Comparison theorem and logarithmic Sobolev inequality

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Let $(M, \mathcal{B}, m)$ be a σ-finite measure space. Suppose we are given a Markovian contraction semigroup $\{T_t\}$ on $L^2(m)$. Further, we are given a semigroup $\{\vec{T}_t\}$ acting on Hilbert space valued functions. We denote the Hilbert space by H. We denote the norm of the Hilbert space by $|\cdot|$. We are interested in when

$$|\vec{T}_t u| \leq T_t|\theta|, \quad {}^\forall\theta \in L^2(m; H), \tag{1}$$

holds. We can give a sufficient condition for (1) in terms of square field operator. We denote the generators and bilinear forms associated with $\{T_t\}$, $\{\vec{T}_t\}$ by A, $\vec{A}$ and $\mathcal{E}$, $\vec{\mathcal{E}}$. We assume that the square field operator

$$\Gamma(f, g) = \frac{1}{2}\{A(fg) - (Af) - f(Ag)\} \tag{2}$$

is well-defined and satisfies the derivation property.

We introduce the following property for $\{\vec{T}_t\}$:

$(\vec{\Gamma}_\lambda)$ For θ, $\eta \in \mathrm{Dom}(\vec{A}_2)$, we have $(\theta|\eta)_H \in \mathrm{Dom}(A_1)$ and there exists $\lambda \in \mathbb{R}$ such that

$$A_1|\theta|^2 - 2(\vec{A}_2\theta|\theta)_H + 2\lambda|\theta|^2 \geq 0. \tag{3}$$

Here A_1 denotes the generator in L^1 and $\vec{A}_2$ denotes the generator in L^2. We define $\vec{\Gamma}$ by

$$\vec{\Gamma}(\theta, \eta) = \frac{1}{2}\{A_1(\theta|\eta)_H - (\vec{A}_2\theta|\eta)_H - (\theta|\vec{A}_2\eta)_H\}. \tag{4}$$

$(\vec{D})$ For $\theta, \eta \in \mathrm{Dom}(\vec{\mathcal{E}}) \cap L^\infty$, $f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$, it holds that $(\theta|\eta)_H \in \mathrm{Dom}(\mathcal{E})$, $f\theta \in \mathrm{Dom}(\vec{\mathcal{E}})$ and

$$2f\vec{\Gamma}(\theta, \eta) = -\Gamma(f, (\theta|\eta)_H) + \vec{\Gamma}(\theta, f\eta) + \vec{\Gamma}(f\theta, \eta). \tag{5}$$

Under conditions of $(\vec{\Gamma}_\lambda)$ and $(\vec{D})$, we have $|\vec{T}_t\theta| \leq e^{\lambda t}T_t|\theta|$.

Typical example is the Hodge-Kodaira Laplacian acting on p-forms on a Riemannian manifold. If the manifold has no boundary, the above criterion works well. But it doesn't work if the manifold has a boundary. We attempt to extend the above criterion.

To define $\vec{\Gamma}$, the condition $(\vec{\Gamma}_\lambda)$ must be satisfied. But we notice the following identity.

$$-\mathcal{E}((\theta, \eta), f) + \vec{\mathcal{E}}(f\theta, \eta) + \vec{\mathcal{E}}(\theta, f\eta) = 2\int_M \vec{\Gamma}(\theta, \eta) f\,dm$$

The left had side is well-defined without $(\vec{\Gamma}_\lambda)$. For example, we consider a Riemannian manifold with a boundary. The Hodge-Kodaira Laplacian acting on 1-forms satisfies the following:

$$\begin{aligned}&-\mathcal{E}((\theta, \eta), f) + \mathcal{E}^{B_a}_{(1)}(f\theta, \eta) + \mathcal{E}^{B_a}_{(1)}(\theta, f\eta)\\ &\quad= 2\int_M (\nabla\theta, \nabla\eta) f\,dm - 2\int_{\partial M} f(\alpha(\theta, \eta), N)\,d\sigma + 2\int_M f\,\mathrm{Ric}(\theta, \eta)\,dm.\end{aligned}$$

Here

$$\mathcal{E}^{B_a}_{(1)}(\theta, \eta) = \int_M \{(d\theta, d\theta) + (d^*\theta, d^*\eta)\}dm$$

and B_a denotes the absolute boundary condition. α is the second fundamental form on the boundary ∂M, N is the inner unit normal vector field and Ric is the Ricci tensor. In this case, we have the following comparison theorem.

Theorem 1. *We assume that* $(\alpha(\theta,\theta),N)\le 0$ *and* $\mathrm{Ric}(\theta,\theta)\ge\lambda|\theta|^2$. *Then we have*

$$|\vec{T}_t\theta|\le e^{-\lambda t}T_t|\theta|. \tag{6}$$

We can also obtain similar criterion for p-forms. Further combining this result with the commutation theorem, we can reformulate Bakry-Emery criterion for the logarithmic Sobolev inequality.

References

[1] E. Ouhabaz, Invariance of closed convex sets and domination criteria for semigroups *Potential Analysis*, **5** (1996), 611–625.

[2] B. Simon, An abstract Kato's inequality for generators of positivity preserving semigroups, *Indiana Univ. Math. J.*, **26** (1997), 1069–1073.

[3] B. Simon, Kato's inequality and the comparison of semigroups, *J. Funct. Anal.*, **32** (1979), 97–101.

[4] I. Shigekawa, L^p contraction semigroups for vector valued functions, *J. Funct. Anal.*, **147** (1997), 69–108.