Exponential convergence of Markovian semigroups and their spectra on L^p-spaces

Seiichiro Kusuoka[a*] and Ichiro Shigekawa[b†]

February 14, 2013

[a] Mathematical Institute, Tohoku University
6-3, Aramaki Aza-Aoba, Aoba-ku, Sendai 980-8578, Japan

[b] Department of Mathematics, Graduate School of Science,
Kyoto University,
Kita-Shirakawa, Sakyo-ku, Kyoto 606-8502, Japan

Abstract

Markovian semigroups on L^2-space with suitable conditions can be regarded as Markovian semigroups on L^p-spaces for $p \in [1, \infty)$. When we additionally assume the ergodicity of the Markovian semigroups, the rate of convergence on L^p-space for each p is considerable. However, the rate of convergence depends on the norm of the space. The purpose of this paper is to investigate the relation between the rates on L^p-spaces for different p, to obtain some sufficient condition for the rates to be independent of p, and to give an example that the rates depend on p. We also consider spectra of Markovian semigroups on L^p-spaces, because the rate of convergence is closely related to the spectra.

2010 AMS Classification Numbers: 60J25, 46E30, 47A10.

Key words: Markovian semigroup, rate of convergence, hypercontractivity, ultracontractivity, logarithmic Sobolev inequality, spectra on L^p-spaces.

1 Introduction

Let $(M, \mathscr{B})$ be a measurable space, m a probability measure on $(M, \mathscr{B})$ and $L^p(m)$ the L^p-space of $\mathbb{C}$-valued functions with respect to m. We denote the L^p-norm by $|| \cdot ||_p$, $\int f dm$ by $\langle f \rangle$ for $f \in L^1(m)$, and the constant function which takes values 1 by $\mathbf{1}$. A semigroup $\{T_t\}$ on $L^2(m)$ is called a Markovian semigroup if $0 \le T_t f \le 1$ m-almost everywhere whenever $f \in L^2(m)$ and $0 \le f \le 1$ m-almost everywhere.

*e-mail: kusuoka@math.tohoku.ac.jp
†e-mail: ichiro@math.kyoto-u.ac.jp

In this paper, we always assume that $T_t\mathbf{1} = \mathbf{1}$ for all $t \geq 0$. Let $\{T_t\}$ be a strongly continuous Markovian semigroup. We assume that $T_t^*\mathbf{1} = \mathbf{1}$ where T_t^* is the dual operator of T_t on $L^2(m)$. Then, as we will see in Section 2, the semigroup $\{T_t\}$ can be extended or restricted to a semigroups on $L^p(m)$ for $p \in [1, \infty]$. Moreover, $\{T_t\}$ is strongly continuous for $p \in [1, \infty)$. Let

$$\gamma_{p\to q} := -\limsup_{t\to\infty} \frac{1}{t} \log ||T_t - m||_{p\to q} \tag{1.1}$$

where m means the linear operator $f \mapsto \langle f \rangle \mathbf{1}$ and $||\cdot||_{p\to q}$ the operator norm from $L^p(m)$ to $L^q(m)$ for $p, q \in [1, \infty]$. Consider the case that $T_t f$ converges to $\langle f \rangle$ for sufficiently many f. In this case $\gamma_{p\to q}$ means the exponential rate of the convergence. Generally $\gamma_{p\to q}$ depends on $p, q \in [1, \infty]$. In this paper we consider the properties of $\gamma_{p\to q}$, relation among $\{\gamma_{p\to q}; p, q \in [1, \infty]\}$, some sufficient conditions that $\gamma_{p\to q}$ is to be independent of p and q, and give some examples that they depend on p and q. We also consider spectra of Markovian semigroups with respect to L^p-spaces, because the rate of convergence is closely related to the spectra.

The organization of this paper is as follows. In Section 2 we consider properties on $\gamma_{p\to q}$ which are obtained by general argument. We also discuss the relation between the spectra of Markovian semigroups and $\gamma_{p\to q}$. In Section 3 we consider properties of hyperbounded Markovian semigroups and the relation of $\gamma_{p\to q}$ between different pairs (p, q). We also consider the cases of hypercontractive Markovian semigroups and ultracontractive Markovian semigroups. In Section 4 we consider a sufficient condition for $\gamma_{p\to p}$ to be independent of p. Precisely speaking, we consider a hyperbounded Markovian semigroup whose generator is a normal operator on L^2-space, and show the p-independence of the spectra of the generator. In particular, this implies that $\gamma_{p\to p}$ is independent of p. In Section 5 we give a sufficient condition for non-symmetric Markovian semigroups to be hyperbounded by using the logarithmic Sobolev inequality, and consider a diffusion process on a manifold as an example. Non-symmetric diffusion semigroups on manifolds are also considered in [7]. In the paper, equivalent conditions to contractivity conditions are obtained. In Section 6 we consider the relation of the spectra of linear operators which are consistent on L^p-spaces for p. Markovian semigroups and their generators are examples of consistent operators on L^p-spaces. We remark that self-adjointness of the operator on L^2-space is additionally assumed in Section 6. In Section 7 we give an example of Markovian semigroup that $\gamma_{p\to p}$ depends on p. More precisely we give a generator on half line, which is a second order differential operator with boundary condition. By investigating the spectra of the generator, we will show that $\gamma_{p\to p}$ depends on p.

In the rest of this section, we give some notations used through this paper. For $z \in \mathbb{C}$, we denote the conjugate complex number of z by $\bar{z}$, and for $p \in [1, \infty]$ we denote by p^* the conjugate exponent, i.e. $1/p + 1/p^* = 1$.

Let (M, m) be a measure space and $L^p(m)$ be the L^p-space with respect to m for $p \in [1, \infty]$. For $p \in [1, \infty]$, $f \in L^p(m)$ and $g \in L^{p^*}(m)$, define $\langle f, g \rangle$ by $\int f(x)\overline{g(x)}m(dx)$. This notation is standard for $p = 2$, because $\langle \cdot, \cdot \rangle$ is the inner product on $L^2(m)$. On the other hand, the notation may not standard for $p \neq 2$, because $\langle \cdot, \cdot \rangle$ is not bilinear on $L^p(m) \times L^{p^*}(m)$. In this paper, we consider L^p-spaces and L^2-space at the same time. So, we use the notation $\langle \cdot, \cdot \rangle$ as defined above. Let A_p be a linear operator on $L^p(m)$ and $\mathrm{Dom}(A_p)$ the domain of A_p. We define the dual operator $(A_p)^*$ as follows. Let $\mathrm{Dom}((A_p)^*)$ be the total set of $f \in L^{p^*}(m)$ such that there exists $h \in L^{p^*}(m)$ satisfying

$$\langle A_p g, f \rangle = \langle g, h \rangle, \quad g \in \mathrm{Dom}(A_p), \tag{1.2}$$

and for $f \in \mathrm{Dom}((A_p)^*)$ define $(A_p)^* f := h$ where h is the element of $L^{p^*}(m)$ appearing in (1.2).

We define the point spectra of A_p by the total set of $\lambda \in \mathbb{C}$ such that $\lambda - A_p$ is not injective on $L^p(m)$, and denote the point spectra of A_p by $\sigma_{\mathrm{p}}(A_p)$. We define the continuous spectra of A_p by the total set of $\lambda \in \mathbb{C}$ such that $\lambda - A_p$ is injective, but is not onto map, and the range of $\lambda - A_p$ is dense in $L^p(m)$. We denote the continuous spectra of A_p by $\sigma_{\mathrm{c}}(A_p)$. We define the residual spectra of A_p by the total set of $\lambda \in \mathbb{C}$ such that $\lambda - A_p$ is injective, but is not onto map, and the range of $\lambda - A_p$ is not dense in $L^p(m)$. We denote the residual spectra of A_p by $\sigma_{\mathrm{r}}(A_p)$. Let $\sigma(A_p) := \sigma_{\mathrm{p}}(A_p) \cup \sigma_{\mathrm{c}}(A_p) \cup \sigma_{\mathrm{r}}(A_p)$. We define the resolvent set of A_p by the total set of $\lambda \in \mathbb{C}$ such that $\lambda - A_p$ is bijective, and denote it by $\rho(A_p)$. By the definition, $\sigma_{\mathrm{p}}(A_p)$, $\sigma_{\mathrm{c}}(A_p)$, $\sigma_{\mathrm{r}}(A_p)$ and $\rho(A_p)$ are disjoint set of $\mathbb{C}$ and their union is equal to $\mathbb{C}$.

In this paper $1/0$ and $1/\infty$ are often regarded as ∞ and 0, respectively.

2 Relation between spectra and the exponential rate of convergence for semigroups

In this section we consider immediate consequences on $\gamma_{p \to q}$ obtained by general theories.

Let (M, m) be a probability space and $\{T_t\}$ a strongly continuous Markovian semigroup on $L^2(m)$. We assume that $T_t^* \mathbf{1} = \mathbf{1}$ where T_t^* is the dual operator of T_t on $L^2(m)$. Then, it is easy to see that m is an invariant measure of both $\{T_t\}$ and $\{T_t^*\}$. By Jensen's inequality, for $p \in [1, \infty)$ we have

$$\int |T_t f|^p dm \leq \int T_t(|f|^p) dm = \int |f|^p dm.$$

This implies that T_t is contractive on $L^p(m)$ for $p \in [1, \infty)$. Since $\{T_t\}$ is positivity preserving on $L^2(m)$ (i.e. $T_t f \geq 0$ if $f \in L^2(m)$ and $f \geq 0$), it is easy to see that T_t is also contractive on $L^\infty(m)$. Hence, $\{T_t\}$ can be extended or restricted to a Markovian semigroup on $L^p(m)$ for $p \in [1, \infty]$. Let $p \in (1, \infty)$. For given $f \in L^p(m)$ and $\varepsilon > 0$, take a bounded measurable function g such that $||f - g||_p < \varepsilon$. Then, by Hölder's inequality

$$||T_t f - f||_p \leq ||T_t f - T_t g||_p + ||T_t g - g||_p + ||g - f||_p$$

$$
\begin{aligned}
&\leq 2||f-g||_p + \left(\int |T_t g - g| \cdot |T_t g - g|^{p-1} dm \right)^{1/p} \\
&\leq 2\varepsilon + ||T_t g - g||_2^{1/p} ||T_t g - g||_\infty^{1-1/p} \\
&\leq 2\varepsilon + 2||g||_\infty^{1-1/p} ||T_t g - g||_2^{1/p}.
\end{aligned}
$$

Hence, $\limsup_{t \to 0} ||T_t f - f||_p \leq 2\varepsilon$. This implies that $\{T_t\}$ is strongly continuous on $L^p(m)$ for $p \in (1, \infty)$. Trivially $\{T_t\}$ is strongly continuous on $L^1(m)$. Therefore, $\{T_t\}$ is strongly continuous for $p \in [1, \infty)$. Define $\mathfrak{A}_p$ be the generator of $\{T_t\}$ on $L^p(m)$ for $p \in [1, \infty)$. We regard $\{T_t\}$ as a semigroup on $L^p(m)$ for all $p \in [1, \infty]$. Define $\gamma_{p \to q}$ by (1.1) for $p, q \in [1, \infty]$.

Proposition 2.1. *Let $p_1, p_2, q_1, q_2 \in [1, \infty]$. Let r_1 and r_2 be a real number in $[1, \infty]$ such that there exists $\theta \in [0, 1]$ such that*

$$
\frac{1}{r_1} = \frac{1-\theta}{p_1} + \frac{\theta}{q_1} \quad \text{and} \quad \frac{1}{r_2} = \frac{1-\theta}{p_2} + \frac{\theta}{q_2}. \tag{2.1}
$$

Then,

$$
\gamma_{r_1 \to r_2} \geq (1-\theta)\gamma_{p_1 \to p_2} + \theta \gamma_{q_1 \to q_2}.
$$

In particular, the function $s \mapsto \gamma_{1/s \to 1/s}$ on $[0, 1]$ is concave.

Proof. By Riesz-Thorin's interpolation theorem (see Theorem 2.2.14 in [2]),

$$
||T_t - m||_{r_1 \to r_2} \leq ||T_t - m||_{p_1 \to p_2}^{1-\theta} ||T_t - m||_{q_1 \to q_2}^{\theta}.
$$

Hence, by the definition of $\gamma_{p \to q}$ we have the assertion. . □

Proposition 2.1 gives us some nice properties on $\gamma_{p \to p}$. We state the properties in the theorems below.

Theorem 2.2. *The function $p \mapsto \gamma_{p \to p}$ on $[1, \infty]$ is continuous on $(1, \infty)$. If $\gamma_{p \to p} > 0$ for some $p \in [1, \infty]$, then $\gamma_{p \to p} > 0$ for all $p \in (1, \infty)$.*

Proof. The equation (2.1) implies that $s \mapsto \gamma_{1/s \to 1/s}$ on $[0, 1]$ is concave, hence $s \mapsto \gamma_{1/s \to 1/s}$ is continuous on $(0, 1)$. Hence, the first assertion holds. Since $||T_t - m||_{p \to p} \leq 2$ for $p \in [1, \infty]$, $\gamma_{p \to p} \geq 0$ for $p \in [1, \infty]$. This fact and the concavity conclude the second assertion. □

Remark 2.3. *The function $\gamma_{p \to p}$ may not be continuous at $p = 1, \infty$. Indeed, let m be the probability measure with the standard normal distribution and $\{T_t\}$ be the Ornstein-Uhlembeck semigroup. Then, $\gamma_{p \to p} = 1$ for $p \in (1, \infty)$, however $\gamma_{p \to p} = 0$ for $p = 1, \infty$.*

Theorem 2.4. *Assume that $\{T_t\}$ is self-adjoint on $L^2(m)$. Then, $\gamma_{p \to p} = \gamma_{p^* \to p^*}$ for $p \in [1, \infty]$ and the function $p \mapsto \gamma_{p \to p}$ on $[1, \infty]$ is non-decreasing on $[1, 2]$ and non-increasing on $[2, \infty]$. In particular, the maximum is attained at $p = 2$.*

Proof. Let $f(s) := \gamma_{1/s \to 1/s}$ for $s \in [0,1]$. In view of Proposition 2.1 we have already known that f is concave on $[0,1]$. On the other hand, the symmetry of $\{T_t\}$ on $L^2(m)$ implies that $||T_t^* - m||_{p\to p} = ||T_t - m||_{p\to p}$. Since the operator-norm of the dual operator is equal to that of the original operator, we have $||T_t - m||_{p^*\to p^*} = ||T_t - m||_{p\to p}$. Hence, $\gamma_{p\to p} = \gamma_{p^*\to p^*}$ for $p \in [1,\infty]$. This fact and the concavity conclude the other assertions. $\square$

Remark 2.5. *In Theorem 2.4 we obtain that $p \mapsto \gamma_{p\to p}$ is non-decreasing on $[1,2]$, non-increasing on $[2,\infty]$, and the maximum is attained by $p = 2$. This assertion also follows from (2.2) and Remark 6.3 below.*

Next we consider the relation between $\gamma_{p\to p}$ and the radius of spectra. When we regard T_t as an operator on $L^p(m)$, we denote $T_t : L^p(m) \to L^p(m)$ by $T_t^{(p)}$. For a bounded linear operator A on a Banach space, define the radius of spectra $\mathrm{Rad}(A)$ by

$$\mathrm{Rad}(A) := \sup\{|\lambda|; \lambda \in \sigma(A)\}.$$

It is well-known that the limit

$$\lim_{t\to\infty} \frac{1}{t} \log ||T_t - m||_{p\to p}$$

exists (see e.g. Theorem 1.22. in Chapter 1 of [1]), and of course, the limit equals to $-\gamma_{p\to p}$. Moreover, it holds

$$\mathrm{Rad}(T_t^{(p)} - m) = e^{-\gamma_{p\to p} t} \tag{2.2}$$

(see e.g. Theorem 1.22. in Chapter 1 of [1] and Theorem 4.1.3 in of [2]). Hence, to see $\gamma_{p\to p}$ it is sufficient to see the spectra of $T_t^{(p)}$. There is also some relation between the spectra of semigroups and that of their generators. Let $\mathfrak{A}_p$ the generator of $\{T_t^{(p)}\}$ for $[1,\infty)$. Then, it is known that

$$e^{t\sigma(\mathfrak{A}_p)\setminus\{0\}} \subset \sigma(T_t^{(p)} - m) \setminus \{0\} \tag{2.3}$$

for $t \in [0,\infty)$ (see e.g. Theorem 2.16 in Chapter 2 of [1]). In general setting, the inclusion cannot be replaced by equality (see Theorem 2.17 in Chapter 2 of [1]). Sufficient conditions for the inclusion in (2.3) to be replaced by equality are known (see Corollary 3.12 in Chapter IV of [4]). For example, if $\{T_t^{(p)}\}$ is an analytic semigroup, then

$$e^{t\sigma(\mathfrak{A}_p)\setminus\{0\}} = \sigma(T_t^{(p)} - m) \setminus \{0\}, \quad t \in [0,\infty). \tag{2.4}$$

On the other hand, in general setting the two equalities

$$e^{t\sigma_{\mathrm{p}}(\mathfrak{A}_p)\setminus\{0\}} = \sigma_{\mathrm{p}}(T_t^{(p)} - m) \setminus \{0\}$$
$$e^{t\sigma_{\mathrm{r}}(\mathfrak{A}_p)\setminus\{0\}} = \sigma_{\mathrm{r}}(T_t^{(p)} - m) \setminus \{0\}$$

hold for $t \in [0,\infty)$ (see Theorem 3.7 in Chapter IV of [4]). Note that the definition of residual spectra in [4] is different from that in this paper. However, it is easy to see that the equality above still holds.

Consider the case that $\{T_t\}$ is a Markovian semigroup on (M,m) such that $\{T_t^{(2)}\}$ is symmetric on $L^2(m)$. By Theorem 1 in Section 2 of Chapter III of [10] $\{T_t^{(p)}\}$ is an analytic semigroup on $L^p(m)$ for $p \in (1,\infty)$. Hence, (2.4) holds. Moreover, by Corollary 3.12 in Chapter IV of [4] we obtain

$$\sup\{\mathrm{Re}\lambda; \lambda \in \sigma(\mathfrak{A}_p) \setminus \{0\}\} = \lim_{t\to\infty} \frac{1}{t} \log ||T_t - m||_{p\to p} \tag{2.5}$$

for $p \in (1,\infty)$. We will use this equality in Section 7.

Now we introduce a property of spectra of real operators on a general Banach space. Let B be a complex Banach space and A a linear operator on B. If there exists a bounded linear operator J on B satisfying that

$$\begin{gathered} J(\alpha x + \beta y) = \bar{\alpha} Jx + \bar{\beta} Jy,\ \alpha,\beta \in \mathbb{C},\ x,y \in B, \\ J^2 = I, \quad ||Jx|| = ||x||,\ x \in B, \quad AJ = JA, \end{gathered} \tag{2.6}$$

then A is called a real operator. Denote the resolvent operator with respect to $\lambda \in \rho(A)$ by R_λ.

Lemma 2.6. *If A is a real operator, then $\sigma_{\mathrm{p}}(A) = \overline{\sigma_{\mathrm{p}}(A)}$, $\sigma_{\mathrm{c}}(A) = \overline{\sigma_{\mathrm{c}}(A)}$, $\sigma_{\mathrm{r}}(A) = \overline{\sigma_{\mathrm{r}}(A)}$ and $\rho(A) = \overline{\rho(A)}$ where $\overline{\Lambda} := \{\bar{\lambda}; \lambda \in \Lambda\}$ for $\Lambda \subset \mathbb{C}$. Moreover, $R_{\bar{\lambda}} = JR_\lambda J$ for $\lambda \in \rho(A)$.*

Proof. If $\lambda x = Ax$ holds for $x \in \mathrm{Dom}(A) \setminus \{0\}$, then $\bar{\lambda} Jx = AJx$ and $Jx \neq 0$. Hence, $\sigma_{\mathrm{p}}(A) = \overline{\sigma_{\mathrm{p}}(A)}$. If there exists a sequence $\{x_n\} \subset B$ such that $||x_n|| = 1$ and $\lim_{n\to\infty} ||\lambda x_n - Ax_n|| = 0$, then $||Jx_n|| = 1$ and $\lim_{n\to\infty} ||\bar{\lambda} Jx_n - AJx_n|| = 0$. This implies that the conjugate of approximate point spectrum is also an approximate point spectrum. Hence, $\sigma_{\mathrm{p}}(A) \cup \sigma_{\mathrm{c}}(A) = \overline{\sigma_{\mathrm{p}}(A)} \cup \overline{\sigma_{\mathrm{c}}(A)}$. Since $\sigma_{\mathrm{p}}(A)$ and $\sigma_{\mathrm{c}}(A)$ are disjoint each other and $\sigma_{\mathrm{p}}(A) = \overline{\sigma_{\mathrm{p}}(A)}$, we have $\sigma_{\mathrm{c}}(A) = \overline{\sigma_{\mathrm{c}}(A)}$. For $\lambda \in \rho(A)$

$$JR_\lambda J(\bar{\lambda} - A) = I \text{ on } \mathrm{Dom}(A) \quad \text{and} \quad (\bar{\lambda} - A)JR_\lambda J = I \text{ on } B.$$

This implies that $\bar{\lambda} \in \rho(A)$ and $R_{\bar{\lambda}} = JR_\lambda J$. Since $\sigma_{\mathrm{p}}(A) = \overline{\sigma_{\mathrm{p}}(A)}$, $\sigma_{\mathrm{c}}(A) = \overline{\sigma_{\mathrm{c}}(A)}$ and $\rho(A) = \overline{\rho(A)}$, disjointness of $\sigma_{\mathrm{p}}(A)$, $\sigma_{\mathrm{c}}(A)$, $\sigma_{\mathrm{r}}(A)$, and $\rho(A)$ implies that $\sigma_{\mathrm{r}}(A) = \overline{\sigma_{\mathrm{r}}(A)}$. □

Consider the following property for a linear operator A on a $\mathbb{C}$-valued function space B:

$$\begin{aligned} &\text{if } f \in \mathrm{Dom}(A) \text{ and } f \text{ is a real-valued function,} \\ &\text{then } Af \text{ is also a real-valued function.} \end{aligned} \tag{2.7}$$

It is easy to see that an operator A satisfying (2.7) is a real operator by letting $Jf := \bar{f}$ for B. Since Markovian semigroups are positivity preserving, they satisfy (2.7). Hence, so are the generators of strong continuous Markovian semigroups. Consider $\{T_t\}$ and $\mathfrak{A}_p$ defined in the beginning of this section. Then, $\{T_t\}$ and $\mathfrak{A}_p$ are real operators on $L^p(m)$ for $p \in [1,\infty)$. Hence, by Lemma 2.6 we have that each kind of spectra of $\{T_t\}$ on $L^p(m)$ and $\mathfrak{A}_p$ are symmetric with respect to the real axis.

3 Hyperboundedness and p-independence of $\gamma_{p\to p}$

In this section we discuss the relation between hyperboundedness and $\gamma_{p\to q}$. Hyperboundedness enables us to compare $\{\gamma_{p\to q};\ p,q \in (1,\infty)\}$ with each other and hyperboundedness and $\{\gamma_{p\to q};\ p,q \in (1,\infty)\}$ characterize each other. In particular, we obtain the p-independence of $\gamma_{p\to p}$ for $p \in (1,\infty)$ from hyperboundedness. Hence, the results in this section give some sufficient conditions for $\gamma_{p\to p}$ to be p-independent. We also discuss the relation between hypercontractivity and $\gamma_{p\to p}$.

Let (M,m) and $\{T_t\}$ be the same as in Section 2. However, the assumption "$T_t^*\mathbf{1} = \mathbf{1}$" is not needed on the results before Proposition 3.3. For $p,q \in (1,\infty)$ such that $p<q$, $\{T_t\}$ is called (p,q)-hyperbounded if there exist $K \geq 0$ and $C > 0$ such that

$$||T_K f||_q \leq C||f||_p, \quad f \in L^p(m), \tag{3.1}$$

and $\{T_t\}$ is called (p,q)-hypercontractive if there exists $K \geq 0$ such that (3.1) holds with $C = 1$.

First we prepare the following lemma.

Lemma 3.1. *Let $p,q \in (1,\infty)$ such that $p<q$. If there exist non-negative constants K and C such that $||T_K f||_q \leq C||f||_p$ for $f \in L^p(m)$, then for $n_1, n_2 \in \mathbb{N}$ such that $q^{-n_1}/p^{-n_1-1} > 1$,*

$$||T_{(n_1+n_2)K} f||_{q^{n_2}/p^{n_2-1}} \leq C^{\alpha(n_1,n_2)}||f||_{q^{-n_1}/p^{-n_1-1}}, \quad f \in L^{q^{-n_1}/p^{-n_1-1}}(m),$$

where $\alpha(n_1,n_2) = \sum_{k=-n_1}^{n_2-1} p^k/q^k$.

Proof. Let $f \in L^{q^{n_1+1}/p^{n_1}}(m)$. By the positivity of $\{T_t\}$, Jensen's inequality and the assumption, for $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ such that $q^{m-1}/p^{m-2} > 1$ we have

$$\begin{aligned}
||T_{nK} f||_{q^m/p^{m-1}} &\leq \left[\int \left(T_K\left(|T_{(n-1)K} f|^{q^{m-1}/p^{m-1}}\right)\right)^q dm\right]^{p^{m-1}/q^m} \\
&= \left|\left|T_K\left(|T_{(n-1)K} f|^{q^{m-1}/p^{m-1}}\right)\right|\right|_q^{p^{m-1}/q^{m-1}} \\
&\leq C^{p^{m-1}/q^{m-1}} \left|\left||T_{(n-1)K} f|^{q^{m-1}/p^{m-1}}\right|\right|_p^{p^{m-1}/q^{m-1}} \\
&= C^{p^{m-1}/q^{m-1}} ||T_{(n-1)K} f||_{q^{m-1}/p^{m-2}}.
\end{aligned}$$

Iterating this calculation, we have the conclusion. □

Next we give the following theorem on hyperboundedness and hypercontractivity.

Theorem 3.2. *If $\{T_t\}$ is (p,q)-hyperbounded for some $p,q \in (1,\infty)$ such that $p<q$, then $\{T_t\}$ is (p,q)-hyperbounded for any $p,q \in (1,\infty)$ such that $p<q$. Moreover, if $\{T_t\}$ is (p,q)-hypercontractive for some $p,q \in (1,\infty)$ such that $p<q$, then $\{T_t\}$ is (p,q)-hypercontractive for any $p,q \in (1,\infty)$ such that $p<q$.*

Proof. Assume that $\{T_t\}$ is (p_1, q_1)-hyperbounded for $p_1 < q_1$. It is easy to see that $\{T_t\}$ is (p_2, q_2)-hyperbounded for $p_1 \leq p_2 < q_2 \leq q_1$. Let $p, q \in (1, \infty)$ such that $p < q$. Choose p_2 and q_2 so that $p_1 \leq p_2 < q_2 \leq q_1$ and that $1 < {p_2}^{n_1+1}/{q_2}^{n_1} < p$ with some $n_1 \in \mathbb{N}$. Take $n_2 \in \mathbb{N}$ such that ${q_2}^{n_2}/{p_2}^{n_2-1} > q$. Then, by applying Lemma 3.1 we have $\{T_t\}$ is $({q_2}^{n_2}/{p_2}^{n_2-1}, {p_2}^{n_1+1}/{q_2}^{n_1})$-hyperbounded, and therefore, $\{T_t\}$ is (p, q)-hyperbounded. Similarly, we obtain the second assertion. □

This theorem says that (p, q)-hyperboundedness for some $p, q \in (1, \infty)$ such that $p < q$ implies (p, q)-hyperboundedness for all $p, q \in (1, \infty)$ such that $p < q$ and the same assertion holds for hypercontractivity. Hence, we simply say that $\{T_t\}$ is hyperbounded and hypercontractive instead that $\{T_t\}$ is (p, q)-hyperbounded and (p, q)-hypercontractive respectively.

In the rest of this section we consider the relation between hypercontractivity (or hyperboundedness) and the exponential rate of convergence $\gamma_{p\to p}$. Note that the assumption "$T_t^*\mathbf{1} = \mathbf{1}$" is needed from now. First we show the following proposition, which is an extension of the first assertion of Lemma 6.1.5 in [3].

Proposition 3.3. *Assume that*

$$||T_K f||_r \leq ||f||_2, \quad f \in L^2(m) \tag{3.2}$$

for some $K > 0$ and $r > 2$. Then, we have

$$||T_K f - \langle f\rangle||_2 \leq (r-1)^{-1/2}||f||_2, \quad f \in L^2(m). \tag{3.3}$$

and

$$||T_t f - \langle f\rangle||_2 \leq \sqrt{r-1}\exp\left\{-\frac{t}{K}\log\sqrt{r-1}\right\}||f||_2, \; f \in L^2(m), \; t \in [0,\infty). \tag{3.4}$$

Proof. Let $f \in L^\infty(m)$ such that $\langle f\rangle = 0$ and $||f||_\infty \leq a_0$ with a nonnegative constant a_0 and let a be a positive constant such that $a > a_0$. From (3.2) we have

$$(a^2 + ||f||_2^2)^{r/2} = ||a+f||_2^r \geq ||T_K(a+f)||_r^r = \int |a + T_K f(x)|^r m(dx). \tag{3.5}$$

By the Taylor theorem there exists $\theta \in [0, 1]$ such that

$$(a^2 + ||f||_2^2)^{r/2} = a^r + \frac{r}{2}a^{r-2}||f||_2^2 + \frac{1}{2}\frac{r(r-2)}{4}(a^2 + \theta||f||_2^2)^{r/2-2}||f||_2^4. \tag{3.6}$$

Since $\{T_t\}$ is a Markovian semigroup, $||T_K f||_\infty \leq a_0$. Hence, by the Taylor theorem again, for each x there exists $\eta_x \in [0, 1]$ such that

$$\begin{aligned}(a + T_K f)^r(x) = a^r + ra^{r-1}T_K f(x) + \frac{r(r-1)}{2}a^{r-2}(T_K f)^2(x) \\ + \frac{r(r-1)(r-2)}{6}(a + \eta_x T_K f)^{r-3}(x)(T_K f)^3(x).\end{aligned}$$

By integrating both sides we have

$$\begin{aligned}\int (a+T_K f)^r dm = a^r + \frac{r(r-1)}{2}a^{r-2}||T_K f||_2^2 \\ + \frac{r(r-1)(r-2)}{6}\int (a+\eta_x T_K f)^{r-3}(T_K f)^3 dm.\end{aligned} \tag{3.7}$$

From (3.5), (3.6) and (3.7)

$$\begin{aligned}&\frac{r}{2}a^{r-2}||f||_2^2 + \frac{1}{2}\frac{r(r-2)}{4}(a^2+\theta||f||_2^2)^{r/2-2}||f||_2^4 \\ &\geq \frac{r(r-1)}{2}a^{r-2}||T_K f||_2^2 + \frac{r(r-1)(r-2)}{6}\int (a+\eta_x T_K f)^{r-3}(x)(T_K f)^3(x)m(dx).\end{aligned}$$

Dividing both sides by a^{r-2} and taking limit as $a \to \infty$, we have

$$\frac{r}{2}||f||_2^2 \geq \frac{r(r-1)}{2}||T_K f||_2^2.$$

Hence, (3.3) follows.

To show (3.4), for given $t \geq 0$ take $n \in \mathbb{N} \cup \{0\}$ and $\rho \in [0, K)$ such that $t = nK + \rho$. Then, by (3.3)

$$\begin{aligned}&||T_t f - \langle f \rangle||_2 = ||T_{nK}T_\rho f - \langle T_\rho f \rangle||_2 \leq (r-1)^{-n/2}||T_\rho f||_2 \\ &\leq (r-1)^{-\frac{1}{2}\left(\frac{t}{K}-1\right)}||f||_2 \leq \sqrt{r-1}\exp\left\{-\frac{t}{K}\log\sqrt{r-1}\right\}||f||_2.\end{aligned}$$

Hence, we have (3.4). □

Next we show the following theorem, which tells us the relation between hyperboundedness and $\gamma_{p\to q}$.

Theorem 3.4. *The following conditions are equivalent:*

(i) $\{T_t\}$ *is hyperbounded.*

(ii) $\gamma_{p\to q} \geq 0$ *for some* $1 < p < q < \infty$.

(iii) $\gamma_{p\to q} = \gamma_{2\to 2}$ *for all* $p, q \in (1, \infty)$.

Proof. First we show (ii) implies (i). By the definition of $\gamma_{p\to q}$ there exists $K > 0$ such that $||T_K - m||_{p\to q} < \infty$. Hence, $||T_K||_{p\to q} < \infty$. Therefore, we obtain (i) by Theorem 3.2. Immediately (ii) follows from (iii), since $\gamma_{2\to 2} \geq 0$.

Finally we show that (i) implies (iii). For given $p, q, r, s \in (1, \infty)$ take $K > 0$ and $C > 0$ such that $||T_K||_{p\to r} \leq C$ and $||T_K||_{s\to q} \leq C$. Then, it is easy to see that

$$||T_K - m||_{p\to r} \leq C + 1 \quad \text{and} \quad ||T_K - m||_{s\to q} \leq C + 1. \tag{3.8}$$

Since

$$||T_{t+2K} - m||_{p\to q} \leq ||T_K - m||_{p\to r}||T_t - m||_{r\to s}||T_K - m||_{s\to q},$$

we have

$$
\begin{aligned}
&-\frac{1}{t}\log ||T_{t+2K}-m||_{p\to q}\\
&\geq -\frac{1}{t}\log ||T_K-m||_{p\to r}-\frac{1}{t}\log ||T_t-m||_{r\to s}-\frac{1}{t}\log ||T_K-m||_{s\to q}.
\end{aligned}
$$

In view of (3.8), letting $t\to\infty$, we obtain $\gamma_{p\to q}\geq\gamma_{r\to s}$. Since $p,q,r,s\in(1,\infty)$ are arbitrary, (iii) follows. □

Finally we show the following theorem, which tells us the relation between hypercontractivity and $\gamma_{p\to q}$, and some criterion for $\{T_t\}$ to be hypercontractive.

Theorem 3.5. *The following conditions are equivalent:*

(i) $\{T_t\}$ *is hypercontractive.*

(ii) $\gamma_{p\to q}>0$ *for some* $1<p<q<\infty$.

(iii) $\gamma_{p\to q}=\gamma_{2\to 2}$ *for all* $p,q\in(1,\infty)$ *and* $\gamma_{2\to 2}>0$.

(iv) *There exist* $K>0$ *and* $r>0$ *such that*

$$
||T_K||_{2\to r}<\infty \quad \textit{and} \quad ||T_K-m||_{2\to 2}<1.
$$

Proof. By Theorem 3.4 we have that (ii) implies (iii). Trivially (ii) follows from (iii).

By Theorem 3.4, (i) implies that $\gamma_{p\to q}=\gamma_{2\to 2}$ for all $p,q\in(1,\infty)$. On the other hand, by Proposition 3.3 we obtain from (i) that $\gamma_{2\to 2}>0$. Hence, (i) implies (iii). Lemma 6.1.5 in [3] and Theorem 3.2 tells that (iv) implies (i).

To finish the proof, it is sufficient to prove that (iii) implies (iv). Assume (iii). As we have seen in Theorem 3.4, there exists $K>0$ and $r>0$ such that $||T_K||_{2\to r}<\infty$. Since $\gamma_{2\to 2}>0$, by the definition of $\gamma_{p\to q}$ it holds that there exists $K>0$ such that $||T_K-m||_{2\to 2}<1$. Thus, we obtain (iv). □

Remark 3.6. *We introduce the defective logarithmic Sobolev inequality and the logarithmic Sobolev inequality in Section 5 below. It is known that hyperboundedness and hypercontractivity are equivalent to the defective logarithmic Sobolev inequality and the logarithmic Sobolev inequality, respectively (See Theorem 6.1.14 in [3]).*

4 Sufficient conditions for spectra to be p-independent

In Section 3 we showed that when hyperboundedness holds, the exponential rate of convergence $\{\gamma_{p\to p};p\in(1,\infty)\}$ are independent of p. However, hyperboundedness gives us the further information that the spectra of $\{-\mathfrak{A}_p;p\in(1,\infty)\}$ are independent of p. Recall that $-\mathfrak{A}_p$ and $\gamma_{p\to p}$ are closely related to each other (see Section 2). In this section we show the assertion.

Let (M, m), $\{T_t\}$ be the same as in Section 2. Let $p \in (2, \infty)$ and fix p. Assume that there exists positive constants K and C such that

$$||T_K f||_p \le C||f||_2, \quad f \in L^2(m). \tag{4.1}$$

By Theorem 3.2 this assumption is equivalent to hyperboundedness on $\{T_t\}$. Hence, if necessary taking another pair (K, C), both (4.1) and

$$||T_K f||_2 \le C||f||_{p^*}, \quad f \in L^{p^*}(m) \tag{4.2}$$

hold. We choose a pair (K, C) such that both (4.1) and (4.2) hold, and fix it. Let $\mathfrak{A}_p$ be the generator of $\{T_t\}$ on $L^p(m)$ for $p \in [1, \infty)$ and assume that $\mathfrak{A}_2$ is a *normal* operator, i.e. $(\mathfrak{A}_2)^*\mathfrak{A}_2 = \mathfrak{A}_2(\mathfrak{A}_2)^*$. Then, we can consider the spectral decomposition of $-\mathfrak{A}_2$ (see [8]) as follows:

$$-\mathfrak{A}_2 = \int_{\mathbb{C}} \lambda dE_\lambda.$$

For a bounded $\mathbb{C}$-valued measurable function ϕ on $\mathbb{C}$, define a operator $\phi(-\mathfrak{A}_2)$ on $L^2(m)$ by

$$\phi(-\mathfrak{A}_2) = \int_{\mathbb{C}} \phi(\lambda) dE_\lambda.$$

Note that it is sufficient that ϕ is defined only on $\sigma(-\mathfrak{A}_2)$. Since $L^p(m) \subset L^2(m)$ and $L^2(m)$ is dense in $L^{p^*}(m)$ in our setting, $\phi(-\mathfrak{A}_2)$ can be regarded as a linear operator on $L^p(m)$ and on $L^{p^*}(m)$. So, we denote $\phi(-\mathfrak{A}_2)$ by $\phi(-\mathfrak{A})$ simply and regard $\phi(-\mathfrak{A})$ as a linear operator on $L^2(m)$, on $L^p(m)$, and on $L^{p^*}(m)$.

It is easy to see that $\phi(-\mathfrak{A})$ is a bounded operator on $L^2(m)$ if and only if ϕ is bounded on $\sigma(-\mathfrak{A}_2)$. However, it is not easy to obtain sufficient conditions for $\phi(-\mathfrak{A})$ to be a bounded operator on $L^p(m)$ and on $L^{p^*}(m)$. Now we consider a sufficient condition for the boundedness of $\phi(-\mathfrak{A})$ on $L^p(m)$ and on $L^{p^*}(m)$ under the assumption (4.1). Define a function χ on $\mathbb{C}$ by

$$\chi(\lambda) := \begin{cases} 0, & \mathrm{Re}\lambda < 0, \\ 1, & \mathrm{Re}\lambda \ge 0, \end{cases}$$

and let $\chi_n(\lambda) := \chi(\lambda - n)$.

Proposition 4.1. *The followings hold.*

(i) *If ϕ is bounded and the real part of the support of ϕ is bounded, then $\phi(-\mathfrak{A})$ is a bounded operator on $L^p(m)$ and also on $L^{p^*}(m)$.*

(ii) *There exists a positive constant $c = c(p, n)$ satisfying*

$$||T_t\chi_n(-\mathfrak{A})||_{p\to p} \le ce^{-nt}, \tag{4.3}$$

$$||T_t\chi_n(-\mathfrak{A})||_{p^*\to p^*} \le ce^{-nt}, \tag{4.4}$$

for $t \in [0, \infty)$.

Proof. To show (i) let $\psi(\lambda) := \phi(\lambda)e^{K\lambda}$ where K is the constant which appeared in (4.1). Since the real part of the support of ϕ is bounded, $\psi(-\mathfrak{A})$ is a bounded operator on $L^2(m)$. By using the fact that $\phi(-\mathfrak{A}) = T_K\psi(-\mathfrak{A})$ and (4.1), we have

$$||\phi(-\mathfrak{A})||_{2\to p} \le ||T_K||_{2\to p}||\psi(-\mathfrak{A})||_{2\to 2} \le C||\psi(-\mathfrak{A})||_{2\to 2}.$$

Hence, by the continuity of the embedding $L^p(m) \hookrightarrow L^2(m)$ we have $\phi(-\mathfrak{A})$ is a bounded operator on $L^p(m)$. Similar argument is available to estimate $||\phi(-\mathfrak{A})||_{p^*\to 2}$ and we have $\phi(-\mathfrak{A})$ is a bounded operator on $L^{p^*}(m)$. Thus, we obtain (i).

Next we show (ii). Since $\sup\limits_{\mathrm{Re}\lambda\ge 0} |e^{-t\lambda}|\chi_n(\lambda) \le e^{-nt}$, we have

$$||T_t\chi_n(-\mathfrak{A})||_{2\to 2} \le e^{-nt}.$$

Hence, by (4.1), for $t \ge 0$

$$||T_{t+K}\chi_n(-\mathfrak{A})||_{2\to p} \le ||T_K||_{2\to p}||T_t\chi_n(-\mathfrak{A})||_{2\to 2} \le Ce^{-nt} \le Ce^{nK}e^{-n(t+K)}.$$

Therefore, choosing $c \ge Ce^{nK}$, (4.3) holds for $t \ge K$.

Since $\mathbb{I}_{\{\mathrm{Re}\lambda\ge 0\}} - \chi_n$ is bounded and the real part of its support is bounded, (i) implies that $I - \chi_n(-\mathfrak{A})$ is a bounded operator on $L^p(m)$. Here, note that $\sigma(-\mathfrak{A}_2) \subset \{z \in \mathbb{C}; \mathrm{Re}z \ge 0\}$. Thus, for $t \in [0, K]$

$$\begin{aligned} ||T_t\chi_n(-\mathfrak{A})||_{p\to p} &= ||T_t(I - I + \chi_n(-\mathfrak{A}))||_{p\to p} \\ &\le ||T_t||_{p\to p} + ||T_t(I - \chi_n(-\mathfrak{A}))||_{p\to p} \\ &\le 1 + ||(I - \chi_n(-\mathfrak{A}))||_{p\to p} \\ &\le (1 + ||(I - \chi_n(-\mathfrak{A}))||_{p\to p})e^{nK}e^{-nt}. \end{aligned}$$

Therefore, by taking $c \ge 1 + ||(I - \chi_n(-\mathfrak{A}))||_{p\to p}$ (4.3) holds for $t \in [0, K]$. Consequently, letting $c = \max\{Ce^{nK}, 1 + ||(I - \chi_n(-\mathfrak{A}))||_{p\to p}\}$ (4.3) holds for $t \in [0, \infty)$.

We are able to prove (4.4) by similar way. Hence, we omit it. □

By using Proposition 4.1 we can show a sufficient condition for $\phi(-\mathfrak{A})$ to be a bounded linear operator on $L^p(m)$ and on $L^{p^*}(m)$. The following theorem is an extension of the result by Meyer [5].

Theorem 4.2. *Assume (4.1). Let h be a $\mathbb{C}$-valued bounded measurable function on $\mathbb{C}$ which is analytic on the neighborhood around 0 and define a $\mathbb{C}$-valued bounded function ϕ on $\mathbb{C}$ by $\phi(\lambda) = h(1/\lambda)$. Then, $\phi(-\mathfrak{A})$ is a bounded operator on $L^p(m)$ and also on $L^{p^*}(m)$.*

Proof. The proofs for boundedness of $\phi(-\mathfrak{A})$ on $L^p(m)$ and for that on $L^{p^*}(m)$ are the same. So, we only prove that $\phi(-\mathfrak{A})$ is a bounded operator on $L^p(m)$. Choose $n \in \mathbb{N}$ such that h is analytic on $\{z \in \mathbb{C}; |z| \le 1/n\}$ and let

$$\phi^{(1)} := \phi(1 - \chi_n) \text{ and } \phi^{(2)} := \phi\chi_n.$$

Then, ϕ is decomposed as

$$\phi = \phi^{(1)} + \phi^{(2)}.$$

Since $\sigma(-\mathfrak{A}_2) \subset \{z \in \mathbb{C}; \mathrm{Re} z \geq 0\}$, (i) of Proposition 4.1 implies that $\phi^{(1)}(-\mathfrak{A})$ is a bounded operator on $L^p(m)$. Hence, it is sufficient to show that $\phi^{(2)}(-\mathfrak{A})$ is a bounded operator on $L^p(m)$.

Let

$$R := \int_0^\infty T_t \chi_n(-\mathfrak{A}) dt.$$

Since for $k \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} R^k &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty T_{t_1}\chi_n(-\mathfrak{A}) T_{t_2}\chi_n(-\mathfrak{A}) \cdots T_{t_k}\chi_n(-\mathfrak{A}) dt_1 dt_2 \cdots dt_k \\ &= \int_0^\infty \int_0^\infty \cdots \int_0^\infty T_{t_1+t_2+\cdots+t_k}\chi_n(-\mathfrak{A}) dt_1 dt_2 \cdots dt_k, \end{aligned}$$

by (ii) of Proposition 4.1 we have

$$||R^k||_{p\to p} \leq c n^{-k}, \quad k \in \mathbb{N} \cup \{0\}. \tag{4.5}$$

By using spectral argument on L^2-space

$$R = \int_0^\infty \int_{\{\mathrm{Re}\lambda \geq n\}} e^{-\lambda t} dE_\lambda dt = \int_{\{\mathrm{Re}\lambda \geq n\}} \lambda^{-1} dE_\lambda,$$

and hence

$$R^k = \int_{\{\mathrm{Re}\lambda \geq n\}} \lambda^{-k} dE_\lambda. \tag{4.6}$$

On the other hand, since h is analytic on $\{z \in \mathbb{C}; |z| \leq 1/n\}$, by using Taylor expansion we have

$$h(z) = \sum_{k=0}^\infty a_k z^k, \quad |z| \leq \frac{1}{n}.$$

Note that $\sum_{k=0}^\infty |a_k| n^{-k} < \infty$. Hence, by (4.6) we obtain

$$\phi^{(2)}(-\mathfrak{A}) = \int_{\{\mathrm{Re}\lambda \geq n\}} h(\lambda^{-1}) dE_\lambda = \sum_{k=0}^\infty a_k \int_{\{\mathrm{Re}\lambda \geq n\}} \lambda^{-k} dE_\lambda = \sum_{k=0}^\infty a_k R^k.$$

Therefore, (4.5) implies that $\phi^{(2)}(-\mathfrak{A})$ is a bounded operator on $L^p(m)$. $\square$

Theorem 4.2 enables us to show that the spectra of $\mathfrak{A}_p$ are independent of p under the condition (4.1) as follows.

Theorem 4.3. *Assume that (4.1) holds for some $p \in (2, \infty)$ and positive numbers K and C. Then, $\sigma(-\mathfrak{A}_q) = \sigma(-\mathfrak{A}_2)$ for $q \in (1, \infty)$.*

Proof. As mentioned in the beginning of this section, in view of Theorem 3.2 the assumption that (4.1) holds for some $p \in (2, \infty)$, $K > 0$ and $C > 0$ implies that for any $p \in (2, \infty)$ there exists $K > 0$ and $C > 0$ such that (4.1) and (4.2) hold.

First we show that $\sigma(-\mathfrak{A}_q) \supset \sigma(-\mathfrak{A}_2)$ for $q \in (1, \infty)$. For given $p \in (2, \infty)$, take positive numbers K and C such that (4.1) and (4.2) hold, and fix them. Let $\alpha \in \sigma(-\mathfrak{A}_2)$. For $n \in \mathbb{N}$ define $U_n := \{z \in \mathbb{C}; |z - \alpha| \leq 1/n\}$ and $S_n := \left\{\int_{U_n} dE_\lambda f; f \in L^2(m)\right\}$. Then, S_n is a closed linear subspace of $L^2(m)$ and $S_n \neq \{0\}$ for $n \in \mathbb{N}$. Take $f_n \in S_n$ such that $||f_n||_2 = 1$. Then, it is easy to see that $\lim_{n\to\infty} ||\mathfrak{A} f_n + \alpha f_n||_2 = 0$. Since

$$\begin{aligned}
\mathfrak{A} f_n + \alpha f_n &= -\int_{U_n} \lambda dE_\lambda f_n + \alpha f_n \\
&= -\int_{U_n} e^{-K\lambda} e^{K\lambda} \lambda dE_\lambda f_n + \int_{U_n} e^{-K\lambda} e^{K\lambda} \alpha dE_\lambda f_n \\
&= \left(\int_{U_n} e^{-K\lambda} dE_\lambda\right) \left(\int_{U_n} e^{K\lambda} (\alpha - \lambda) dE_\lambda f_n\right) \\
&= T_K \int_{U_n} e^{K\lambda} (\alpha - \lambda) dE_\lambda f_n,
\end{aligned}$$

by (4.1) we have

$$\begin{aligned}
||\mathfrak{A} f_n + \alpha f_n||_p &\leq C || \int_{U_n} e^{K\lambda} (\alpha - \lambda) dE_\lambda f_n||_2 \\
&\leq \frac{C}{n} e^{K(\mathrm{Re}\alpha + 1/n)} ||f_n||_2.
\end{aligned}$$

Hence, $\lim_{n\to\infty} ||\mathfrak{A} f_n + \alpha f_n||_p = 0$. On the other hand, $||f_n||_p \geq ||f_n||_2 = 1$. These yield that $\alpha \in \sigma(-\mathfrak{A}_p)$. Similarly to the argument above,

$$\begin{aligned}
\mathfrak{A} f_n + \alpha f_n &= -\int_{U_n} e^{K\lambda} e^{-K\lambda} \lambda dE_\lambda f_n + \int_{U_n} e^{K\lambda} e^{-K\lambda} \alpha dE_\lambda f_n \\
&= \left(\int_{U_n} e^{K\lambda} (\alpha - \lambda) dE_\lambda\right) \left(\int_{U_n} e^{-K\lambda} dE_\lambda f_n\right) \\
&= \int_{U_n} e^{K\lambda} (\alpha - \lambda) dE_\lambda (T_K f_n).
\end{aligned}$$

Hence, by (4.2) we have

$$\begin{aligned}
||\mathfrak{A} f_n + \alpha f_n||_{p^*} &\leq || \int_{U_n} e^{K\lambda} (\alpha - \lambda) dE_\lambda (T_K f_n)||_2 \\
&\leq \frac{1}{n} e^{K(\mathrm{Re}\alpha + 1/n)} ||T_K f_n||_2 \\
&\leq \frac{C}{n} e^{K(\mathrm{Re}\alpha + 1/n)} ||f_n||_{p^*}.
\end{aligned}$$

Letting $\tilde{f}_n := f_n/||f_n||_{p^*}$, we have $||f_n||_{p^*} = 1$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} ||\mathfrak{A} \tilde{f}_n + \alpha \tilde{f}_n||_p = 0$. This yields that $\alpha \in \sigma(-\mathfrak{A}_{p^*})$. Thus, we have $\sigma(-\mathfrak{A}_2) \subset \sigma(-\mathfrak{A}_q)$ for $q \in (1, \infty)$.

Next we show that $\sigma(-\mathfrak{A}_q) \subset \sigma(-\mathfrak{A}_2)$ for $q \in (1,\infty)$. It is sufficient to show that $\rho(-\mathfrak{A}_q) \supset \rho(-\mathfrak{A}_2)$ for $q \in (1,\infty)$. For given $p \in (2,\infty)$, take positive numbers K and C such that (4.1) and (4.2) hold, and fix them. Let $\alpha \in \rho(-\mathfrak{A}_2)$ and $\phi(z) := 1/(\alpha + z)$. Then,

$$(\alpha - \mathfrak{A})^{-1} = \int_{\mathbb{C}} \phi(\lambda) dE_\lambda \tag{4.7}$$

$$\phi\left(\frac{1}{z}\right) = \frac{z}{\alpha z + 1}. \tag{4.8}$$

The equality (4.8) implies that $\phi(1/z)$ is analytic on a neighborhood around $z = 0$. Since $\alpha \in \rho(-\mathfrak{A}_2)$, the integral on the right-hand side of (4.7) is not changed by replacing $\phi(\lambda)$ by 0 on a neighborhood around $\lambda = -\alpha$. This implies that we can regard ϕ as a bounded function. Hence, applying Theorem 4.2, we have that $(\alpha - \mathfrak{A})^{-1}$ is a bounded operator on $L^p(m)$. Therefore, $\alpha \in \rho(-\mathfrak{A}_p)$. We also have $\alpha \in \rho(-\mathfrak{A}_{p^*})$ in the same manner. Thus, we have $\rho(-\mathfrak{A}_2) \subset \rho(-\mathfrak{A}_q)$ for $q \in (1,\infty)$. □

By using Theorem 4.3, we are able to know a little more information on the spectra of $\{T_t\}$ satisfying hyperboundedness.

Theorem 4.4. *If $\{T_t\}$ is hyperbounded, then $\sigma_{\mathrm{p}}(-\mathfrak{A}_2) = \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$, $\sigma_{\mathrm{c}}(-\mathfrak{A}_2) = \sigma_{\mathrm{c}}(-\mathfrak{A}_p)$ and $\sigma_{\mathrm{r}}(-\mathfrak{A}_p) = \emptyset$ for $p \in (1,\infty)$.*

Proof. Let $p, q \in (1,\infty)$. Let $\alpha \in \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$. Then, there exists $f \in \mathrm{Dom}(-\mathfrak{A}_p) \setminus \{0\}$ such that $\alpha f + \mathfrak{A} f = 0$. Hence, $\alpha T_t f + \mathfrak{A} T_t f = 0$ for $t \in [0,\infty)$. Since $\{T_t\}$ is hyperbounded, there exists a sufficiently large $t \in [0,\infty)$ such that $T_t f \in \mathrm{Dom}(-\mathfrak{A}_q) \setminus \{0\}$. This implies that $\alpha \in \sigma_{\mathrm{p}}(-\mathfrak{A}_q)$ and $T_t f$ is an eigenfunction with respect to α. Hence $\sigma_{\mathrm{p}}(-\mathfrak{A}_p) \subset \sigma_{\mathrm{p}}(-\mathfrak{A}_q)$. Since this holds for arbitrary $p, q \in (1,\infty)$, we have $\sigma_{\mathrm{p}}(-\mathfrak{A}_2) = \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$ for $p \in (1,\infty)$.

Let $p, q \in (1,\infty)$ such that $p < q$. By using dual argument we have

$$||T_t||_{p \to q} = ||T_t^*||_{q^* \to p^*}, \quad t \in [0,\infty).$$

Note that T_t^* is also a normal operator, the generator of T_t^* on $L^{p^*}(m)$ is $(\mathfrak{A}_p)^*$, and $q^* < p^*$. In view of Theorem 3.2, the hyperboundedness of $\{T_t\}$ implies that of $\{T_t^*\}$. Applying the argument above to $\{T_t^*\}$, we have

$$\sigma_{\mathrm{p}}(-(\mathfrak{A}_2)^*) = \sigma_{\mathrm{p}}(-(\mathfrak{A}_p)^*), \quad p \in (1,\infty). \tag{4.9}$$

Now assume $\alpha \in \sigma_{\mathrm{r}}(-\mathfrak{A}_p)$ for some $p \in (1,\infty)$, and we will make contradiction. Since there exists $f \in L^{p^*}(m)$ such that $\langle (\alpha + \mathfrak{A}_p) g, f \rangle = 0$ for $g \in \mathrm{Dom}(\mathfrak{A}_p)$, $f \in \mathrm{Dom}((\mathfrak{A}_p)^*)$ and $-(\mathfrak{A}_p)^* f = \bar{\alpha} f$. Hence, $\alpha \in \overline{\sigma_{\mathrm{p}}(-(\mathfrak{A}_p)^*)}$. Since $\mathfrak{A}_2$ is a normal operator, it is easy to see that $||(z + \mathfrak{A}_2) f||_2 = ||(\bar{z} + (\mathfrak{A}_2)^*) f||_2$ for $f \in \mathrm{Dom}(\mathfrak{A}_2)$ and $z \in \mathbb{C}$. In particular, $\overline{\sigma_{\mathrm{p}}(-(\mathfrak{A}_2)^*)} = \sigma_{\mathrm{p}}(-\mathfrak{A}_2)$. Hence, by (4.9) we have

$$\overline{\sigma_{\mathrm{p}}(-(\mathfrak{A}_p)^*)} = \overline{\sigma_{\mathrm{p}}(-(\mathfrak{A}_2)^*)} = \sigma_{\mathrm{p}}(-\mathfrak{A}_2).$$

Since $\sigma_{\mathrm{p}}(-\mathfrak{A}_2) = \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$, we have $\alpha \in \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$. However, this conflicts with the disjointness of $\sigma_{\mathrm{r}}(-\mathfrak{A}_p)$ and $\sigma_{\mathrm{p}}(-\mathfrak{A}_p)$. Hence, $\sigma_{\mathrm{r}}(-\mathfrak{A}_p) = \emptyset$.

By Theorem 4.3 and the disjointness of $\sigma_{\mathrm{c}}(-\mathfrak{A}_p)$ and $\sigma_{\mathrm{p}}(-\mathfrak{A}_p)$, we have $\sigma_{\mathrm{c}}(-\mathfrak{A}_2) = \sigma_{\mathrm{c}}(-\mathfrak{A}_p)$ for $p \in (1, \infty)$. □

In Section 5 we consider a sufficient condition for hyperboundedness via logarithmic Sobolev inequalities. It is to be obtained that spectra are the same for $p \in (1, \infty)$ if generators are normal (not necessarily symmetric) and the assumptions hold in Theorem 5.1.

Now we consider the relation between ultracontractivity and $\{\gamma_{p\to p}; p \in [1, \infty]\}$. If there exists positive constants K and C such that

$$||T_K f||_\infty \le C||f||_1, \quad f \in L^1(m),$$

then $\{T_t\}$ is called ultracontractive. In the case that $\{T_t\}$ is symmetric, we have the following proposition.

Proposition 4.5. *If $\{T_t\}$ is symmetric on $L^2(m)$, then $\{T_t\}$ is ultracontractive if and only if there exist $q \in [1, \infty)$ such that*

$$||T_K f||_\infty \le C||f||_q, \quad f \in L^q(m). \tag{4.10}$$

with some positive constants K and C.

Proof. It is sufficient to show that ultracontractivity holds if (4.10) holds for some q, K and C. It is immediately obtained that $\{T_t\}$ is (p, q)-hyperbounded for any $p \in (1, \infty)$. Hence, by Theorem 3.2 there exists $K' > 0$ such that $||T_{K'}||_{q^*\to q} < \infty$. Symmetry of $\{T_t\}$ on $L^2(m)$ implies that $||T_t||_{1\to q^*} = ||T_t^*||_{1\to q^*}$. On the other hand, by the duality we have $||T_t^*||_{1\to q^*} = ||T_t||_{q\to\infty}$. Hence, (4.10) implies that $||T_K||_{1\to q^*} = ||T_K||_{q\to\infty} < \infty$. Thus, we have

$$||T_{2K+K'}||_{1\to\infty} \le ||T_K||_{1\to q^*}||T_{K'}||_{q^*\to q}||T_K||_{q\to\infty} < \infty.$$

□

When $\{T_t\}$ is ultracontractive, we can discuss p-independence of the spectra of the generator of $\{T_t\}$ for $p \in [1, \infty)$ in the same way as in the case of hyperbounded Markovian semigroups.

Theorem 4.6. *Assume that $\{T_t\}$ is ultracontractive and that $\mathfrak{A}_2$ is a normal operator. Then, $\sigma(-\mathfrak{A}_p) = \sigma(-\mathfrak{A}_2)$ for $p \in [1, \infty)$. Moreover, $\sigma_{\mathrm{p}}(-\mathfrak{A}_2) = \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$, $\sigma_{\mathrm{c}}(-\mathfrak{A}_2) = \sigma_{\mathrm{c}}(-\mathfrak{A}_p)$ and $\sigma_{\mathrm{r}}(-\mathfrak{A}_p) = \emptyset$ for $p \in [1, \infty)$.*

Note that $\{T_t\}$ is not necessarily symmetric (or equivalently $\mathfrak{A}_2$ is not) in Theorem 4.6.

Remark 4.7. *If $\{T_t\}$ is symmetric on $L^2(m)$ and ultracontractive, the compactness of T_t on $L^p(m)$ for $p \in (1, \infty)$ and $t \geq K$ is to be obtained (See Theorem 13.4.2 in [2]).*

Remark 4.8. *When $T_t f(x) = \int f(y)p_t(x, y)m(dy)$ and*

$$\int\int |p_K(x, y)|^2 \, m(dy)m(dx) < \infty$$

holds for some $K > 0$, we have the compactness of T_K on $L^2(m)$ by Theorem 4.2.16 in [2]. Therefore, p-independence of spectra is obtained (See Remark 6.8).

5 Non-symmetric Markovian semigroups and logarithmic Sobolev inequality

In Section 4 we obtain some sufficient conditions for the spectra of a Markovian semigroup $\{T_t\}$ on $L^p(m)$ to be independent of $p \in (1, \infty)$. In this section we consider a sufficient condition for non-symmetric Markovian semigroups to satisfy hyperboundedness.

Let (M, m), $\{T_t\}$ be as same as in Section 2. However, in this section, the finiteness of m is not needed. Let $\mathfrak{A}_p$ be the generator of $\{T_t\}$ on $L^p(m)$. We often denote $\mathfrak{A}_p$ by $\mathfrak{A}$ simply. Let $\{R_\alpha\}$ be the resolvent operator of $\{T_t\}$ on $L^2(m)$ and define

$$\mathscr{D} := R_1(L^1(m) \cap L^\infty(m)).$$

Then, $\mathscr{D} \subset \mathrm{Dom}(\mathfrak{A}_p)$ for $p \in [1, \infty]$ and $\mathscr{D} \subset L^1(m) \cap L^\infty(m)$.

We prepare another supplementary symmetric semigroup $\{S_t\}$ on $L^2(m)$. Let $\mathscr{E}$ the Dirichlet form associated with $\{S_t\}$. Let $\alpha \in (0, \infty)$ and $\beta \in [0, \infty)$ and assume that

$$\int |f(x)|^2 \log(|f(x)|^2/||f||_2^2)m(dx) \leq \alpha\mathscr{E}(f, f) + \beta||f||_2^2, \quad f \in L^2(m). \tag{5.1}$$

This inequality is called a defective logarithmic Sobolev inequality. In the case that $\alpha > 0$ and $\beta = 0$, (5.1) is called a logarithmic Sobolev inequality. Additionally assume the following:

$$\begin{aligned} &\text{For } p > 1 \text{ and } f \in \mathscr{D},\ |f|^{p/2} \in \mathrm{Dom}(\mathscr{E}) \text{ and} \\ &\frac{4(p-1)}{p^2}\mathscr{E}(|f|^{p/2}, |f|^{p/2}) \leq -(\mathfrak{A}f, |f|^{p-1}\mathrm{sgn}(f)). \end{aligned} \tag{5.2}$$

When T_t is symmetric on $L^2(m)$, by taking S_t by T_t we have (5.2) (see the proof of Theorem 6.1.14 in [3]).

Theorem 5.1. *Assume (5.1) and (5.2). Then, we have*

$$||T_t||_{p\to q} \leq \exp\left\{\beta\left(\frac{1}{p} - \frac{1}{q}\right)\right\}$$

for $t > 0$ and $1 < p \le q < \infty$ such that $e^{4t/\alpha} \ge (q-1)/(p-1)$. Hence, $\{T_t\}$ is hyperbounded. Moreover, $\{T_t\}$ is hypercontractive if $\beta = 0$.

Proof. The proof is just the same as the proof of Theorem 6.1.14 in [3]. Let $f \in \mathscr{D}$ and denote $T_t f$ by f_t. Let $q(t) := 1+(p-1)e^{4t/\alpha}$. By following the proof of Theorem 6.1.14 in [3] we have

$$\begin{aligned}&||f_t||_{q(t)}^{q(t)-1}\frac{d}{dt}||f_t||_{q(t)}\\&= \int |f_t|^{q(t)-1}\mathrm{sgn}(f_t)\mathfrak{A}f_t dm + \frac{q'(t)}{q(t)^2}\int |f_t|^{q(t)}\log\left(|f_t|^{q(t)}/||f_t||_{q(t)}^{q(t)}\right)dm.\end{aligned}$$

By (5.2) we obtain

$$\begin{aligned}&||f_t||_{q(t)}^{q(t)-1}\frac{d}{dt}||f_t||_{q(t)}\\&\le -\frac{4(q(t)-1)}{q(t)^2}\mathscr{E}(|f_t|^{q(t)/2},|f_t|^{q(t)/2}) + \frac{q'(t)}{q(t)^2}\int |f_t|^{q(t)}\log\left(|f_t|^{q(t)}/||f_t||_{q(t)}^{q(t)}\right)dm.\end{aligned}$$

Hence, we can continue our proof in the same way as the proof of Theorem 6.1.14 in [3] and obtain the conclusion. □

In Theorem 5.1 we assumed (5.1) and (5.2). Now, we give an example of a non-symmetric Markovian semigroup $\{T_t\}$ satisfying (5.1) and (5.2).

Let M be a complete Riemannian manifold and m be the volume measure on M. Denote the total set of vector fields on M by D. We define the basis measure ν on M by $\nu := e^{-U}m$ where U is a C^∞-function on M such that $\int_M e^{-U}dm = 1$. Let ∇ be an affine connection. Then, the dual ∇_ν^* of ∇ on $L^2(\nu)$ is characterized by $\nabla_\nu^*\theta = \nabla^*\theta + (\nabla U, \theta)$ for $\theta \in D$ where ∇^* is the dual of ∇ on $L^2(m)$.

Let $b \in D$ and consider the generator $\mathfrak{A}$ defined by

$$\mathfrak{A} = -\frac{1}{2}\nabla_\nu^*\nabla + b. \tag{5.3}$$

Then, the dual $\mathfrak{A}_\nu^*$ of $\mathfrak{A}$ on $L^2(\nu)$ satisfies

$$\mathfrak{A}_\nu^* = -\frac{1}{2}\nabla_\nu^*\nabla - b - \mathrm{div}_\nu b$$

where div_ν is the divergence on $L^2(\nu)$, i.e. div_ν is the linear operator on D which is characterized by

$$\int Xf d\nu = -\int f\mathrm{div}_\nu X d\nu, \quad f \in C_0^1(M).$$

Let $\mathfrak{B} := -\frac{1}{2}\nabla_\nu^*\nabla$ and $\mathscr{E}$ the Dirichlet form associated with $\mathfrak{B}$. Then,

$$\mathscr{E}(f,g) = -\frac{1}{2}\int(\mathrm{grad}f, \mathrm{grad}g)d\nu, \quad f,g \in C_0^\infty(M)$$

where grad$f$ is the gradient of $f \in C^\infty(M)$. For $\mathfrak{B}$ to be a generator of a Markovian semigroup, we assume that the closure of $\mathfrak{B}$ defined on $C_0^\infty(M)$ is m-dissipative on $L^p(m)$ for $p \in [0, \infty]$. Sufficient conditions for the assumption is found in [9]. Additionally, we assume

$$\mathrm{div}_\nu b \geq 0. \tag{5.4}$$

Under these assumptions we show (5.2). Since $\mathfrak{B}$ is symmetric on $L^2(\nu)$, (5.2) holds for $\mathfrak{B}$ and $\mathscr{E}$ (See a remark just after (5.2)). Hence, letting $\{G_\alpha\}$ be the resolvent associated with $\mathfrak{B}$, we have for $f \in G_1(L^1 \cap L^\infty)$

$$\frac{4(p-1)}{p^2}\mathscr{E}(|f|^{p/2}, |f|^{p/2}) \leq \frac{1}{2}\left(\nabla_\nu^*\nabla f, |f|^{p-1}\mathrm{sgn}(f)\right). \tag{5.5}$$

In particular, since $C_0^\infty(M) \subset G_1(L^1 \cap L^\infty)$, (5.5) holds for $f \in C_0^\infty(M)$. For $f \in C_0^\infty(M)$ we have

$$\begin{aligned}
-(\mathfrak{A}f, |f|^{p-1}\mathrm{sgn}(f)) &= \int \left(\frac{1}{2}\nabla_\nu^*\nabla f - bf\right)|f|^{p-1}\mathrm{sgn}(f)d\nu \\
&= \frac{1}{2}\int \left(\nabla_\nu^*\nabla f\right)|f|^{p-1}\mathrm{sgn}(f)d\nu - \int (bf)|f|^{p-1}\mathrm{sgn}(f)d\nu.
\end{aligned}$$

By using (5.4)

$$-\int (bf)|f|^{p-1}\mathrm{sgn}(f)d\nu = -\frac{1}{p}\int b\left(|f|^p\right)d\nu = \frac{1}{p}\int \left(\mathrm{div}_\nu b\right)|f|^p d\nu \geq 0.$$

Hence, by (5.5) we obtain

$$-(\mathfrak{A}f, |f|^{p-1}\mathrm{sgn}(f)) \geq \frac{4(p-1)}{p^2}\mathscr{E}(|f|^{p/2}, |f|^{p/2}), \quad f \in C_0^\infty(M). \tag{5.6}$$

Since each function f which belongs to $\mathrm{Dom}(\mathfrak{A}_p)$ can be approximated by a sequence $\{f_n\}$ in $C_0^\infty(M)$ with respect to the graph-norm of $\mathfrak{A}_p$, (5.6) implies that $\sup_n \mathscr{E}(|f_n|^{p/2}, |f_n|^{p/2}) < \infty$. Hence, there exists a subsequence of $\{f_n\}$ which converges weakly with respect to the norm given by the inner product $\mathscr{E}_1(\cdot,\cdot) := (\cdot,\cdot) + \mathscr{E}(\cdot,\cdot)$. Denote the subsequence by $\{f_n\}$ again. Clearly, the limit of $\{f_n\}$ is f. By (5.6) we have

$$\begin{aligned}
\frac{4(p-1)}{p^2}\mathscr{E}(|f|^{p/2}, |f|^{p/2}) &\leq \liminf_{n\to\infty} \frac{4(p-1)}{p^2}\mathscr{E}(|f_n|^{p/2}, |f_n|^{p/2}) \\
&\leq -\liminf_{n\to\infty}(\mathfrak{A}f_n, |f_n|^{p-1}\mathrm{sgn}(f_n)) \\
&\leq -(\mathfrak{A}f, |f|^{p-1}\mathrm{sgn}(f)).
\end{aligned}$$

Therefore, (5.2) holds.

For (5.1) we additionally assume that

$$\mathrm{Ric} + \mathrm{Hess}U \geq \varepsilon I$$

for some $\varepsilon > 0$. Then it is known that the logarithmic Sobolev inequality holds for $\mathfrak{B}$ (see Theorem 6.2.42 in [3]). Hence, (5.1) holds.

By Theorem 5.1, the hyperboundedness holds. Furthermore, when we apply the results in Section 4, we need the conditions that ν is the invariant measure with respect to the semigroup generated by $\mathfrak{A}$ and $\mathfrak{A}$ is normal on $L^2(\nu)$.

Example 5.2. *Let $M := \mathbb{R}^2$, $\nu(dx) := (1/2\pi)e^{-|x|^2/2}dx$, and*

$$b = b_1(x)\frac{\partial}{\partial x_1} + b_2(x)\frac{\partial}{\partial x_2} := -cx_2\frac{\partial}{\partial x_1} + cx_1\frac{\partial}{\partial x_2}$$

where c is a positive constant. Then,

$$\mathfrak{A} = -\frac{1}{2}\nabla_\nu^*\nabla + b = \frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + b.$$

Hence, the diffusion associated with $\mathfrak{A}$ is the Ornstein-Uhlembeck diffusion with rotation. In this case, by explicit calculation we have that ν is the invariant measure and $\mathfrak{A}$ is normal on $L^2(\nu)$.

6 Properties on spectra of operators on L^p-spaces

In this section we consider consistent linear operators on L^p-spaces and discuss its spectra with respect to L^p-spaces. Let (M, m) be a probability space and $L^p(m)$ the L^p-space of $\mathbb{C}$-valued functions with respect to m. For $p \in [1, \infty)$ let A_p be a densely defined closed linear operator on $L^p(m)$ and assume $\{A_p; p \in [1, \infty)\}$ are consistent, i.e. if $p > q$, then $\mathrm{Dom}(A_p) \subset \mathrm{Dom}(A_q)$ and $A_p f = A_q f$ for $f \in \mathrm{Dom}(A_p)$. Moreover, assume that A_p is a real operator for some $p \in [1, \infty)$. Note that A_p is a real operator for all $p \in [1, \infty)$ by this assumption. A Markovian semigroup $\{T_t\}$ and its generators $\{\mathfrak{A}_p; p \in [1, \infty)\}$ defined in Section 2 satisfy the assumption on $\{A_p; p \in [1, \infty)\}$. Since the argument below is applicable to both $\{T_t\}$ and $\{\mathfrak{A}_p; p \in [1, \infty)\}$, so we prepare $\{A_p; p \in [1, \infty)\}$ as a unified notation. Also note that, when we consider a Markovian semigroup $\{T_t\}$ as $\{A_p\}$, the results below include the case that $p = \infty$.

In this section, we additionally assume that A_2 is self-adjoint on $L^2(m)$, i.e. $A_2 = A_2^*$. By using consistency it is easy to see that $(A_p)^* = A_{p^*}$ for $p \in [1, \infty)$. We denote A_p by A simply when confusion does not occur.

Lemma 6.1. *$\sigma_{\mathrm{r}}(A_p) = \emptyset$ for $p \leq 2$.*

Proof. Assuming that there exists $\lambda \in \sigma_{\mathrm{r}}(A_p)$, we will make a contradiction. Then, there exists $f \in L^{p^*}(m)\backslash\{0\}$ such that $\langle(\lambda - A)g, f\rangle = 0$ for $g \in \mathrm{Dom}(A_p)$. Since $g \mapsto \langle Ag, f\rangle = \langle g, \bar{\lambda}f\rangle$ is a bounded linear functional on $\mathrm{Dom}(A_p)$, $f \in \mathrm{Dom}((A_p)^*) = \mathrm{Dom}(A_{p^*})$ and $Af = \bar{\lambda}f$. On the other hand, $f \in \mathrm{Dom}(A_{p^*}) \setminus \{0\} \subset \mathrm{Dom}(A_p) \setminus \{0\}$. This implies that f is an eigenfunction of A_p with respect to the eigenvalue $\bar{\lambda}$. By Lemma 2.6 we have $\lambda \in \sigma_{\mathrm{p}}(A_p)$. This conflicts with the disjointness of $\sigma_{\mathrm{p}}(A_p)$ and $\sigma_{\mathrm{r}}(A_p)$. □

Proposition 6.2. *We have the following.*

(i) $\sigma_{\mathrm{p}}(A_p) \subset \sigma_{\mathrm{p}}(A_q)$ *for* $q \leq p$.

(ii) $\sigma_{\mathrm{r}}(A_q) \subset \sigma_{\mathrm{r}}(A_p)$ *for* $q \leq p$.

(iii) $\sigma_{\mathrm{c}}(A_p) \subset \sigma_{\mathrm{c}}(A_q) \cup \sigma_{\mathrm{p}}(A_q)$ *for* $q \leq p \leq 2$.

(iv) $\rho(A_q) \subset \rho(A_p)$ *for* $q \leq p \leq 2$.

Proof. Let $\lambda \in \sigma_{\mathrm{p}}(A_p)$. Then, there exists $f \in \mathrm{Dom}(A_p) \setminus \{0\}$ such that $\lambda f = Af$. This implies that $\lambda \in \sigma_{\mathrm{p}}(A_q)$, because $f \in \mathrm{Dom}(A_p) \setminus \{0\} \subset \mathrm{Dom}(A_q) \setminus \{0\}$. Therefore, we have (i).

Next we prove (ii). Let $\lambda \in \sigma_{\mathrm{r}}(A_q)$. If $\lambda \in \sigma_{\mathrm{p}}(A_p)$, by (i) we have that $\lambda \in \sigma_{\mathrm{p}}(A_q)$. This conflicts that $\sigma_{\mathrm{p}}(A_q)$ and $\sigma_{\mathrm{r}}(A_q)$ are disjoint each other. Thus, $\lambda \notin \sigma_{\mathrm{p}}(A_p)$. Since $\lambda \in \sigma_{\mathrm{r}}(A_q)$, there exists $f \in L^{q^*}(m) \setminus \{0\}$ and that $\langle (\lambda - A)g, f\rangle = 0$ for $g \in \mathrm{Dom}(A_q)$. Noting that $q^* \geq p^*$, we have that $f \in L^{p^*}(m) \setminus \{0\}$ and that $\langle (\lambda - A)g, f\rangle = 0$ for $g \in \mathrm{Dom}(A_p)$. Hence, $\lambda \in \sigma_{\mathrm{r}}(A_p)$. Thus, (ii) follows.

Now we show (iv). Let $q \leq p \leq 2$. Let $\lambda \in \rho(A_q)$. Note that $\rho(A_q) = \rho(A_{q^*})$. Let $(\lambda - A_q)^{-1}$ and $(\lambda - A_{q^*})^{-1}$ be the resolvent operators of A_q and A_{q^*} with respect to λ, respectively. Define a linear operator $R_\lambda^{(p)}$ on $L^p(m)$ by $R_\lambda^{(p)} f := (\lambda - A_q)^{-1} f$ for $f \in \mathrm{Dom}(R_\lambda^{(p)})$ where $\mathrm{Dom}(R_\lambda^{(p)}) := \{f \in L^p(m); (\lambda - A_q)^{-1} f \in L^p(m)\}$. Then, $R_\lambda^{(p)}$, $(\lambda - A_q)^{-1}$ and $(\lambda - A_{q^*})^{-1}$ are consistent. Hence, $L^{q^*}(m) \subset \mathrm{Dom}(R_\lambda^{(p)})$ and $\mathrm{Dom}(R_\lambda^{(p)})$ is dense in $L^p(m)$. By the Riesz-Thorin theorem we have

$$||R_\lambda^{(p)}||_{p \to p} \leq ||(\lambda - A_q)^{-1}||_{q \to q}^{1-\theta} \; ||(\lambda - A_{q^*})^{-1}||_{q^* \to q^*}^{\theta},$$

where $\theta \in [0,1]$ satisfying $1/p = (1-\theta)/q + \theta/q^*$. This implies that $||R_\lambda^{(p)}||_{p \to p} < \infty$. By the definition of $R_\lambda^{(p)}$ we have

$$\begin{aligned} (\lambda - A_p) R_\lambda^{(p)} &= I, \quad \text{on } \mathrm{Dom}(R_\lambda^{(p)}), \\ R_\lambda^{(p)} (\lambda - A_p) &= I, \quad \text{on } \mathrm{Dom}(A_p), \end{aligned}$$

and therefore the closure of $R_\lambda^{(p)}$ is the resolvent operator of A_p with respect to λ. Hence, $\lambda \in \rho(A_p)$ and we have (iv).

We obtain (iii) by (iv) and Lemma 6.1. □

Remark 6.3. *By* (iv) of Proposition 6.2 we have that $\sigma(A_p)$ is decreasing for $p \in [1,2]$ and increasing for $p \in [2,\infty)$.

Corollary 6.4. *Let* $p \in [2,\infty)$. *Then the followings hold.*

(i) $\sigma_{\mathrm{p}}(A_p) \cup \sigma_{\mathrm{r}}(A_p) = \sigma_{\mathrm{p}}(A_{p^*})$.

(ii) $\sigma_{\mathrm{c}}(A_p) = \sigma_{\mathrm{c}}(A_{p^*})$.

Proof. By (i) of Proposition 6.2, we have $\sigma_{\mathrm{p}}(A_p) \subset \sigma_{\mathrm{p}}(A_{p^*})$. By similar argument to that in the proof of Lemma 6.1, it holds that $\sigma_{\mathrm{r}}(A_p) \subset \sigma_{\mathrm{p}}(A_{p^*})$. Hence, we have

$$\sigma_{\mathrm{p}}(A_p) \cup \sigma_{\mathrm{r}}(A_p) \subset \sigma_{\mathrm{p}}(A_{p^*}). \tag{6.1}$$

Let $\lambda \in \sigma_{\mathrm{p}}(A_{p^*})$ and S the total set of $f \in \mathrm{Dom}(A_{p^*})$ such that $\lambda f = Af$. Since $\lambda \in \sigma_{\mathrm{p}}(A_{p^*})$, $S \neq \{0\}$. If $L^p(m) \cap S \neq \{0\}$, then $\lambda \in \sigma_{\mathrm{p}}(A_p)$. Consider the case that $L^p(m) \cap S = \{0\}$. Then, $\lambda \notin \sigma_{\mathrm{p}}(A_p)$. Take $f \in S \setminus \{0\}$. Then, it holds that $\langle \lambda f, g\rangle = \langle Af, g\rangle$ for $g \in L^p(m)$. Hence, by the symmetry of A we have $\langle f, \bar{\lambda} g\rangle = \langle f, Ag\rangle$ for $g \in \mathrm{Dom}(A_p)$. Here, note the definition of $\langle \cdot,\cdot\rangle$ in Section 1. On the other hand, since $\lambda \notin \sigma_{\mathrm{p}}(A_p)$, we have $\bar{\lambda} \notin \sigma_{\mathrm{p}}(A_p)$ by Lemma 2.6. These facts imply $\bar{\lambda} \in \sigma_{\mathrm{r}}(A_p)$. By Lemma 2.6 again, we have $\lambda \in \sigma_{\mathrm{r}}(A_p)$. Thus,

$$\sigma_{\mathrm{p}}(A_{p^*}) \subset \sigma_{\mathrm{p}}(A_p) \cup \sigma_{\mathrm{r}}(A_p). \tag{6.2}$$

By (6.1) and (6.2) yield (i).

Since $\sigma(A_p) = \sigma(A_{p^*})$, we have (ii). □

Corollary 6.5. $\sigma_{\mathrm{p}}(A_p) \subset \mathbb{R}$ *for* $p \in [2,\infty)$.

Proof. The assertion immediately follows by (i) of Proposition 6.2 and $\sigma(A_2) \subset \mathbb{R}$. □

Remark 6.6. *Since A_2 is a self-adjoint operator, by using the general theory of self-adjoint operators on Hilbert spaces it is obtained that $\sigma(A_2) \subset \mathbb{R}$. However, when $p \neq 2$, it does not always hold. An example that $\sigma(A_p) \not\subset \mathbb{R}$ when $p \neq 2$, is given in Section 7.*

Let $\lambda_p^{\min} := \min\{|\lambda|; \lambda \in \sigma(A_p)\}$ and $\lambda_p^{\max} := \max\{|\lambda|; \lambda \in \sigma(A_p)\}$ for $p \in [1,\infty)$. Note that the minimum and the maximum above exist in $[0,\infty]$, because $\sigma(A_p)$ is closed set in $\mathbb{C}$. The following corollary follows immediately from (iv) of Proposition 6.2.

Corollary 6.7. $\lambda_q^{\min} \geq \lambda_p^{\min}$ *and* $\lambda_q^{\max} \geq \lambda_p^{\max}$ *for* $q \in [1,\min\{p,p^*\}] \cup [\max\{p,p^*\},\infty)$.

This corollary gives the relation of the exponential rate of convergence for Markovian semigroups. For example, let $A_p = \mathfrak{A}_p$, where $\mathfrak{A}_p$ is the generator of the Markovian semigroup on $L^p(m)$ defined in Section 2. Then, $\lambda_p^{\min}$ is the distance between 0 and $\sigma(\mathfrak{A}_p)$. For another example, let A_p be $T_t^{(p)} - m$ for some $t > 0$, where $T_t^{(p)}$ is the Markovian semigroup on $L^p(m)$ defined in Section 2. Then, $\lambda_p^{\max} = \mathrm{Rad}(T_t^{(p)} - m)$. As mentioned in Section 2, these are related to the rate of convergence of the Markovian semigroups.

Remark 6.8. *In Chapter 4 of [2] spectra of consistent bounded operators are considered. When we additionally assume that A_p is bounded for any $p \in [1,\infty)$ and that A_p is compact for some $p \in [1,\infty)$, then the p-independence of spectra of A_p is obtained by using Schauder's theorem (see Theorem 4.2.13 in [2]) and Theorem 4.2.14 in [2].*

7 Example that $\gamma_{p\to p}$ depends on p

In Section 4 we give a sufficient condition for the spectra of a Markovian semigroup as an operator on $L^p(m)$ to be independent of p. However, generally the spectra depend on p. We give an example so that the spectra depend on p in this section.

Let $p \in [1,\infty)$. Define a measure ν on $[0,\infty)$ by $\nu(dx) := e^{-x}dx$ and a differential operator $\mathfrak{A}_p^\circ$ with its domain $\mathrm{Dom}(\mathfrak{A}_p^\circ)$ by

$$\begin{aligned}\mathrm{Dom}(\mathfrak{A}_p^\circ) &:= \left\{ f \in C_0^2([0,\infty);\mathbb{C}); f'(0) = 0 \right\},\\ \mathfrak{A}_p^\circ &:= \frac{d^2}{dx^2} - \frac{d}{dx}.\end{aligned}$$

Consider a generator $\mathfrak{A}_p$ by the closed extension of $\mathfrak{A}_p^\circ$ on $L^p(\nu)$. Note that $\mathfrak{A}_2$ is a self-adjoint operator on $L^2(\nu)$. This is an example that the spectra $\sigma(\mathfrak{A}_p)$ depend on p and $\gamma_{q\to q} < \gamma_{p\to p}$ for $q < p \le 2$. Now, we show them by investigating $\sigma(\mathfrak{A}_p)$ explicitly.

Let $p \in [1,2]$. Consider the linear transformation I defined by

$$(If)(x) := e^{-x/2} f(x). \tag{7.1}$$

Then, we have

$$\int_0^\infty |If(x)|^p e^{\left(\frac{p}{2}-1\right)x} dx = \int_0^\infty |f(x)|^p \nu(dx),$$

and $f'(0) = 0$ if and only if $\frac{1}{2}(If)(0) + (If)'(0) = 0$ for $f \in C^1([0,\infty);\mathbb{C})$. Hence, I is an isometric transformation from $L^p(\nu)$ to $L^p(\tilde{\nu}_p)$, where $\tilde{\nu}_p := e^{\left(\frac{p}{2}-1\right)x}dx$. Define a linear operator $\tilde{\mathfrak{A}}_p$ on $L^p(\tilde{\nu}_p)$ by

$$\begin{aligned}\mathrm{Dom}(\tilde{\mathfrak{A}}_p) &:= \left\{ \tilde{f} \in W^{2,p}(\tilde{\nu}_p);\ \frac{1}{2}\tilde{f}(0) + \tilde{f}'(0) = 0 \right\},\\ \tilde{\mathfrak{A}}_p \tilde{f} &:= \frac{d^2}{dx^2}\tilde{f} - \frac{1}{4}\tilde{f}.\end{aligned} \tag{7.2}$$

Then, we have for $\tilde{f} \in C_0^\infty([0,\infty);\mathbb{C})$

$$\begin{aligned}\left(I \circ \mathfrak{A}_p \circ I^{-1}\right)\tilde{f}(x) &= e^{-x/2}\left(\frac{d^2}{dx^2} - \frac{d}{dx}\right) e^{x/2}\tilde{f}(x)\\ &= \tilde{f}''(x) + \tilde{f}'(x) + \frac{1}{4}\tilde{f}(x) - \tilde{f}'(x) - \frac{1}{2}\tilde{f}(x)\\ &= \tilde{f}''(x) - \frac{1}{4}\tilde{f}(x).\end{aligned}$$

Thus, we have the following commutative diagram.

$$\begin{array}{ccc} L^p(\nu) & \xrightarrow{\mathfrak{A}_p} & L^p(\nu)\\ I\downarrow & & \downarrow I\\ L^p(\tilde{\nu}_p) & \xrightarrow{\tilde{\mathfrak{A}}_p} & L^p(\tilde{\nu}_p)\end{array}$$

By this diagram we have

$$\sigma_{\mathrm{p}}(\mathfrak{A}_p) = \sigma_{\mathrm{p}}(\tilde{\mathfrak{A}}_p),\ \sigma_{\mathrm{c}}(\mathfrak{A}_p) = \sigma_{\mathrm{c}}(\tilde{\mathfrak{A}}_p) \text{ and } \sigma_{\mathrm{r}}(\mathfrak{A}_p) = \sigma_{\mathrm{r}}(\tilde{\mathfrak{A}}_p). \tag{7.3}$$

Hence, to see the spectra of $\mathfrak{A}_p$, it is sufficient to see the spectra of $\tilde{\mathfrak{A}}_p$.

From now we cannot discuss the cases that $1 \le p < 2$ and that $p = 2$ in the same way. First we consider the case that $1 \le p < 2$. Let $\sqrt{z} := \sqrt{r}e^{i\theta/2}$ for $z \in \mathbb{C}$ where $z = re^{i\theta}$ such that $r \ge 0$ and $\theta \in (-\pi, \pi]$.

Lemma 7.1. *If* $1 \le p < 2$, *then*

$$\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_p) = \{0\} \cup \left\{ x + iy;\ x, y \in \mathbb{R},\ x > \frac{p-1}{p^2},\ |y| < \left(\frac{2}{p} - 1\right)\sqrt{x - \frac{p-1}{p^2}} \right\}.$$

Proof. Let $u(x) = x - 2$ for $x \in [0, \infty)$. Then, $u \in L^p(\tilde{\nu}_p)$,

$$-\frac{d^2}{dx^2}u + \frac{1}{4}u = \frac{1}{4}u \quad \text{and} \quad \frac{1}{2}u(0) + u'(0) = 0.$$

Hence,

$$\frac{1}{4} \in \sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_p). \tag{7.4}$$

Let $\lambda \in \mathbb{C} \setminus \left\{\frac{1}{4}\right\}$. Consider the differential equation:

$$-\frac{d^2}{dx^2}u + \frac{1}{4}u = \lambda u \tag{7.5}$$

where $u : [0, \infty) \to \mathbb{C}$. Then, u is the solution of (7.5) if and only if

$$u(x) = C_1 e^{x\sqrt{-\lambda+1/4}} + C_2 e^{-x\sqrt{-\lambda+1/4}}$$

where C_1, C_2 are constants in $\mathbb{C}$. Note that $\frac{1}{2}u(0) + u'(0) = 0$ if and only if $C_1(1/2 + \sqrt{-\lambda + 1/4}) + C_2(1/2 - \sqrt{-\lambda + 1/4}) = 0$. Hence, u is the solution of the boundary-value problem on $[0, \infty)$:

$$\begin{cases} -\dfrac{d^2}{dx^2}u + \dfrac{1}{4}u = \lambda u \\ \dfrac{1}{2}u(0) + u'(0) = 0, \end{cases}$$

if and only if

$$\begin{cases} u(x) = C_1 e^{x\sqrt{-\lambda+1/4}} + C_2 e^{-x\sqrt{-\lambda+1/4}} \\ C_1 \left(\dfrac{1}{2} + \sqrt{-\lambda + \dfrac{1}{4}}\right) + C_2 \left(\dfrac{1}{2} - \sqrt{-\lambda + \dfrac{1}{4}}\right) = 0. \end{cases} \tag{7.6}$$

When u satisfies (7.6),

$$|C_1|^p \int_0^\infty e^{\left(\mathrm{Re}\sqrt{-\lambda+1/4}\right)px} e^{(p/2-1)x} dx - |C_2|^p \int_0^\infty e^{-\left(\mathrm{Re}\sqrt{-\lambda+1/4}\right)px} e^{(p/2-1)x} dx$$

$$\leq \int_0^\infty |u(x)|^p e^{(p/2-1)x} dx$$
$$\leq |C_1|^p \int_0^\infty e^{\left(\mathrm{Re}\sqrt{-\lambda+1/4}\right)px} e^{(p/2-1)x} dx + |C_2|^p \int_0^\infty e^{-\left(\mathrm{Re}\sqrt{-\lambda+1/4}\right)px} e^{(p/2-1)x} dx.$$

This implies that

$$u \in L^p(\tilde{\nu}_p) \text{ if and only if } \text{“}p\,\mathrm{Re}\sqrt{-\lambda+1/4} + \frac{p}{2} - 1 < 0 \text{ or } C_1 = 0\text{”}. \tag{7.7}$$

By (7.6), if $C_1 = 0$, then $\lambda = 0$ or $C_2 = 0$. Therefore, (7.4) and (7.7) imply that $\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_p) = \{0\} \cup \left\{\lambda \in \mathbb{C};\ \mathrm{Re}\sqrt{-\lambda+1/4} < \frac{1}{p} - \frac{1}{2}\right\}$. □

Lemma 7.2. *If* $1 \leq p < 2$, *then*

$$\rho(-\tilde{\mathfrak{A}}_p) \supset \left\{x + iy;\ x, y \in \mathbb{R},\ y^2 > \left(\frac{2}{p} - 1\right)^2 \left(x - \frac{p-1}{p^2}\right)\right\} \setminus \{0\}.$$

Proof. It is sufficient to show that $\left\{z \in \mathbb{C} \setminus \{0\};\ \mathrm{Re}\sqrt{-z+1/4} > \frac{1}{p} - \frac{1}{2}\right\} \subset \rho(-\tilde{\mathfrak{A}}_p)$. For $\lambda \in \left\{z \in \mathbb{C} \setminus \{0\};\ \mathrm{Re}\sqrt{-z+1/4} > \frac{1}{p} - \frac{1}{2}\right\}$ let

$$\begin{aligned}
\phi_\lambda(x) &:= \left(\frac{1}{2} - \sqrt{-\lambda + \frac{1}{4}}\right) e^{x\sqrt{-\lambda+\frac{1}{4}}} - \left(\frac{1}{2} + \sqrt{-\lambda + \frac{1}{4}}\right) e^{-x\sqrt{-\lambda+\frac{1}{4}}}, \quad && x \in [0,\infty) \\
\psi_\lambda(x) &:= e^{-x\sqrt{-\lambda+\frac{1}{4}}}, && x \in [0,\infty) \\
W_\lambda &:= -2\sqrt{-\lambda+\frac{1}{4}}\left(\frac{1}{2} - \sqrt{-\lambda+\frac{1}{4}}\right)
\end{aligned}$$

and define a $\mathbb{C}$-valued function g_λ on $[0,\infty) \times [0,\infty)$ by

$$g_\lambda(x,y) := \begin{cases} \dfrac{1}{W_\lambda}\phi_\lambda(x)\psi_\lambda(y), & x \leq y \\ \dfrac{1}{W_\lambda}\phi_\lambda(y)\psi_\lambda(x), & y \leq x \end{cases}$$

Let $G_\lambda f(x) := \int_0^\infty g_\lambda(x,y) f(y) dy$ for $f \in C_0([0,\infty);\mathbb{C})$. Then, by explicit calculation, we have for $f \in C_0([0,\infty);\mathbb{C})$

$$\left\{\lambda - (-\tilde{\mathfrak{A}}_p)\right\} G_\lambda f = f, \quad \text{and} \quad \frac{1}{2} G_\lambda f(0) + (G_\lambda f)'(0) = 0.$$

In view of Lemmas 6.1 and 7.1, to show that $\lambda \in \rho(-\tilde{\mathfrak{A}}_p)$, it is sufficient to prove the boundedness of the operator G_λ on $L^p(\tilde{\nu}_p)$. Let

$$\begin{aligned}
C_\lambda(\varepsilon) &:= \sup_{y \in [0,\infty)} e^{(1-p/2)y}\left(\int_0^\infty |g_\lambda(x,y)|^{(1-\varepsilon)p} e^{(p/2-1)x} dx\right), \\
C'_\lambda(\varepsilon) &:= \sup_{x \in [0,\infty)} \left(\int_0^\infty |g_\lambda(x,y)|^{\varepsilon p^*} dy\right)^{p/p^*}
\end{aligned}$$

for $\varepsilon \in (0,1)$. By explicit calculation, we have $C_\lambda(\varepsilon) < \infty$ when $\mathrm{Re}\sqrt{-\lambda+1/4} > \frac{1}{1-\varepsilon}\left(\frac{1}{p}-\frac{1}{2}\right)$, and $C'_\lambda(\varepsilon) < \infty$. By Hölder's inequality we have

$$
\begin{aligned}
&||G_\lambda f||^p_{L^p(\tilde{\nu}_p)} \\
&= \int_0^\infty \left| \int_0^\infty g_\lambda(x,y) f(y) dy \right|^p e^{(p/2-1)x} dx \\
&\leq \int_0^\infty \left[\int_0^\infty |g_\lambda(x,y)|^{1-\varepsilon} |f(y)| \cdot |g_\lambda(x,y)|^\varepsilon \, dy \right]^p e^{(p/2-1)x} dx \\
&\leq \int_0^\infty \left(\int_0^\infty |g_\lambda(x,y)|^{(1-\varepsilon)p} |f(y)|^p dy \right) \left(\int_0^\infty |g_\lambda(x,y)|^{\varepsilon p^*} \, dy \right)^{p/p^*} e^{(p/2-1)x} dx \\
&\leq C'_\lambda(\varepsilon) \int_0^\infty \left(\int_0^\infty |g_\lambda(x,y)|^{(1-\varepsilon)p} e^{(p/2-1)x} dx \right) |f(y)|^p dy \\
&\leq C'_\lambda(\varepsilon) C_\lambda(\varepsilon) ||f||^p_{L^p(\tilde{\nu}_p)}.
\end{aligned}
$$

Since this estimate holds for all $\varepsilon \in (0,1)$, $\left\{ z \in \mathbb{C} \setminus \{0\};\ \mathrm{Re}\sqrt{-z+1/4} > \frac{1}{p} - \frac{1}{2} \right\} \subset \rho(\tilde{\mathfrak{A}}_p)$. $\square$

By the lemmas above, the spectra of $-\tilde{\mathfrak{A}}_p$ are determined exactly.

Theorem 7.3. *Followings hold for* $1 \leq p < 2$.

(i) $\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_p) = \{0\} \cup \left\{ x+iy;\ x,y \in \mathbb{R},\ x > \frac{p-1}{p^2} \text{ and } |y| < \left(\frac{2}{p}-1\right)\sqrt{x - \frac{p-1}{p^2}} \right\}$,

(ii) $\sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_p) = \left\{ x+iy;\ x,y \in \mathbb{R},\ x \geq \frac{p-1}{p^2},\ \text{ and } |y| = \left(\frac{2}{p}-1\right)\sqrt{x - \frac{p-1}{p^2}} \right\} \setminus \{0\}$,

(iii) $\rho(-\tilde{\mathfrak{A}}_p) = \left\{ x+iy;\ x,y \in \mathbb{R},\ y^2 > \left(\frac{2}{p}-1\right)^2 \left(x - \frac{p-1}{p^2}\right) \right\} \setminus \{0\}$.

Proof. The assertion (i) is obtained in Lemma 7.1. Since any limit point of point spectra are either a point spectrum or a continuous spectrum, by (i) and Lemma 7.2, we have (ii). By (i), (ii) and Lemma 6.1, we obtain (iii). $\square$

By (7.3) we have the following theorem.

Theorem 7.4. *Followings hold for* $1 \leq p < 2$.

(i) $\sigma_{\mathrm{p}}(-\mathfrak{A}_p) = \{0\} \cup \left\{ x+iy;\ x,y \in \mathbb{R},\ x > \frac{p-1}{p^2} \text{ and } |y| < \left(\frac{2}{p}-1\right)\sqrt{x - \frac{p-1}{p^2}} \right\}$,

(ii) $\sigma_{\mathrm{c}}(-\mathfrak{A}_p) = \left\{ x+iy;\ x,y \in \mathbb{R},\ x \geq \frac{p-1}{p^2} \text{ and } |y| = \left(\frac{2}{p}-1\right)\sqrt{x - \frac{p-1}{p^2}} \right\} \setminus \{0\}$,

(iii) $\rho(-\mathfrak{A}_p) = \left\{ x+iy;\ x,y \in \mathbb{R},\ y^2 > \left(\frac{2}{p}-1\right)^2 \left(x - \frac{p-1}{p^2}\right) \right\} \setminus \{0\}$.

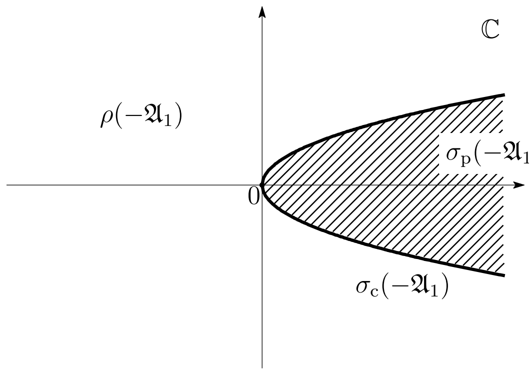

Figure 1: $p = 1$

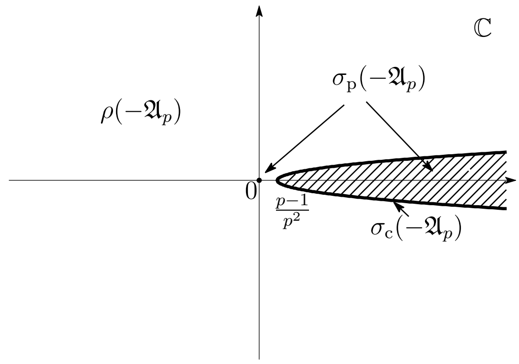

Figure 2: $1 < p < 2$

The pictures of $\sigma_{\mathrm{p}}(-\mathfrak{A}_p)$, $\sigma_{\mathrm{c}}(-\mathfrak{A}_p)$ and $\rho(-\mathfrak{A}_p)$ for $p = 1$ and for $1 < p < 2$ are described in Figures 1 and 2.

Next we check $\sigma(-\tilde{\mathfrak{A}}_2)$. Note that $\tilde{\nu}_p$ is equal to the Lebesgue measure dx when $p = 2$. Since $\sigma(-\tilde{\mathfrak{A}}_2)$ is self-adjoint and non-negative definite on $L^2(dx)$, we know that $\sigma(-\tilde{\mathfrak{A}}_2) \subset [0, \infty)$ and $\sigma_{\mathrm{r}}(-\tilde{\mathfrak{A}}_2) = \emptyset$ (see Lemma 6.1). The purpose of the argument below is to investigate both $\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2)$ and $\sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_2)$ explicitly.

Lemma 7.5.

$$\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2) = \{0\}.$$

Proof. The assertion follows by almost the same way as the proof of Lemma 7.5 except the part of checking whether $\frac{1}{4}$ is a point spectrum or not. Let u be the unique solution of the differential equation:

$$-\frac{d^2}{dx^2}u + \frac{1}{4}u = \frac{1}{4}u \quad \text{and} \quad \frac{1}{2}u(0) + u'(0) = 0.$$

Then $u(x) = x - 2$. Since $u \notin L^2(dx)$, $\frac{1}{4} \notin \sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2)$. The rest of the proof is same as that of Lemma 7.5. □

We have already obtained $\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2)$ and $\sigma_{\mathrm{r}}(-\tilde{\mathfrak{A}}_2)$ explicitly in Lemmas 6.1 and 7.5. Now we investigate $\sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_2)$. Since any limit point of point spectra is either a point spectrum or a continuous spectrum, it was easy to see $\sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_p)$ for $1 \le p < 2$. However, in the case that $p = 2$ it is impossible to discuss continuous spectra in a similar way to the case that $1 \le p < 2$. Recall that by (7.3) it is sufficient to check the spectra of $\tilde{\mathfrak{A}}_2$ on $L^2(dx)$ defined on (7.2).

Let $\mathscr{E}$ and $\tilde{\mathscr{E}}$ be the bilinear forms associated with $\mathfrak{A}_2$ and $\tilde{\mathfrak{A}}_2$ respectively. Then, for $f, g \in C_b^2([0, \infty))$ such that $f(x) = g(x) = 0$ for $x > M$ with some $M > 0$, we have

$$\begin{aligned} \tilde{\mathscr{E}}(f, g) &= \mathscr{E}(I^{-1}f, I^{-1}g) \\ &= \int_0^\infty (e^{x/2}f(x))'(e^{x/2}g(x))'e^{-x}dx \end{aligned} \tag{7.8}$$

$$
\begin{aligned}
&= \int_0^\infty \left(f'(x)g'(x) + \frac{1}{2} f'(x)g(x) + \frac{1}{2} f(x)g'(x) + \frac{1}{4} f(x)g(x) \right) dx \\
&= \int_0^\infty f'(x)g'(x)dx + \frac{1}{4}\int_0^\infty f(x)g(x)dx + \frac{1}{2}\int_0^\infty (f(x)g(x))'dx \\
&= \int_0^\infty f'(x)g'(x)dx + \frac{1}{4}\int_0^\infty f(x)g(x)dx - \frac{1}{2} f(0)g(0).
\end{aligned}
$$

Denote the Sobolev space on $[0,\infty)$ with measure dx and indices k,p by $W^{k,p}(dx)$ where k is the index for differentiability and p is the index for integrability. Let

$$
\begin{aligned}
\mathrm{Dom}(\tilde{\mathfrak{A}}_2^{(0)}) &:= \left\{ f \in W^{2,2}(dx); f'(0) = 0 \right\}, \\
\tilde{\mathfrak{A}}_2^{(0)} &:= \frac{d^2}{dx^2} - \frac{1}{4},
\end{aligned}
$$

and $\tilde{\mathscr{E}}^{(0)}$ the bilinear form associated with $\tilde{\mathfrak{A}}_2^{(0)}$. Then, by using integration by parts, we have for $f,g \in W^{2,2}(dx) \cap \{f \in C_b^2([0,\infty)); f'(0) = 0 \text{ and } \lim_{x\to\infty} f(x) = 0\}$

$$
\begin{aligned}
\tilde{\mathscr{E}}^{(0)}(f,g) &= -\int_0^\infty \left(\tilde{\mathfrak{A}}_2^{(0)} f \right)(x)g(x)dx \\
&= -\int_0^\infty f''(x)g(x)dx + \frac{1}{4}\int_0^\infty f(x)g(x)dx \\
&= \int_0^\infty f'(x)g'(x)dx + \frac{1}{4}\int_0^\infty f(x)g(x)dx.
\end{aligned} \tag{7.9}
$$

Define a norm $||\cdot||_{\tilde{\mathscr{E}}_1^{(0)}}$ by

$$
||f||^2_{\tilde{\mathscr{E}}_1^{(0)}} = \tilde{\mathscr{E}}^{(0)}(f,f) + \int_0^\infty |f(x)|^2 dx.
$$

Then, by standard calculation we have that the closure of $\mathrm{Dom}(\tilde{\mathfrak{A}}_2^{(0)})$ with respect to $||\cdot||_{\tilde{\mathscr{E}}_1^{(0)}}$ is equal to $W^{1,2}(dx)$. Hence, $\mathrm{Dom}(\tilde{\mathscr{E}}^{(0)}) = W^{1,2}(dx)$. Now we have the following proposition.

Proposition 7.6. $\mathrm{Dom}(\tilde{\mathscr{E}}^{(0)}) = \mathrm{Dom}(\tilde{\mathscr{E}})$.

Proof. Since $\tilde{\mathscr{E}}(f,f) \le \mathscr{E}^{(0)}(f,f)$ for $f \in C_0^\infty([0,\infty))$, $\mathrm{Dom}(\tilde{\mathscr{E}}^{(0)}) \subset \mathrm{Dom}(\tilde{\mathscr{E}})$. To show $\mathrm{Dom}(\tilde{\mathscr{E}}^{(0)}) \supset \mathrm{Dom}(\tilde{\mathscr{E}})$, it is sufficient to show that $f \mapsto f(0)$ is a continuous linear functional on $W^{1,2}(dx)$. Let $f \in C_0^\infty([0,\infty))$. Since $f(x) = f(0) + \int_0^x f'(y)dy$, we have

$$
\begin{aligned}
|f(0)|^2 &= \int_0^1 \left| f(x) - \int_0^x f'(y)dy \right|^2 dx \\
&\le 2\int_0^1 |f(x)|^2 dx + 2\int_0^1 \left| \int_0^x f'(y)dy \right|^2 dx \\
&\le 2\int_0^\infty |f(x)|^2 dx + 2\int_0^1 \sqrt{x} \left(\int_0^x |f'(y)|^2 dy \right) dx \\
&\le 2\int_0^\infty |f(x)|^2 dx + 2\int_0^\infty |f'(y)|^2 dy.
\end{aligned}
$$

Hence, $f \mapsto f(0)$ is a continuous linear functional $W^{1,2}(dx)$. □

Now we extend the operators $\tilde{\mathfrak{A}}_2$ and $\tilde{\mathfrak{A}}_2^{(0)}$ by the same way as argument written in Section 2.2 of [11]. Let $H := L^2(dx)$, $V := \mathrm{Dom}(\tilde{\mathscr{E}}^{(0)}) = \mathrm{Dom}(\tilde{\mathscr{E}})$ and V^* the dual space of V. By the Riesz theorem, the dual of H can be identified with H^*. By this identification, we can regard $V \subset H = H^* \subset V^*$. Noting that V and H are dense subsets of H and V^* respectively, the operator $\tilde{\mathfrak{A}}_2$ can be extended to a operator from V to V^*. Denote the extension of $\tilde{\mathfrak{A}}_2$ by $\mathfrak{B}$. For $\lambda \in (0, \infty)$ $\lambda - \mathfrak{B}$ is a bijection from V to V^* and the inverse $(\lambda - \mathfrak{B})^{-1} : V^* \to V$ is an extension of the resolvent $(\lambda - \tilde{\mathfrak{A}}_2)^{-1} : H \to \mathrm{Dom}(\tilde{\mathfrak{A}}_2)$. We also define $\mathfrak{B}^{(0)}$ from $\tilde{\mathfrak{A}}_2^{(0)}$ similarly. Note that $\mathfrak{B}^{(0)}$ has same properties as $\mathfrak{B}$.

Denote the essential spectra of a linear operator A by $\sigma_{\mathrm{ess}}(A)$. The definition of essential spectra is in Section 2 of Chapter XII in [6]. Then, we have the following proposition.

Proposition 7.7. $\sigma_{\mathrm{ess}}(-\tilde{\mathfrak{A}}_2) = \sigma_{\mathrm{ess}}(-\tilde{\mathfrak{A}}_2^{(0)}) = \big[\frac{1}{4}, \infty\big)$.

Proof. It is well-known that $\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2^{(0)}) = \emptyset$ and $\sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_2^{(0)}) = \big[\frac{1}{4}, \infty\big)$. Since $-\tilde{\mathfrak{A}}_2$ and $-\tilde{\mathfrak{A}}_2^{(0)}$ are non-negative definite, $-1 \in \rho(-\tilde{\mathfrak{A}}_2) \cap \rho(-\tilde{\mathfrak{A}}_2^{(0)})$. Once we have the compactness of the bounded linear operator $(1 - \tilde{\mathfrak{A}}_2)^{-1} - (1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1}$ on H, we obtain the conclusion by Weyl's theorem (see Theorem XIII.14 in [6]).

$$\begin{aligned}
&(1 - \tilde{\mathfrak{A}}_2)^{-1} - (1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1} \\
&= (1 - \tilde{\mathfrak{A}}_2)^{-1}(1 - \tilde{\mathfrak{A}}_2^{(0)})(1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1} - (1 - \tilde{\mathfrak{A}}_2)^{-1}(1 - \tilde{\mathfrak{A}}_2)(1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1} \\
&= (1 - \mathfrak{B})^{-1}(1 - \mathfrak{B}^{(0)})(1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1} - (1 - \mathfrak{B})^{-1}(1 - \mathfrak{B})(1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1} \\
&= (1 - \mathfrak{B})^{-1}(\mathfrak{B} - \mathfrak{B}^{(0)})(1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1}.
\end{aligned}$$

The linear operator $(1 - \mathfrak{B})^{-1}(\mathfrak{B} - \mathfrak{B}^{(0)})(1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1}$ is the mapping as follows.

$$H \xrightarrow{(1-\tilde{\mathfrak{A}}_2^{(0)})^{-1}} \mathrm{Dom}(\tilde{\mathfrak{A}}_2^{(0)}) \hookrightarrow V \xrightarrow{\mathfrak{B}-\mathfrak{B}^{(0)}} V^* \xrightarrow{(1-\mathfrak{B})^{-1}} V \hookrightarrow H.$$

Since $(1 - \tilde{\mathfrak{A}}_2^{(0)})^{-1}$ and $(1 - \mathfrak{B})^{-1}$ are continuous, it is sufficient to show the compactness of the operator $\mathfrak{B} - \mathfrak{B}^{(0)}$ form V to V^*. By (7.8) and (7.9) we have for $f, g \in V$

$${}_{V^*}\langle (\mathfrak{B} - \mathfrak{B}^{(0)})f, g \rangle_V = \frac{1}{2} f(0)g(0).$$

This implies that $\mathfrak{B} - \mathfrak{B}^{(0)}$ is a mapping $f \mapsto f(0)\delta$ where $\delta \in V^*$ is a bounded linear operator on V defined by $\delta(g) = g(0)$ for V. Hence, the range of $\mathfrak{B} - \mathfrak{B}^{(0)}$ is one-dimensional. This concludes the compactness of $\mathfrak{B} - \mathfrak{B}^{(0)}$. □

By Lemma 7.5 and Proposition 7.7 we obtain the explicit information of spectra of $\tilde{\mathfrak{A}}_2$ as follows.

Theorem 7.8. *It holds that*

$$\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2) = \{0\}, \quad \sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_2) = \left[\frac{1}{4}, \infty\right).$$

Proof. We have already obtained that $\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2) = \{0\}$ in Lemma 7.5. Noting that $\sigma_{\mathrm{p}}(-\tilde{\mathfrak{A}}_2) \cap \sigma_{\mathrm{ess}}(-\tilde{\mathfrak{A}}_2) = \emptyset$, by the definition of essential spectra we have $\sigma_{\mathrm{c}}(-\tilde{\mathfrak{A}}_2) = \sigma_{\mathrm{ess}}(-\tilde{\mathfrak{A}}_2) = \left[\frac{1}{4}, \infty\right)$. □

By (7.3) we have the following theorem.

Theorem 7.9. *It holds that*

$$\sigma_{\mathrm{p}}(-\mathfrak{A}_2) = \{0\}, \quad \sigma_{\mathrm{c}}(-\mathfrak{A}_2) = \left[\frac{1}{4}, \infty\right).$$

The picture of $\sigma_{\mathrm{p}}(-\mathfrak{A}_2)$, $\sigma_{\mathrm{c}}(-\mathfrak{A}_2)$ and $\rho(-\mathfrak{A}_2)$ is described in Figure 3.

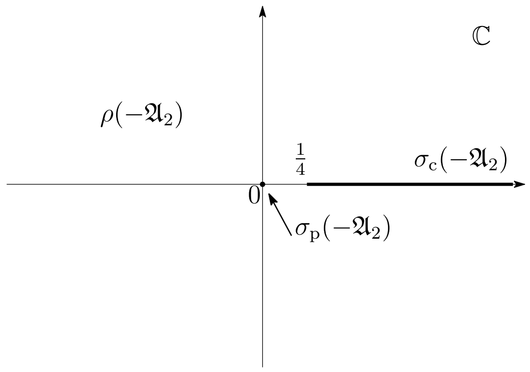

Figure 3: $p = 2$

By Theorems 7.4 and 7.9 we obtain the spectra of $-\mathfrak{A}_p$ exactly for $p \in [1, 2]$ as described in Figures 1, 2 and 3.

We have considered only the case that $1 \leq p \leq 2$. We also obtain $\sigma_{\mathrm{p}}(-\mathfrak{A}_p)$, $\sigma_{\mathrm{c}}(-\mathfrak{A}_p)$ and $\sigma_{\mathrm{r}}(-\mathfrak{A}_p)$ explicitly for $p \in (2, \infty)$ by using Proposition 6.2, Corollary 6.4 and Theorems 7.4 and 7.9.

Theorem 7.10. *For $p \in (2, \infty)$, we have the following.*

(i) $\sigma_{\mathrm{p}}(-\mathfrak{A}_p) = \{0\}$,

(ii) $\sigma_{\mathrm{c}}(-\mathfrak{A}_p) = \left\{x + iy;\ x, y \in \mathbb{R},\ x \geq \frac{p^*-1}{p^{*2}} \text{ and } |y| = \left(\frac{2}{p^*} - 1\right)\sqrt{x - \frac{p^*-1}{p^{*2}}}\right\} \backslash \{0\}$,

(iii) $\sigma_{\mathrm{r}}(-\mathfrak{A}_p) = \left\{x + iy;\ x, y \in \mathbb{R},\ x > \frac{p^*-1}{p^{*2}} \text{ and } |y| < \left(\frac{2}{p^*} - 1\right)\sqrt{x - \frac{p^*-1}{p^{*2}}}\right\}$,

(iv) $\rho(-\mathfrak{A}_p) = \left\{x + iy;\ x, y \in \mathbb{R},\ y^2 > \left(\frac{2}{p^*} - 1\right)^2 \left(x - \frac{p^*-1}{p^{*2}}\right)\right\} \setminus \{0\}$.

Proof. Let $p \in (2, \infty)$. Since $\sigma(-\mathfrak{A}_p) = \sigma(-\mathfrak{A}_{p^*})$, we have (iv). By Theorem 7.4 and Corollary 6.4 we obtain (ii). By Corollary 6.4 again, we have

$$\sigma_{\mathrm{p}}(-\mathfrak{A}_p) \cup \sigma_{\mathrm{r}}(-\mathfrak{A}_p) = \sigma_{\mathrm{p}}(-\mathfrak{A}_{p^*}). \tag{7.10}$$

On the other hand, applying Proposition 6.2 for $q = 2$, we have $\sigma_{\mathrm{p}}(-\mathfrak{A}_p) \subset \sigma_{\mathrm{p}}(-\mathfrak{A}_2)$. Hence, Theorem 7.9 implies that $\sigma_{\mathrm{p}}(-\mathfrak{A}_p) \subset \{0\}$. Since $\sigma_{\mathrm{p}}(-\mathfrak{A}_p) \supset \{0\}$, we obtain (i). By (7.10), (i) and Theorem 7.4, we have (iii). □

This operator $-\mathfrak{A}_p$ is an example that the spectra depend on p, the spectra are not included by $\mathbb{R}$ for $p \neq 2$, and $\sigma_{\mathrm{c}}(-\mathfrak{A}_q) \subset \sigma_{\mathrm{p}}(-\mathfrak{A}_p)$ for some $p < q \leq 2$ even if $-\mathfrak{A}_p$ is a diffusion operator, consistent on $L^p(\nu)$ for $p \in [1, \infty)$, self-adjoint when $p = 2$, and ergodic.

In view of the argument in Section 2, the exact information on the spectra of $-\mathfrak{A}_p$ give the explicit value of $\gamma_{p\to p}$ as follows.

Corollary 7.11.

$$\gamma_{p\to p} = \frac{p-1}{p^2}, \ p \in [1, \infty].$$

Proof. Since $-\mathfrak{A}_2$ is self-adjoint on $L^2(\nu)$, the argument in Section 2 is available and (2.5) holds. By (2.5) we have $\gamma_{p\to p} = \frac{p-1}{p^2}$ for $p \in (1, 2]$. By Theorem 2.4 we have $0 \leq \gamma_{1\to 1} \leq \inf\{\gamma_{p\to p}; p \in [1, 2]\} = 0$. Hence, $\gamma_{1\to 1} = 0$. By Theorem 2.4 again $\gamma_{p\to p} = \gamma_{p^*\to p^*}$ for $p \in [1, \infty]$. Therefore, the assertion holds. □

Thus, we obtain an example that the exponential rate of convergence $\{\gamma_{p\to p}; p \in [1, \infty]\}$ depend on p.

References

[1] E. B. Davies, *One-parameter semigroups*, London Mathematical Society Monographs **15**, Academic Press, London-New York, 1980.

[2] E. B. Davies, *Linear operators and their spectra*, Cambridge Studies in Advanced Mathematics **106**, Cambridge University Press, Cambridge, 2007.

[3] J.-D. Deuschel and D. W. Stroock, *Large deviations*, Academic Press, San Diego, 1989.

[4] K. -J. Engel and R. Nagel,*One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics **194**, Springer-Verlag, New York, 2000.

[5] P. A. Meyer, Notes sur les processus d'Ornstein-Uhlenbeck, *Séminaire de Prob.* XVI, ed. par J. Azema et M. Yor, Lecture Notes in Math., vol. 920 (1982), 95-133, Springer-Verlag, Berlin-Heidelberg-New York.

[6] M. Reed and B. Simon, *Method of modern mathematical physics IV. Analysis of operator*, Academic Press, New York-London, 1978.

[7] M. Röckner and F.-Y. Wang, Supercontractivity and ultracontractivity for (non-symmetric) diffusion semigroups on manifolds, *Forum Math.*, 15 (2003), no. 6, 893—921.

[8] W. Rudin, *Functional analysis*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York-Dusseldorf-Johannesburg, 1973.

[9] I. Shigekawa, Non-symmetric diffusions on a Riemannian manifold, *Probabilistic approach to geometry*, Adv. Stud. Pure Math. **57**, 437-461, Math. Soc. Japan, Tokyo, 2010.

[10] E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Annals of Mathematics Studies **63**, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1970.

[11] H. Tanabe, *Equations of evolution*, Monographs and Studies in Mathematics **6**, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1979.