\mathfrak{g}$. The above stochastic differential equation is rewritten in matrix notation as follows:

$$\begin{cases} d\gamma_t = \gamma_t \circ dB_t, \\ \gamma_0 = e. \end{cases} \tag{3}$$

$(B_t)$ induces a measure on the path space $P\mathfrak{g}$, which is called the Wiener measure and is denoted by $P^W$. There exists the unique strong solution to (3), i.e., there exists a measurable function $I\colon P\mathfrak{g} \to PG$ such that $\gamma = I(B)$ is the unique solution to (3). We call this map $I$ the Itô map. We denote the image measure of $P^W$ under $I$ by $\mu$. Then $(P\mathfrak{g}, P^W) \cong (PG, \mu)$ as measure spaces. We sometimes regard a function on $(PG, \mu)$ as a function on $(P\mathfrak{g}, P^W)$.

The solution to (3) is called a left Brownian motion on $G$. We may regard $(\gamma_t)$ as a right Brownian motion on $G$. To see this, we need another Brownian motion $(b_t)$ that is defined by

$$b_t = \int_0^t \mathrm{Ad}(\gamma_s) dB_s. \tag{4}$$

Here the integral is the Itô stochastic integral. It is easy to see that $(b_t)$ is a Brownian motion on $\mathfrak{g}$, since the inner product $(\cdot|\cdot)_{\mathfrak{g}}$ is $\mathrm{Ad}(G)$-invariant.

**Proposition 2.1** *We have*

$$b_t = \int_0^t \mathrm{Ad}(\gamma_s) dB_s = \int_0^t \mathrm{Ad}(\gamma_s) \circ dB_s. \tag{5}$$

*Proof.* We use the matrix notation with respect to the basis $\{X_1, \ldots, X_d\}$. Note that

$$\mathrm{Ad}(\gamma_t)^i_j \circ dB^j_t = \mathrm{Ad}(\gamma_t)^i_j dB^j_t + \frac{1}{2}\langle d\mathrm{Ad}(\gamma_t)^i_j, dB^j_t\rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the quadratic variation. Hence it is enough to show

$$\langle d\mathrm{Ad}(\gamma_t)^i_j, dB^j_t\rangle = 0.$$

Since $(\gamma_t)$ is the solution to (2), $(\mathrm{Ad}(\gamma_t))$ satisfies the following system of stochastic differential equations:

$$d\mathrm{Ad}(\gamma_t)^i_j = \mathrm{Ad}(\gamma_t)^i_k \mathrm{ad}(X_l)^k_j \circ dB^l_t.$$

Therefore

$$\begin{aligned}
\langle d\mathrm{Ad}(\gamma_t)^i_j, dB^j_t\rangle &= \langle \mathrm{Ad}(\gamma_t)^i_k \mathrm{ad}(X_l)^k_j \circ dB^l_t, dB^j_t\rangle \\
&= \langle \mathrm{Ad}(\gamma_t)^i_k \mathrm{ad}(X_l)^k_j dB^l_t, dB^j_t\rangle \\
&= \sum_{j=1}^d \mathrm{Ad}(\gamma_t)^i_k \mathrm{ad}(X_j)^k_j dt \\
&= \sum_{j,k=1}^d \mathrm{Ad}(\gamma_t)^i_k ([X_j, X_j]|X_k)_{\mathfrak{g}} \\
&= 0.
\end{aligned}$$

This completes the proof. ∎

Now we easily have

$$B_t = \int_0^t \mathrm{Ad}(\gamma_s^{-1}) db_s = \int_0^t \mathrm{Ad}(\gamma_s^{-1}) \circ db_s. \tag{6}$$

Therefore we can rewrite (3) as

$$d\gamma_t = \gamma_t \circ dB_t = \gamma_t \circ (\mathrm{Ad}(\gamma_t^{-1}) \circ db_t) = \gamma_t \gamma_t^{-1} \circ db_t \gamma_t = \circ db_t \gamma_t.$$

This shows that $(\gamma_t)$ is a right Brownian motion on $G$.

The Cameron-Martin space $H \subseteq P\mathfrak{g}$ is defined by

$$H = \{h \in P\mathfrak{g} : h \text{ is absolutely continuous and } \dot{h} \in L^2([0,T]\to\mathfrak{g})\}.$$

Here $\dot{h}$ denotes the Radon-Nikodym derivative of $h$. The inner product in $H$ is defined as follows:

$$(h|k)_H := \int_0^T (\dot{h}(t)|\dot{k}(t))_{\mathfrak{g}} dt.$$

For $h \in H$ we define a one-parameter subgroup $\{\varphi_u; u \in \mathbb{R}\}$ of $PG$ as follows:

$$\varphi_u(t) = \exp\{uh(t)\},$$

where $\exp\colon \mathfrak{g} \to G$ is the exponential map. $\{\varphi(u)\}$ defines a one-parameter transformation group on $PG$ as

$$\Phi_u(\gamma) = \varphi_u\gamma.$$

Finally, we can obtain a vector field $\mathbf{X}^h$ as an infinitesimal transformation of $\{\Phi_u\}$, i.e.,

$$\mathbf{X}^h f(\gamma) = \frac{d}{du} f(\Phi_u(\gamma))\Big|_{u=0} \quad \text{for } f \in \mathcal{F}C^\infty(PG),$$

where $\mathcal{F}C^\infty(PG)$ is a set of all functions $f\colon PG \to \mathbb{R}$ of the form

$$f(\gamma) = F(\gamma_{t_1}, \ldots, \gamma_{t_n}), \quad 0 < t_1 < \cdots < t_n \leq T,\ F \in C^\infty(G^n).$$

Notice that $\mathbf{X}^h$ is a *right invariant* vector field. We set

$$T(PG)_\gamma := \{\mathbf{X}^h_\gamma; h \in H\}.$$

We regard $T(PG)_\gamma$ as a tangent space of $PG$ at $\gamma$, and so the tangent bundle is defined by

$$T(PG) = \bigcup_{\gamma \in PG} T(PG)_\gamma.$$

Since $T(PG)_\gamma$ is isomorphic to $H$, we sometimes identify $T(PG)_\gamma$ with $H$ in the sequel. Furthermore $T(PG)$ is isomorphic to $PG \times H$. The isomorphism is given by $PG \times H \ni (\gamma, h) \mapsto \mathbf{X}^h_\gamma \in T(PG)$. Using this isomorphism, we can introduce an inner product in $T(PG)_\gamma$ as follows: for $h$, $k \in H$

$$(\mathbf{X}^h_\gamma|\mathbf{X}^k_\gamma) := (h|k)_H.$$

Hence $PG$ can be regarded as a Riemannian manifold with a *right invariant metric*. Similarly we can define the cotangent bundle $T^*(PG)$, and $T^*(PG)$ can be identified with $PG \times H^*$.

For $f \in \mathcal{F}C^\infty(PG)$, $h \mapsto \mathbf{X}^h f$ is a bounded linear map, and so we denote it by $\hat{\nabla} f$:

$$\langle \hat{\nabla} f, h \rangle = \mathbf{X}^h f.$$

($\nabla$ is reserved for the covariant derivation on the based loop space $\Omega G$). As is well-known (see, e.g. Gross [17, Lemma 4.15]), the differential of the Itô map $I$ is given by

$$I_*(h) = \gamma \cdot h \tag{7}$$

where

$$(\gamma \cdot h)(t) = \int_0^t \mathrm{Ad}(\gamma_s)\dot{h}(s)ds. \tag{8}$$

Here (7) means that

$$\langle D(f \circ I)_B, h \rangle = \langle \hat{\nabla} f, \gamma \cdot h \rangle \tag{9}$$

where $D$ denotes the Gross $H$-differentiation operator and $\gamma = I(B)$. Thus we have the following diagram:

$$\begin{array}{ccc} T(P\mathfrak{g}) \cong P\mathfrak{g} \times H & \xrightarrow{I_*} & PG \times H \cong T(PG) \\ \downarrow & & \downarrow \\ P\mathfrak{g} & \xrightarrow{I} & PG \end{array} \tag{10}$$

The important thing is that $I_*$ preserves the inner product and therefore $I$ is an isomorphism in the sense of 'Riemannian manifolds.'

By the way, on the Wiener space $P\mathfrak{g}$, we can introduce the set of all $H$-valued smooth function in the sense of Malliavin, which we denote by $W^{\infty,\infty-}(H)$. We may regard an element of $W^{\infty,\infty-}(H)$ as a smooth section of $T(P\mathfrak{g}) \cong P\mathfrak{g} \times H$, and so we denote it by $\Gamma(T(P\mathfrak{g}))$. The associated set of smooth section of $T(PG)$ is defined to be the image of $\Gamma(T(P\mathfrak{g}))$ under $I$. We denote it by $\Gamma(T(PG))$. We can define $\Gamma(T^*(PG))$ similarly and it is easy to see that $\hat{\nabla} f \in \Gamma(T^*(PG))$ for $f \in \mathcal{F}C^\infty(PG)$.

We introduce a homomorphism $\pi \colon PG \to G$ by

$$\pi(\gamma) = \gamma_T. \tag{11}$$

The kernel of $\pi$ is $\Omega G$. The differential of $\pi$ is also given by

$$\pi_*(h) = \mathrm{Ad}(\gamma_T^{-1})h_T \tag{12}$$

and we also have the following diagram:

$$\begin{array}{ccc} T(PG) \cong PG \times H & \xrightarrow{\pi_*} & G \times \mathfrak{g} \cong T(G) \\ \Big\downarrow & & \Big\downarrow \\ PG & \xrightarrow{\pi} & G \end{array} \tag{13}$$

As is well-known in the Malliavin calculus, the $H$-derivative of $\gamma_t$ is given by

$$\gamma_t^{-1}\langle D\gamma_t, h\rangle = \mathrm{Ad}(\gamma_t^{-1})\int_0^t \mathrm{Ad}(\gamma_s)\dot{h}(s)ds. \tag{14}$$

In general, $D\gamma_t \in \mathcal{L}_{(2)}(H, T(G)_{\gamma_t})$ where $\mathcal{L}_{(2)}(H, T(G)_{\gamma_t})$ denotes the set of all Hilbert-Schmidt class operators from $H$ into $T(G)_{\gamma_t}$. $\gamma_t^{-1}$ in the left hand gives rise to an isomorphism $T(G)_{\gamma_t} \cong \mathfrak{g}$. To be more precise, $\gamma_t^{-1} = (L_{\gamma_t^{-1}})_* : T(G)_{\gamma_t} \to \mathfrak{g}$.

We see (14) quickly. Using the matrix notation, we have by (3),

$$dB_t = \gamma_t^{-1}\circ d\gamma_t$$

Differentiating both hands, we have

$$\begin{aligned} \dot{h}_t dt &= \langle D(\gamma_t^{-1}\circ d\gamma_t), h\rangle \\ &= -\gamma_t^{-1}\langle D\gamma_t, h\rangle\gamma_t^{-1}\circ d\gamma_t + \gamma_t^{-1}\circ d\langle D\gamma_t, h\rangle \\ &= -\gamma_t^{-1}\langle D\gamma_t, h\rangle\circ dB_t + \gamma_t^{-1}\circ d\langle D\gamma_t, h\rangle. \end{aligned}$$

Hence

$$d\langle D\gamma_t, h\rangle = \langle D\gamma_t, h\rangle\circ dB_t + \gamma_t\dot{h}(t)dt.$$

Now set

$$Y_t = \int_0^t \mathrm{Ad}(\gamma_s)\dot{h}(s)ds = \int_0^t \gamma_s\dot{h}(s)\gamma_s^{-1}dt.$$

Then,

$$\begin{aligned} d(Y_t\gamma_t) = dY_t\circ\gamma_t + Y_t\circ d\gamma_t &= \gamma_t\dot{h}(t)\gamma_t^{-1}dt\circ\gamma_t + Y_t\circ(\gamma_t\circ dB_t) \\ &= \gamma_t\dot{h}(t)dt + (Y_t\gamma_t)\circ dB_t. \end{aligned}$$

By the uniqueness of the solution, we have

$$\langle D\gamma_t, h\rangle = Y_t\gamma_t$$

and therefore

$$\gamma_t^{-1}\langle D\gamma_t, h\rangle = \gamma_t^{-1}Y_t\gamma_t = \mathrm{Ad}(\gamma_t^{-1})\int_0^t \mathrm{Ad}(\gamma_s)\dot{h}(s)ds$$

which is (14).

Using the notation (8), we can rewrite (14) as

$$\gamma_t^{-1}\langle D\gamma_t, h\rangle = \mathrm{Ad}(\gamma_t^{-1})(\gamma\cdot h)(t). \tag{15}$$

Since $\gamma_T = \pi\circ I$, we have

$$\gamma_T^{-1}\langle D\gamma_T, h\rangle = \mathrm{Ad}(\gamma_T^{-1})(\gamma\cdot h)(T) = \pi_* I_*(h). \tag{16}$$

We have had a covariant differentiation on $P\mathfrak{g}$, we can transfer it to $PG$, since $P\mathfrak{g}$ and $PG$ are isomorphic (up to constant) to each other as Riemannian manifolds. Let us calculate the covariant derivative.

**Proposition 2.2** *Let* $\mathbf{X}^h$, $\mathbf{X}^k$ *be right invariant vector fields associated with* $h$, $k \in H$. *Then we have*

$$\hat{\nabla}_{\mathbf{X}^h}\mathbf{X}^k = \mathbf{X}^l \tag{17}$$

*where*

$$l(t) = -\int_0^t [h(s), \dot{k}(s)]ds.$$

*Moreover we have*

$$[\mathbf{X}^h, \mathbf{X}^k] := \hat{\nabla}_{\mathbf{X}^h}\mathbf{X}^k - \hat{\nabla}_{\mathbf{X}^k}\mathbf{X}^h = -\mathbf{X}^{[h,k]}. \tag{18}$$

*Here* $[h,k](t) := [h(t), k(t)]$ *and* $[\ ,\ ]$ *is the Lie bracket.*

*Proof.* First we note that, for $\gamma = I(B)$,

$$I_*^{-1}(\mathbf{X}^k) = \gamma^{-1}\cdot k.$$

Therefore

$$D(\gamma^{-1}\cdot k)(t) \quad = \quad D(\int_0^t \mathrm{Ad}(\gamma_s^{-1})\dot{k}(s)ds)$$

$$
\begin{aligned}
&= D(\int_0^t \gamma_s^{-1}\dot{k}(s)\gamma_s ds) \\
&= -\int_0^t \gamma_s^{-1}D\gamma_s\gamma_s^{-1}\dot{k}(s)\gamma_s ds + \int_0^t \gamma_s^{-1}\dot{k}(s)\gamma_s\gamma_s^{-1}D\gamma_s ds \\
&= -\int_0^t [\gamma_s^{-1}D\gamma_s, \mathrm{Ad}(\gamma_s^{-1})\dot{k}(s)]ds.
\end{aligned}
$$

On the other hand

$$
\gamma_s^{-1}\langle D\gamma_s, \gamma^{-1}\cdot h\rangle = \mathrm{Ad}(\gamma_s^{-1})\int_0^s \mathrm{Ad}(\gamma_u)\mathrm{Ad}(\gamma_u^{-1})\dot{h}(u)du = \mathrm{Ad}(\gamma_s^{-1})h(s).
$$

Therefore

$$
\begin{aligned}
\langle D(\gamma^{-1}\cdot k)(t), \gamma^{-1}\cdot h\rangle &= -\int_0^t [\mathrm{Ad}(\gamma_s^{-1})h(s), \mathrm{Ad}(\gamma_s^{-1})\dot{k}(s)]ds \\
&= -\int_0^t \mathrm{Ad}(\gamma_s^{-1})[h(s), \dot{k}(s)]ds.
\end{aligned}
$$

This means

$$
\langle D(\gamma^{-1}\cdot h), \gamma^{-1}\cdot k\rangle = -\int_0^{\cdot} \mathrm{Ad}(\gamma_s^{-1})[h(s), \dot{k}(s)]ds.
$$

We eventually arrive at

$$
\begin{aligned}
I_*\langle D(\gamma^{-1}\cdot h), \gamma^{-1}\cdot k\rangle &= -\int_0^{\cdot} \mathrm{Ad}(\gamma_s)\mathrm{Ad}(\gamma_s^{-1})[h(s), \dot{k}(s)]ds \\
&= -\int_0^{\cdot} [h(s), \dot{k}(s)]ds.
\end{aligned}
$$

Then the above calculation shows that

$$
\hat{\nabla}_{\mathbf{X}^h}\mathbf{X}^k = \mathbf{X}^l.
$$

Next we show (18). Note that

$$
-\int_0^t [h(s), \dot{k}(s)]ds + \int_0^t [k(s), \dot{h}(s)]ds = -\int_0^t \frac{d}{ds}[h(s), k(s)]ds = -[h(t), k(t)].
$$

Now we easily have

$$
[\mathbf{X}^h, \mathbf{X}^k] = \hat{\nabla}_{\mathbf{X}^h}\mathbf{X}^k - \hat{\nabla}_{\mathbf{X}^k}\mathbf{X}^h = -\mathbf{X}^{[h,k]}
$$

as desired. ∎

The Lie bracket $[\mathbf{X}^k, \mathbf{X}^h]$ agrees with the usual definition. For the reason why the minus sign appears, see, e.g., Helgason [20, Lemma II.3.5].

Lastly we calculate the divergence. We denote the dual operator of $D$ (with respect to $P^W$) by $\delta$. For a vector field $\mathbf{X}$, we define $\mathrm{div}\mathbf{X} = -\delta\mathbf{X}^\flat$, $\mathbf{X}^\flat$ being the 1-form associated with $\mathbf{X}$, i.e.,

$$\mathbf{X}^\flat(\mathbf{Y}) = (\mathbf{X}|\mathbf{Y}).$$

For $h \in H$, we denote $(\mathbf{X}^h)^\flat$ by $\boldsymbol{\omega}^{\boldsymbol{h}}$. Then we have the following:

**Proposition 2.3** *For $h \in H$,*

$$\mathrm{div}\mathbf{X}^h = -\delta\boldsymbol{\omega}^{\boldsymbol{h}} = -\int_0^T (\dot{h}(t)|db_t)_{\mathfrak{g}}. \tag{19}$$

*Here $(b_t)$ is the Brownian motion defined by (4).*

*Proof.* The vector field corresponding to $\mathbf{X}^h$ is $\gamma^{-1} \cdot h$ that is given by

$$(\gamma^{-1} \cdot h)(t) = \int_0^t \mathrm{Ad}(\gamma_s^{-1})\dot{h}(s)ds.$$

Note that the integrand is adapted. Then it is well-known that $\delta(\gamma^{-1} \cdot h)$ is the stochastic integral:

$$\delta(\gamma^{-1} \cdot h) = \int_0^T (\mathrm{Ad}(\gamma_s^{-1})\dot{h}(s)|dB_s)_{\mathfrak{g}}.$$

Noticing that $db_s = \mathrm{Ad}(\gamma_s)dB_s$ and $(\cdot|\cdot)_{\mathfrak{g}}$ is $\mathrm{Ad}(G)$-invariant, we have

$$\begin{aligned}
\mathrm{div}\mathbf{X}^h &= -\delta(\gamma^{-1} \cdot h) \\
&= -\int_0^T (\mathrm{Ad}(\gamma_s^{-1})\dot{h}(s)|dB_s)_{\mathfrak{g}} \\
&= -\int_0^T (\dot{h}(s)|\mathrm{Ad}(\gamma_s)dB_s)_{\mathfrak{g}} \\
&= -\int_0^T (\dot{h}(s)|db_s)_{\mathfrak{g}}
\end{aligned}$$

which is the desired result. ∎

## 3 Based loop group

Now we proceed to a pinned space. Our interest is in a based loop space, but we consider more general situations. For a fixed $g \in G$, set

$$(PG)_g := \pi^{-1}(g) = \{\gamma \in PG\,; \gamma_T = g\}. \tag{20}$$

In particular, $(PG)_e = \Omega G$. The Cameron-Martin space associated to $(PG)_g$ is given by

$$H_0 = \{h \in H\,; h(T) = 0\}. \tag{21}$$

We can define right invariant vector fields as in the path space case. For $h \in H_0$, let $\{\varphi_u; u \in \mathbb{R}\}$ be a one-parameter subgroup in $(PG)_g$ defined by

$$\varphi_u(t) = \exp\{uh(t)\}.$$

$\{\varphi_u\}$ defines a one-parameter transformation group on $(PG)_g$ as

$$\Phi_u(\gamma) = \varphi_u \gamma.$$

Now

$$\mathbf{X}^h f(\gamma) = \frac{d}{du} f(\Phi_u(\gamma))\Big|_{u=0} \quad \text{for } f \in \mathcal{F}C^\infty((PG)_g),$$

where $\mathcal{F}C^\infty((PG)_g)$ is the set of all functions $f\colon (PG)_g \to \mathbb{R}$ of the form

$$f(\gamma) = F(\gamma_{t_1}, \dots, \gamma_{t_n}), \quad 0 < t_1 < \dots < t_n < 1,\ F \in C^\infty(G^n).$$

We use the same notation as in the path space case, because $\mathbf{X}^h$ is just a restriction to $(PG)_g$.

We can easily presume that the space $(PG)_g$ corresponds to a submanifold of the Wiener space defined by

$$S_g = \{B \in P\mathfrak{g}; \gamma_T = g\} \tag{22}$$

We already have a calculus on a submanifold of the Wiener space (see, e.g. Airault [6] or [22]). First recall the Malliavin covariance. Since $(D\gamma_T)_B\colon H \to T(G)_g \cong \mathfrak{g}$, the Malliavin covariance of $\gamma_T$ is defined by

$$\sigma = \sum_\lambda \langle D\gamma_T, h_\lambda\rangle \otimes \langle D\gamma_T, h_\lambda\rangle \in T(G)_{\gamma_T} \otimes T(G)_{\gamma_T} \cong \mathfrak{g} \otimes \mathfrak{g} \tag{23}$$

where $\{h_\lambda\}$ is a c.o.n.s. in $H$. In other words,

$$\sigma = D\gamma_T \circ (D\gamma_T)^* \in \mathrm{Hom}(T^*(G)_{\gamma_T}, T(G)_{\gamma_T}) \cong T(G)_{\gamma_T} \otimes T(G)_{\gamma_T}.$$

Here the right hand side is the composite of the following mappings:

$$T^*(G)_{\gamma_T} \xrightarrow{(D\gamma_T)^*} H^* \cong H \xrightarrow{D\gamma_T} T(G)_{\gamma_T}$$

Let $\{h_\lambda\}$ be a c.o.n.s. of $H$. Now using (16), we have,

$$\begin{aligned}
\sigma &= \sum_\lambda \langle \gamma_T^{-1} D\gamma_T, h_\lambda\rangle \otimes \langle \gamma_T^{-1} D\gamma_T, h_\lambda\rangle \\
&= \sum_\lambda \mathrm{Ad}(\gamma_T^{-1})(\gamma \cdot h_\lambda) \otimes \mathrm{Ad}(\gamma_T^{-1})(\gamma \cdot h_\lambda) \\
&= \sum_\lambda \int_0^T \mathrm{Ad}(\gamma_T^{-1})\mathrm{Ad}(\gamma_s)\dot{h}_\lambda(s)ds \otimes \int_0^T \mathrm{Ad}(\gamma_T^{-1})\mathrm{Ad}(\gamma_s)\dot{h}_\lambda(s)ds \\
&= \sum_\lambda \sum_{i,j=1}^d \int_0^T (\mathrm{Ad}(\gamma_T^{-1}\gamma_s)\dot{h}_\lambda(s)|X_i)_{\mathfrak{g}} ds \\
&\qquad\qquad \times \int_0^T (\mathrm{Ad}(\gamma_T^{-1}\gamma_s)\dot{h}_\lambda(s)|X_j)_{\mathfrak{g}} ds\, X_i \otimes X_j \\
&= \sum_{i,j=1}^d \sum_\lambda \int_0^T (\dot{h}_\lambda(s)|\mathrm{Ad}(\gamma_s^{-1}\gamma_T)X_i)_{\mathfrak{g}} ds \\
&\qquad\qquad \times \int_0^T (\dot{h}_\lambda(s)|\mathrm{Ad}(\gamma_s^{-1}\gamma_T)X_j)_{\mathfrak{g}} ds\, X_i \otimes X_j \\
&= \sum_{i,j=1}^d \int_0^T (\mathrm{Ad}(\gamma_s^{-1}\gamma_T)X_i)|\mathrm{Ad}(\gamma_s^{-1}\gamma_T)X_j)_{\mathfrak{g}} ds\, X_i \otimes X_j \\
&= \sum_{i,j=1}^d \int_0^T (X_i|X_j)_{\mathfrak{g}} ds\, X_i \otimes X_j \\
&= T\sum_{i=1}^d X_i \otimes X_i.
\end{aligned}$$

Here we regard $\sigma$ as an element of $\mathfrak{g} \otimes \mathfrak{g}$ and we used in the fifth line that the inner product in $\mathfrak{g}$ is $\mathrm{Ad}(G)$-invariant, i.e. $\mathrm{Ad}(\gamma_T^{-1}\gamma_s)$ is orthogonal. The inverse of $\sigma$ is therefore given by

$$\sigma^{-1} = T^{-1}\sum_{i=1}^d \omega^i \otimes \omega^i \in \mathfrak{g}^* \otimes \mathfrak{g}^* \cong T^*(G)_{\gamma_T} \otimes T^*(G)_{\gamma_T}$$

where $\{\omega^1, \ldots, \omega^d\}$ is the dual basis of $\{X_1, \ldots, X_d\}$. Then the tangent space of $S_g$ is given by

$$T(S_g) = \{h \in H; \langle D\gamma_T, h\rangle = 0\}. \tag{24}$$

By this definition and (16), it is easy to see that $h \in T(S_g)$ if and only if $(\gamma \cdot h)(T) = I_*(h)(T) = 0$. Therefore $I_*\colon T(S)_B \to H_0$ is again an isomorphism preserving inner products. $S_g$ is isomorphic to $(PG)_g$ as a Riemannian manifold. We can identify $S_g$ and $(PG)_g$ with one another. On the submanifold $S_g$, we can define $\Gamma(T(S_g))$ to be the set of smooth sections of $T(S_g)$ (see, e.g. [22]) and therefore, by using $I$, we can define the set of smooth sections on $T((PG)_g)$ and denote it by $\Gamma(T((PG)_g))$.

It is easy to see that

$$H_0^\perp = \{X\psi; X \in \mathfrak{g}\} \tag{25}$$

where $\psi_t = t/\sqrt{T}$. Let $p\colon H \to H_0$, $q\colon H \to H_0^\perp$ be orthogonal projections. Then they can be written as

$$ph = h - \tfrac{h(T)}{\sqrt{T}}\psi \tag{26}$$

$$qh = \tfrac{h(T)}{\sqrt{T}}\psi. \tag{27}$$

The normal bundle is defined as the image of $q$ and denoted by $N((PG)_g)$. Since $\mathfrak{g} \ni X \mapsto X\psi \in H_0^\perp$ gives an isomorphism, we identify $H_0^\perp$ with $\mathfrak{g}$. Therefore $N((PG)_g) \cong (PG)_g \times \mathfrak{g}$

Now the covariant differentiation on $(PG)_g$ is defined by

$$\nabla_{\mathbf{X}}\mathbf{Y} := p\hat{\nabla}_{\mathbf{X}}\mathbf{Y}, \qquad \mathbf{X}, \mathbf{Y} \in \Gamma(T((PG)_g)).$$

The second fundamental form $A$ is given by

$$A(\mathbf{X}, \mathbf{Y}) = q\hat{\nabla}_{\mathbf{X}}\mathbf{Y} = \hat{\nabla}_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{X}}\mathbf{Y}.$$

**Proposition 3.1** *For $h, k \in H_0$,*

$$A(\mathbf{X}^h, \mathbf{X}^k) = -\frac{1}{\sqrt{T}}\int_0^T [h(s), \dot{k}(s)]ds \tag{28}$$

*Proof.* By Proposition 2.2, $\hat{\nabla}_{\mathbf{X}^h}\mathbf{X}^k = \mathbf{X}^l$ where

$$l(t) = -\int_0^t [h(s), \dot{k}(s)]ds.$$

Hence

$$A(\mathbf{X}^h, \mathbf{X}^k) = ql = -\frac{1}{\sqrt{T}} \int_0^T [h(s), \dot{k}(s)] ds,$$

which is the desired result. ∎

By using the second fundamental form, we can express the Riemannian curvature as follows (see, e.g. [31, Proposition 26], [22, Theorem 3.5]):

$$R(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) = -(A(\mathbf{X}, \mathbf{Z})|A(\mathbf{Y}, \mathbf{W}))_{\mathfrak{g}} + (A(\mathbf{Y}, \mathbf{Z})|A(\mathbf{X}, \mathbf{W}))_{\mathfrak{g}}. \tag{29}$$

Let us introduce the area measure $m$ on $(PG)_g$ as follows (for the area measure, see [7]):

$$m = \delta_g(\gamma_T)\sqrt{\det \sigma}.$$

Since $\det \sigma = T^d$, we have

$$m = T^{d/2}\, p_T(e, g) E[\,\cdot\, |\gamma_T = g]$$

Here $p_T(e, g)$ denotes the probability density function of the law of $\gamma_T$. Thus $m$ differs from the pinned measure $E[\,\cdot\, |\gamma_T = g]$ up to constant. Let $\nabla^*$ be the dual operator of $\nabla$ with respect to $m$. Then $\nabla^*$ can be written as

$$\nabla^* = \delta - \frac{1}{2} i((D_{\gamma_T} \log \det \sigma)^\sharp)$$

(see, [23, §2]) where $i(\cdot)$ denotes the interior product. Therefore we have $\nabla^* = \delta$ because $\det \sigma$ is constant. For a vector field $\mathbf{X}$, we define $\mathrm{div}\mathbf{X} = -\nabla^* \mathbf{X}^\flat$, $\mathbf{X}^\flat$ being the 1-form associated with $\mathbf{X}$, i.e.,

$$\mathbf{X}^\flat(\mathbf{Y}) = (\mathbf{X}|\mathbf{Y}).$$

In the sequel, we denote $(\mathbf{X}^h)^\flat$ by $\boldsymbol{\omega}^{\boldsymbol{h}}$. By Proposition 2.3, we easily have

**Proposition 3.2** *For $h \in H_0$,*

$$\mathrm{div}\mathbf{X}^h = -\nabla^* \boldsymbol{\omega}^{\boldsymbol{h}} = -\int_0^T (\dot{h}(t)|db_t)_{\mathfrak{g}}. \tag{30}$$

*Here $(b_t)$ is the Brownian motion defined by (4).*

Let us proceed to define the Ricci curvature. By using the second fundamental form, we can write the Ricci curvature as follows (see, e.g. [15], [23]):

$$\mathrm{Ric}(\cdot, \cdot) = -\sum_\lambda (A(\cdot, \mathbf{X}^{h_\lambda})|A(\cdot, \mathbf{X}^{h_\lambda})) + (A(\cdot, \cdot)|q\delta q) \tag{31}$$

where $\{h_\lambda\}$ is a c.o.n.s. of $H_0$. First we calculate $q\delta q$.

**Lemma 3.3** *It holds that*

$$q\delta q = \frac{b_T}{\sqrt{T}}. \tag{32}$$

*Proof.* We first note that the projection operator to $T(S_g)$ which we denote by $Q$, can be written as follows:

$$\begin{aligned} Qh &= I_*^{-1}(q(I_* h)) \\ &= I_*^{-1}(q(\gamma \cdot h)) \\ &= I_*^{-1}\Big(\frac{1}{\sqrt{T}}(\gamma \cdot h)(T)\psi\Big) \\ &= \frac{1}{T}\int_0^{\cdot} \mathrm{Ad}(\gamma_t^{-1})(\gamma \cdot h)(T)dt. \end{aligned}$$

For a c.o.n.s. $\{h_\lambda\}$ of $H$, we can write

$$\delta Q = \delta\{\sum_\lambda (h_\lambda|Q(\cdot))_H \otimes h_\lambda\} = \sum_\lambda \delta(h_\lambda|Q(\cdot))_H h_\lambda.$$

Here for $h \in H$,

$$\begin{aligned} (h_\lambda|Q(h)) &= \frac{1}{T}\int_0^T \Big(\dot{h}_\lambda(t)\Big|\mathrm{Ad}(\gamma_t^{-1})\int_0^T \mathrm{Ad}(\gamma_s)\dot{h}(s)ds\Big)_{\mathfrak{g}} dt \\ &= \frac{1}{T}\int_0^T\int_0^T (\mathrm{Ad}(\gamma_s^{-1}\gamma_t)\dot{h}_\lambda(t)|\dot{h}(s))_{\mathfrak{g}} ds\, dt. \end{aligned}$$

Hence

$$\begin{aligned} (h_\lambda|Q(\cdot))_H &= \frac{1}{T}\int_0^{\cdot} ds \int_0^T \mathrm{Ad}(\gamma_s^{-1}\gamma_t)\dot{h}_\lambda(t)dt \\ &= \frac{1}{T}\int_0^{\cdot} ds \int_0^s \mathrm{Ad}(\gamma_s^{-1}\gamma_t)\dot{h}_\lambda(t)dt \\ &\quad + \frac{1}{T}\int_0^{\cdot} ds \int_s^T \mathrm{Ad}(\gamma_s^{-1}\gamma_t)\dot{h}_\lambda(t)dt. \end{aligned}$$

Here we identify $H^*$ with $H$. Since the integrand of the first term is adapted, we have

$$\delta\Big(\frac{1}{T}\int_0^{\cdot} ds \int_0^s \mathrm{Ad}(\gamma_s^{-1}\gamma_t)\dot{h}_\lambda dt\Big) = \frac{1}{T}\int_0^T \Big(\int_0^s \mathrm{Ad}(\gamma_s^{-1}\gamma_t)\dot{h}_\lambda(t)dt\Big|dB_s\Big)_{\mathfrak{g}}$$

$$= \frac{1}{T}\int_0^T \Big(\int_0^s \mathrm{Ad}(\gamma_t)\dot{h}_\lambda(t)dt\Big|\mathrm{Ad}(\gamma_s)dB_s\Big)_{\mathfrak{g}}$$
$$= \frac{1}{T}\int_0^T ((\gamma\cdot h_\lambda)(s)|db_s)_{\mathfrak{g}}.$$

We calculate the second term. To do this, we use an approximation argument. Let $\triangle = \{0 = s_0 < s_1 < \cdots < s_N = T\}$ be a partition of $[0,T]$ and we set $|\triangle| = \max\{s_k - s_{k-1}; k = 1,\ldots,N\}$. Then

$$\frac{1}{T}\int_0^{\cdot} ds \int_s^T \mathrm{Ad}(\gamma_s^{-1}\gamma_t)\dot{h}_\lambda(t)dt$$
$$= \lim_{|\triangle|\to 0}\frac{1}{T}\sum_{k=1}^N \int_{s_k}^T \mathrm{Ad}(\gamma_{s_k}^{-1}\gamma_t)\dot{h}_\lambda(t)dt \int_0^{\cdot} 1_{[s_{k-1},s_k]}(s)ds.$$

Here, the convergence in the right hand side is the strong convergence in $L^p(P\mathfrak{g}, P^W)$ for all $p \geq 1$. On the other hand, since $(\gamma_t)$ is a left Brownian motion, it holds that, for $t \geq s$

$$\gamma_s^{-1}\gamma_t = \gamma_{t-s}(\theta_s B)$$

where $\theta_s$ is a shift operator: $(\theta_s B)_u = B_{s+u} - B_s$. We can see that for $t \geq s$

$$\langle D\mathrm{Ad}(\gamma_s^{-1}\gamma_t), h\rangle = 0$$

if $\mathrm{supp}\dot{h} \subseteq [0,s]$. By virtue of this fact, we have

$$\delta\Big\{\frac{1}{T}\sum_{k=1}^N \int_{s_k}^T \mathrm{Ad}(\gamma_{s_k}^{-1}\gamma_t)\dot{h}_\lambda(t)dt\int_0^{\cdot} 1_{[s_{k-1},s_k]}(s)ds\Big\}$$
$$= -\frac{1}{T}\sum_{k=1}^N\sum_{i=1}^d \Big(D\int_{s_k}^T (\mathrm{Ad}(\gamma_{s_k}^{-1}\gamma_t)\dot{h}_\lambda(t)|X_i)_{\mathfrak{g}}dt\Big|\int_0^{\cdot} X_i 1_{[s_{k-1},s_k]}(s)ds\Big)_H$$
$$+\frac{1}{T}\sum_{k=1}^N\sum_{i=1}^d \int_{s_k}^T (\mathrm{Ad}(\gamma_{s_k}^{-1}\gamma_t)\dot{h}_\lambda(t)|X_i)_{\mathfrak{g}}dt\,\delta\Big\{\int_0^{\cdot} X_i 1_{[s_{k-1},s_k]}(s)ds\Big\}$$
$$= \frac{1}{T}\sum_{k=1}^N \Big(\int_{s_k}^T \mathrm{Ad}(\gamma_{s_k}^{-1}\gamma_t)\dot{h}_\lambda(t)dt\Big|B_{s_k} - B_{s_{k-1}}\Big)_{\mathfrak{g}}$$
$$= \frac{1}{T}\sum_{k=1}^N\sum_{l=k}^{N-1}\Big(\int_{s_l}^{s_{l+1}} \mathrm{Ad}(\gamma_t)\dot{h}_\lambda(t)dt\Big|\mathrm{Ad}(\gamma_{s_k})(B_{s_k} - B_{s_{k-1}})\Big)_{\mathfrak{g}}$$

$$= \frac{1}{T}\sum_{l=1}^{N-1}\Big(\int_{s_l}^{s_{l+1}}\mathrm{Ad}(\gamma_t)\dot{h}_\lambda(t)dt\Big|\sum_{k=1}^{l}\mathrm{Ad}(\gamma_{s_k})(B_{s_k}-B_{s_{k-1}})\Big)_{\mathfrak{g}}.$$

We notice that

$$\begin{aligned}\lim_{|\triangle|\to 0}\sum_{k:\, s_k\le t}\mathrm{Ad}(\gamma_{s_k})(B_{s_k}-B_{s_{k-1}}) &= 2\int_0^t \mathrm{Ad}(\gamma_s)\circ dB_s - \int_0^t \mathrm{Ad}(\gamma_s)dB_s \\ &= b_t.\end{aligned}$$

Here we used Proposition 2.1. Accordingly, by using the Itô formula, we have

$$\begin{aligned}&\lim_{|\triangle|\to 0}\delta\Big\{\frac{1}{T}\sum_{k=1}^{N}\int_{s_k}^{T}\mathrm{Ad}(\gamma_{s_k}^{-1}\gamma_t)\dot{h}_\lambda(t)dt\int_0^{\cdot}1_{[s_{k-1},s_k]}(s)ds\Big\} \\ &= \frac{1}{T}\int_0^T(\mathrm{Ad}(\gamma_t)\dot{h}_\lambda(t)|b_t)_{\mathfrak{g}}dt \\ &= \frac{1}{T}(\int_0^T\mathrm{Ad}(\gamma_t)\dot{h}_\lambda(t)dt|b_T)_{\mathfrak{g}} - \frac{1}{T}\int_0^T(\int_0^t\mathrm{Ad}(\gamma_s)\dot{h}_\lambda(s)ds|db_t)_{\mathfrak{g}} \\ &= \frac{1}{T}((\gamma\cdot h_\lambda)(T)|b_T)_{\mathfrak{g}} - \frac{1}{T}\int_0^T((\gamma\cdot h_\lambda)(t)|db_t)_{\mathfrak{g}}.\end{aligned}$$

Thus we have

$$\begin{aligned}\delta(h_\lambda|Q(\cdot))_H &= \frac{1}{T}\int_0^T((\gamma\cdot h_\lambda)(s)|db_s)_{\mathfrak{g}} + \frac{1}{T}((\gamma\cdot h_\lambda)(T)|b_T)_{\mathfrak{g}} \\ &\quad -\frac{1}{T}\int_0^T((\gamma\cdot h_\lambda)(t)|db_t)_{\mathfrak{g}} \\ &= \frac{1}{T}((\gamma\cdot h_\lambda)(T)|b_T)_{\mathfrak{g}}.\end{aligned}$$

By summing up over $\lambda$ we have

$$\sum_\lambda \delta(h_\lambda|q(\cdot))_H h_\lambda = \frac{1}{T}\sum_\lambda((\gamma\cdot h_\lambda)(T)|b_T)_{\mathfrak{g}}h_\lambda.$$

Now we have

$$\begin{aligned}q\delta q &= I_*(Q\delta Q) \\ &= \frac{1}{T}\sum_\lambda((\gamma\cdot h_\lambda)(T)|b_T)_{\mathfrak{g}}I_*(qh_\lambda)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{T}\sum_{\lambda}((\gamma\cdot h_\lambda)(T)|b_T)_{\mathfrak{g}}\frac{1}{\sqrt{T}}(\gamma\cdot h_\lambda)(T)\\
&= \frac{1}{T^{3/2}}\sum_{\lambda}\sum_{i=1}^{d}(\int_0^T \mathrm{Ad}(\gamma_s)\dot{h}_\lambda(s)ds|b_T)_{\mathfrak{g}}(\int_0^T \mathrm{Ad}(\gamma_s)\dot{h}_\lambda(s)ds|X_i)_{\mathfrak{g}}\,X_i\\
&= \frac{1}{T^{3/2}}\sum_{i=1}^{d}\sum_{\lambda}\int_0^T(\dot{h}_\lambda(s)|\mathrm{Ad}(\gamma_s^{-1})b_T)_{\mathfrak{g}}ds\int_0^T(\dot{h}_\lambda(s)|\mathrm{Ad}(\gamma_s^{-1})X_i)_{\mathfrak{g}}ds\,X_i\\
&= \frac{1}{T^{3/2}}\sum_{i=1}^{d}\int_0^T(\mathrm{Ad}(\gamma_s^{-1})b_T|\mathrm{Ad}(\gamma_s^{-1})X_i)_{\mathfrak{g}}ds\,X_i\\
&= \frac{1}{T^{3/2}}\sum_{i=1}^{d}T(b_T|X_i)_{\mathfrak{g}}\,X_i\\
&= \frac{1}{\sqrt{T}}b_T
\end{aligned}
$$

which completes the proof. ∎

Now we can calculate the Ricci curvature as follows:

**Proposition 3.4** *The Ricci curvature is written as*

$$
\begin{aligned}
\mathrm{Ric}(\mathbf{X}^h,\mathbf{X}^k) &= \frac{1}{T}\int_0^T K(h(s),k(s))ds+\frac{1}{\sqrt{T}}(b_T|A(\mathbf{X}^h,\mathbf{X}^k))_{\mathfrak{g}} \qquad (33)\\
&\quad -\frac{1}{T^2}\int_0^T\int_0^T K(h(s),k(t))dsdt
\end{aligned}
$$

*where $K$ is the Killing form defined by $K(X,Y)=\mathrm{tr}(\mathrm{ad}X\mathrm{ad}Y)$.*

*Proof.* It is enough to calculate $\sum_\lambda(A(\cdot,\mathbf{X}^{h_\lambda})|A(\cdot,\mathbf{X}^{h_\lambda}))$. Since $(\cdot,\cdot)_{\mathfrak{g}}$ is $\mathrm{Ad}(G)$ invariant, $\mathrm{ad}X$ is skew-symmetric for $X\in\mathfrak{g}$. Hence, for $X,Y\in\mathfrak{g}$,

$$
\begin{aligned}
\sum_i(\mathrm{ad}X(X_i)|\mathrm{ad}Y(X_i))_{\mathfrak{g}} &= -\sum_i(X_i|\mathrm{ad}X\mathrm{ad}Y(X_i))_{\mathfrak{g}}\\
&= -\mathrm{tr}(\mathrm{ad}X\mathrm{ad}Y)\\
&= -K(X,Y).
\end{aligned}
$$

Using this and noting that $\{h_\lambda\}\cup\{X_i\psi\}$ forms a c.o.n.s. of $H$, we have

$$
\sum_\lambda(A(\mathbf{X}^h,\mathbf{X}^{h_\lambda})|A(\mathbf{X}^k,\mathbf{X}^{h_\lambda}))_{\mathfrak{g}}
$$

$$
\begin{aligned}
&= \frac{1}{T}\sum_{\lambda}\Big(\int_0^T [h(s),\dot{h}_\lambda(s)]ds\Big| \int_0^T [k(t),\dot{h}_\lambda(t)]dt\Big)_{\mathfrak{g}} \\
&= \frac{1}{T}\sum_{\lambda}\sum_{i=1}^{d}\int_0^T (\mathrm{ad}h(s)(X_i)|\dot{h}_\lambda(s))_{\mathfrak{g}}ds\int_0^T (\mathrm{ad}k(t)(X_i)|\dot{h}_\lambda(s))_{\mathfrak{g}}dt \\
&= \frac{1}{T}\sum_{i=1}^{d}\int_0^T (\mathrm{ad}h(s)(X_i)|\mathrm{ad}k(s)(X_i))_{\mathfrak{g}}ds \\
&\quad -\frac{1}{T}\sum_{i,j=1}^{d}\int_0^T (\mathrm{ad}h(s)(X_i)|X_j\dot{\psi}(s))_{\mathfrak{g}}ds\int_0^T (\mathrm{ad}k(t)(X_i)|X_j\dot{\psi}(t))_{\mathfrak{g}}dt \\
&= -\frac{1}{T}\int_0^T K(h(s),k(s))ds + \frac{1}{T^2}\sum_{i=1}^{d}\int_0^T\int_0^T (\mathrm{ad}h(s)(X_i)|\mathrm{ad}k(t)(X_i))_{\mathfrak{g}}ds\,dt \\
&= -\frac{1}{T}\int_0^T K(h(s),k(s))ds + \frac{1}{T^2}\int_0^T\int_0^T K(h(s),k(t))ds\,dt.
\end{aligned}
$$

This completes the proof. ∎

We set $\Delta = -\nabla^*\nabla$. Here $\Delta$ acts on scalar-valued functions. Recall that the Bakry-Emery $\Gamma_2$ is defined by

$$\Gamma_2(f,g) := \frac{1}{2}\{\Delta(\nabla f|\nabla g) - (\nabla\Delta f|\nabla g) - (\nabla f|\nabla\Delta g)\}.$$

Then the following formula for $\Gamma_2$ can be found in Getzler [15] and Airault [6] (see also [23]).

$$\Gamma_2(f,g) = (\nabla^2 f|\nabla^2 g) + (\nabla f|\nabla g) + \mathrm{Ric}((\nabla f)^\sharp,(\nabla g)^\sharp). \tag{34}$$

We give an estimate for the norm of $A$. Let $M$ be a constant satisfying

$$|[X,Y]|_{\mathfrak{g}} \le M|X|_{\mathfrak{g}}|Y|_{\mathfrak{g}}. \tag{35}$$

Then we have

$$\|A\| := \sup_{|h|_H,\,|k|_H\le 1} |A(\mathbf{X}^h,\mathbf{X}^k)|_{\mathfrak{g}} \le M\sqrt{T}. \tag{36}$$

To see this,

$$|A(\mathbf{X}^h,\mathbf{X}^k)|_{\mathfrak{g}} \quad \le \quad \frac{1}{\sqrt{T}}\int_0^T |[h(t),\dot{k}(t)]|_{\mathfrak{g}}dt$$

$$
\begin{aligned}
&\leq \frac{M}{\sqrt{T}}\int_0^T |h(t)|_{\mathfrak{g}}|\dot{k}(t)|_{\mathfrak{g}}dt \\
&\leq \frac{M}{\sqrt{T}}\sup_{t\in[0,T]}|h(t)|_{\mathfrak{g}}\int_0^T |\dot{k}(t)|_{\mathfrak{g}}dt \\
&\leq \frac{M}{\sqrt{T}}\int_0^T |\dot{h}(t)|_{\mathfrak{g}}dt\int_0^T |\dot{k}(t)|_{\mathfrak{g}}dt \\
&\leq \frac{M}{\sqrt{T}}T\Big\{\int_0^T |\dot{h}(t)|^2_{\mathfrak{g}}dt\Big\}^{1/2}\Big\{\int_0^T |\dot{k}(t)|^2_{\mathfrak{g}}dt\Big\}^{1/2} \\
&= M\sqrt{T}|h|_H|k|_H.
\end{aligned}
$$

This shows (36).

The lower bound of the Ricci curvature is essential in the later argument. Since $G$ is compact, the Killing form $K$ is negative definite. Therefore, we have to estimate the first term in (33). We can see

$$|K(X,Y)| = |\sum_i(\mathrm{ad}X(X_i)|\mathrm{ad}Y(X_i))_{\mathfrak{g}}| \leq \sum_i M^2 d|X|_{\mathfrak{g}}|Y|_{\mathfrak{g}}.$$

Hence

$$
\begin{aligned}
\Big|\frac{1}{T}\int_0^T K(h(s),k(s))ds\Big| &\leq M^2 d\sup_{t\in[0,T]}|h(t)|_{\mathfrak{g}}\sup_{t\in[0,T]}|k(t)|_{\mathfrak{g}} \qquad (37)\\
&\leq M^2 dT|h|_H|k|_H.
\end{aligned}
$$

## 4 Spectral gap

The spectral gap for the operator $\nabla^*\nabla$ is a fundamental problem. In this section we give a sufficient condition for the spectral gap. But unfortunately, the author does not know any example satisfying this sufficient condition. First we change the reference measure. Set $\mu = e^{-2U}dm$. Here $U$ is a smooth function in the sense of Malliavin. We consider the following Dirichlet form in $L^2((PG)_g,\mu)$:

$$\mathcal{E}(f,h) = \int_{(PG)_g}(\nabla f|\nabla h)d\mu. \qquad (38)$$

We denote the norm in $L^p((PG)_g,\mu)$ by $\|\cdot\|_p$. We assume the following logarithmic Sobolev inequality for $\mathcal{E}$: there exists a constant $\lambda > 0$ and a non-negative potential function $V$ such that

$$\int_{(PG)_g} f^2\log(f^2/\|f\|_2^2)d\mu \leq \lambda\mathcal{E}(f,f) + \int_{(PG)_g} Vf^2 d\mu. \qquad (39)$$

The generator associated with $\mathcal{E}$ can be calculated as follows: for $\boldsymbol{\eta} \in \Gamma(T^*((PG)_g))$

$$\begin{aligned}\int_{(PG)_g} (\nabla f|\boldsymbol{\eta})e^{-2U}dm &= \int_{(PG)_g} f(\nabla^*\boldsymbol{\eta}e^{-2U} - (\nabla e^{-2U}|\boldsymbol{\eta})dm \\ &= \int_{(PG)_g} f(\nabla^*\boldsymbol{\eta} + 2(\nabla U|\boldsymbol{\eta}))e^{-2U}dm.\end{aligned}$$

Hence the dual operator of $\nabla$ with respect to $\mu$ is $\nabla^* + 2(\nabla U|\cdot)$. Therefore the generator $A_U$ can be written as

$$A_U = -(\nabla^* + 2(\nabla U|\cdot))\nabla = -\nabla^*\nabla - 2(\nabla U|\nabla\cdot) = \Delta - 2(\nabla U|\nabla\cdot).$$

Then, by (33), the Bakry-Emery $\Gamma_2$ in this case is given by

$$\begin{aligned}\Gamma_2(f,g) &= \frac{1}{2}\{A_U(\nabla f|\nabla g) - (\nabla A_U f|\nabla g) - (\nabla f|\nabla A_U g)\} \\ &= (\nabla^2 f|\nabla^2 g) + (\nabla f|\nabla g) + \mathrm{Ric}((\nabla f)^\sharp, (\nabla g)^\sharp) \\ &\quad + 2\nabla^2 U((\nabla f)^\sharp, (\nabla g)^\sharp).\end{aligned}$$

Clearly $A_U$ is non-negative definite and $A_U 1 = 0$. The following lemma is easy:

**Lemma 4.1** *If there exists a constant $c > 0$ such that*

$$\int_{(PG)_g} \Gamma_2(f,f)d\mu \geq c\int_{(PG)_g} (\nabla f|\nabla f)d\mu. \tag{40}$$

*Then $A_U$ has a spectral gap. To be precise, denoting the set of spectrum of $-A_U$ by $\sigma(-A_U)$, it holds that*

$$\inf\{\sigma(-A_U) \setminus \{0\}\} \geq c. \tag{41}$$

*Proof.* Note that

$$\int_{(PG)_g} \Gamma_2(f,f)d\mu = \int_{(PG)_g} (A_U f)^2 d\mu$$

and

$$\int_{(PG)_g} (\nabla f|\nabla f)d\mu = -\int_{(PG)_g} (A_U f)f d\mu.$$

Then the assertion is an easy consequence of the spectral theorem. ∎

**Remark 4.1** The above proof shows that (40) and (41) are equivalent to one another.

Now we have the following main theorem.

**Theorem 4.2** *Assume that there exists a constant $c > 0$ such that for any $h \in H_0$,*

$$\Big(1 - dM^2T + \frac{1}{\lambda} - \frac{1}{\lambda}\|e^{\lambda M|b_T|}\|_1 - \frac{V}{\lambda} - c\Big)|\mathbf{X}^h|^2 + 2\nabla^2 U(\mathbf{X}^h, \mathbf{X}^h) \geq 0. \tag{42}$$

*Then $A_U$ has a spectral gap.*

*Proof.* We check the assumption of Lemma 4.1. By Proposition 3.4, (36) and (37), we easily have

$$\begin{aligned}\mathrm{Ric}(\mathbf{X}^h, \mathbf{X}^h) &\geq -(|b_T|/\sqrt{T})\|A\||h|_H^2 - dM^2T|h|_H^2 \\ &\geq -M|b_T||h|_H^2 - dM^2T|h|_H^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\Gamma_2(f,f) \\ &\quad= (\nabla^2 f|\nabla^2 f) + (\nabla f|\nabla f) + \mathrm{Ric}((\nabla f)^\sharp, (\nabla f)^\sharp) + 2\nabla^2 U((\nabla f)^\sharp, (\nabla f)^\sharp) \\ &\quad\geq |\nabla^2 f|^2 + |\nabla f|^2 - M|b_T||\nabla f|^2 - dM^2T|\nabla f|^2 + 2\nabla^2 U((\nabla f)^\sharp, (\nabla f)^\sharp).\end{aligned}$$

Now we notice that from (39) we have

$$\int_{(PG)_g} |\nabla f|^2 \log(|\nabla f|^2/\|\nabla f\|_2^2) d\mu \leq \lambda \int_{(PG)_g} |\nabla^2 f|^2 d\mu + \int_{(PG)_g} V|\nabla f|^2 d\mu. \tag{43}$$

Further recall the Young inequality $st \leq s\log s - s + e^t$ $(s > 0, t \in \mathbb{R})$. Combining these inequalities, we have

$$\begin{aligned}&\int_{(PG)_g} \Gamma_2(f,f) d\mu \\ &\quad\geq \int_{(PG)_g} |\nabla^2 f|^2 d\mu + (1 - dM^2T)\int_{(PG)_g} |\nabla f|^2 d\mu \\ &\qquad - \int_{(PG)_g} M|b_T||\nabla f|^2 d\mu + \int_{(PG)_g} 2\nabla^2 U((\nabla f)^\sharp, (\nabla f)^\sharp) d\mu \\ &\quad\geq \int_{(PG)_g} |\nabla^2 f|^2 d\mu + (1 - dM^2T)\int_{(PG)_g} |\nabla f|^2 d\mu\end{aligned}$$

$$
\begin{aligned}
&\quad -\frac{1}{\lambda}\int_{(PG)_g}|\nabla f|^2(\lambda M|b_T|)d\mu+\int_{(PG)_g}2\nabla^2U((\nabla f)^\sharp,(\nabla f)^\sharp)d\mu \\
&\geq \int_{(PG)_g}|\nabla^2 f|^2d\mu+(1-dM^2T)\int_{(PG)_g}|\nabla f|^2d\mu \\
&\quad -\frac{1}{\lambda}\int_{(PG)_g}|\nabla f|^2\log|\nabla f|^2d\mu+\frac{1}{\lambda}\int_{(PG)_g}|\nabla f|^2d\mu \\
&\quad -\frac{1}{\lambda}\int_{(PG)_g}e^{\lambda M|b_T|}d\mu+2\int_{(PG)_g}\nabla^2U((\nabla f)^\sharp,(\nabla f)^\sharp)d\mu \\
&\geq \int_{(PG)_g}|\nabla^2 f|^2d\mu+(1-dM^2T)\int_{(PG)_g}|\nabla f|^2d\mu-\int_{(PG)_g}|\nabla^2 f|^2d\mu \\
&\quad -\frac{1}{\lambda}\int_{(PG)_g}V|\nabla f|^2d\mu-\frac{1}{\lambda}\|\nabla f\|_2^2\log\|\nabla f\|_2^2+\frac{1}{\lambda}\int_{(PG)_g}|\nabla f|^2d\mu \\
&\quad -\frac{1}{\lambda}\int_{(PG)_g}e^{\lambda M|b_T|}d\mu+2\int_{(PG)_g}\nabla^2U((\nabla f)^\sharp,(\nabla f)^\sharp)d\mu \\
&\geq \int_{(PG)_g}\Big(1-dM^2T+\frac{1}{\lambda}-\frac{V}{\lambda}\Big)|\nabla f|^2d\mu-\|\nabla f\|_2^2\log\|\nabla f\|_2^2 \\
&\quad -\frac{1}{\lambda}\int_{(PG)_g}e^{\lambda M|b_T|}d\mu+2\int_{(PG)_g}\nabla^2U((\nabla f)^\sharp,(\nabla f)^\sharp)d\mu.
\end{aligned}
$$

Replacing $f$ by $f/\|\nabla f\|_2$ (we may assume that $\|\nabla f\|_2\neq 0$), we have

$$
\begin{aligned}
\int_{(PG)_g}\Gamma_2(f,f)d\mu &\geq \int_{(PG)_g}\Big\{\Big(1-dM^2T+\frac{1}{\lambda}-\frac{1}{\lambda}\|e^{\lambda M|b_T|}\|_1-\frac{V}{\lambda}\Big)|\nabla f|^2 \\
&\qquad +2\nabla^2U((\nabla f)^\sharp,(\nabla f)^\sharp)\Big\}d\mu \\
&\geq c\int_{(PG)_g}|\nabla f|^2d\mu.
\end{aligned}
$$

This completes the proof. ∎

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