L^p multiplier theorem for Hodge-Kodaira Laplacian

Ichiro Shigekawa (Kyoto University)

We discuss the L^p multiplier thorem. In L^2 setting, it is well known that $\varphi(-L)$ is bounded if and only if φ is bounded where L is a non-positive self-adjoint operator. In L^p setting, the criterion above is no more true in general.

E. M. Stein gave a sufficient condition when L is a generator of a symmetric Markov process. It reads as follows: define a function φ on $[0, \infty)$ by

$$\varphi(\lambda) = \lambda \int_0^\infty e^{-2t\lambda} m(t) dt. \tag{1}$$

Here we assume that m is a bounded function. A typical example is $\varphi(\lambda) = \lambda^{i\alpha}$ ($\alpha \in \mathbb{R}$). Then Stein proved that $\varphi(-L)$is a bounded operator in L^p for $1 < p < \infty$. He also proved that the operator norm of $\varphi(-L)$ depends only on the bound of m and p.

In the meanwhile we consider the Hodge-Kodaira operator on a compact Riemannian manifold M. It is of the form $\vec{L} = -(dd^* + d^*d)$ where d is the exterior differentiation. A Typical feature is that $\vec{L}$ acts on vector valued functions (to be precise, differential forms on M). In this case, we can get the following theorem:

Theorem 1. *For sufficiently large κ, $\varphi(\kappa - \vec{L})$ is a bounded operator in L^p. Further the operator norm is estimated in terms of m and p only.*

To show this theorem, we use the following facts.

1. the semigroup domination.
2. the Littlewood-Paley inequality.

For the first, we can show that

$$|e^{(-\kappa+\vec{L})t}\theta| \le e^{-Lt}|\theta|. \tag{2}$$

Here L is the Laplace-Beltrami operator on M and the inequality holds pointwise. This inequality can be shown by means of Ouhabaz criterion ([1]). To use the criteiron, the following inequality is essential.

$$L|\theta|^2 - 2(\vec{L}\theta, \theta) + \kappa|\theta|^2 \ge 0.$$

For the second, we need the Littlewood-Paley function. This is somehow different from usual one. We may call it the Littlewood-Paley function of parabolic type. It is defined as follows:

$$\mathcal{P}\theta(x) = \left\{ \int_0^\infty |\nabla \vec{T}_t \theta(x)|^2 dt \right\}^{1/2}.$$

Here $\vec{T}_t$ denotes the semigroup $e^{(-\kappa+\vec{L})t}$. We can show the following inequality: there exists positive constant C independent of θ such that

$$\|\mathcal{P}\theta\|_p \le C\|\theta\|_p.$$

This inequality is called the Littlewood-Paley inequality.

Combining these two inequality we can show that

$$|(\varphi(\kappa-\vec{L})\theta,\eta)| \leq C_1\|\mathcal{P}\theta\|_p\|\mathcal{P}\eta\|_q \leq C_2\|\theta\|_p\|\eta\|_q.$$

Here q is the conjugate exponent of p. Now the desired result follows easily.

References

[1] E. Ouhabaz, Invariance of closed convex sets and domination criteria for semigroups *Potential Analysis*, **5** (1996), 611–625.

[2] E. M. Stein, "*Topics in harmonic analysis, related to Littlewood-Paley theory*," Annals of Math. Studies, 63, Princeton Univ. Press, 1974.

[3] N. T. Varopoulos; Aspects of probabilistic Littlewood-Paley theory, *J. Funct. Anal.*, **38** (1980), 25–60