Square root of a Schrödinger operator and its L^p norms

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The following norm equivalence is well-known on the Euclidean space:

$$\|\sqrt{-\triangle}f\|_p \sim \|\nabla f\|_p. \tag{1}$$

Here, $\triangle$ denotes the Laplacian, $\|\cdot\|_p$ denotes the L^p norm $(1 < p < \infty)$ and the notation $A \sim B$ means that $cA \leq B \leq CA$ for some constants $c > 0$ and $C > 0$ which is independent of f. This equivalence leads the L^p-boundedness of the Riesz transform, which is formally expressed by $\nabla\sqrt{-\triangle}^{-1}$. Moreover the equivalence is extended to Riemannian manifolds (at least compace case).

In this talk, we extend this equiavlence to the case of Schrödinger operator $\triangle - V$ on a Riemannian manifold M. Here $\triangle$ is the Laplace-Beltrami operator and V is a scalar function. We assume the following. First, the Ricci curvature is bounded from below. Second, the potential function V is bounded from below. By adding a positive constant, we can and do assume that V is uniformly positive. This is just for notational convenience. We further assume that $\nabla V/\max\{V, 1\}$ and $\triangle V/\max\{V, 1\}$ are bounded. Under these condtions we have the following

Theorem 1. *For $1 < p < \infty$, the following norm equivalence holds*

$$\|\sqrt{V - \triangle}f\|_p \sim \|\nabla f\|_p + \|\sqrt{V}f\|_p, \quad \forall f \in C_0^\infty(M). \tag{2}$$

To show the theorem above, the following two properties are fundamental.

- the intertwining property
- the Littlewood-Paley inequality

The first one takes the following form.

$$\sqrt{V}(\triangle - V) = A\sqrt{V}. \tag{3}$$

The operator A satisfying this condition is given by

$$A = \triangle + b - \frac{1}{2}\nabla^* b + \frac{1}{4}|b|^2 - V \tag{4}$$

where $b = -\nabla V/V$.

To state the second one, we need to introduce the Littlewood-Paley G-functions. They are defined as follows:

$$G^{\rightarrow}f(x) = \left\{\int_0^\infty t|\partial_t e^{-t\sqrt{V-\triangle}}f(x)|^2 dt\right\}^{1/2},$$
$$G^{\uparrow}f(x) = \left\{\int_0^\infty t|\nabla e^{-t\sqrt{V-\triangle}}f(x)|^2 dt\right\}^{1/2},$$
$$G^{V}f(x) = \left\{\int_0^\infty t|\sqrt{V}e^{-t\sqrt{V-\triangle}}f(x)|^2 dt\right\}^{1/2}.$$

We have the following domination which is called the Littlewood-Paley inequality.

Proposition 2. *For $1 < p < \infty$, it holds that*

$$\|f\|_p \lesssim \|G^{\rightarrow} f\|_p, \lesssim \|f\|_p$$
$$\|G^{\uparrow} f\|_p \lesssim \|f\|_p,$$
$$\|G^{V} f\|_p \lesssim \|f\|_p.$$

Here the notation $A \lesssim B$ means that $A \le CB$ for some constants $C > 0$ which is independent of f.

To combine this with the intertwining property, we need to introduce the Littlewood-Paley G-functions for the operator A. To do this, we just replace $\triangle - V$ with A and denote the Littlewood-Paley G-functions by $G_A^{\rightarrow}$, $G_A^{\uparrow}$, etc. Similar inequality holds for A, e.g., $\|f\|_p \lesssim \|G_A^{\rightarrow} f\|_p \lesssim \|f\|_p$. The intertwining property yields that $G_A^{\rightarrow}\sqrt{V} f = G^V f$. Using this relation, we can show that

$$\|\sqrt{V} f\|_p \lesssim \|\sqrt{V - \triangle} f\|_p.$$

Remaining inequality can be shown similarly.

So far, we have considered $\sqrt{V - \triangle}$. If we consider $\triangle - V$ itself, then we have

Theorem 3. *For $1 < p < \infty$, the following norm equivalence holds*

$$\|(\triangle - V) f\|_p \sim \|\triangle f\|_p + \|V f\|_p, \quad \forall f \in C_0^{\infty}(M). \tag{5}$$

We can also extend the above theorem for the Hodeg-Kodaira operator $dd^* + d^*d$ plus the potential V. In this case, we need the positivity of the Riemannian curvature.

References

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