# Kolmogorov-Pearson diffusions and hypergeometric functions

Ichiro Shigekawa (Kyoto University)

October 16, 2017, Ritsumeikan University

Stochastic analysis and related topics

URL: http://www.math.kyoto-u.ac.jp/~ichiro/

# Contents

# 1-dimensional diffusion processes

# Generator

General form of a generator:

$$\mathfrak{A} = a\frac{d^2}{dx^2} + b\frac{d}{dx}. \tag{1}$$

**Definition 1**

If $a$ is quadratic and $b$ is linear, then the associated diffusion process is called a **Kolmogorov-Pearson diffusion**.

The speed measure of a Kolmogorov-Pearson diffusion is of a Pearson distribution.

Several expressions of a generator are known.

## Feller's expression

Suppose we are given two smooth positive functions $a$ and $\rho$ on an interval $I$. Define a speed measure $dm$ and a scale function $s$ by

$$dm = \rho dx \tag{2}$$

and

$$s' = \frac{1}{a\rho}. \tag{3}$$

So two functions $a$, $\rho$ determine a diffusion.

The generator $\mathfrak{A}$ can be written as follows:

$$\begin{aligned}\mathfrak{A} &= \frac{d}{dm}\frac{d}{ds} = \frac{1}{\rho}\frac{d}{dx}a\rho\frac{d}{dx} \\ &= a\frac{d^2}{dx^2} + \frac{(a\rho)'}{\rho}\frac{d}{dx} \\ &= a\frac{d^2}{dx^2} + (a' + a(\log\rho)')\frac{d}{dx}\end{aligned}$$

In expression of (1), $b$ is given as

$$b = \frac{(a\rho)'}{\rho} = a' + a(\log\rho)' \tag{4}$$

# Expressions of the generator

| | generator | dual operator | differentiation |
|---|---|---|---|
| Kolmogorov | $a\frac{d^2}{dx^2} + b\frac{d}{dx}$ | | |
| Feller | $\frac{d}{dm}\frac{d}{ds}$ | $\frac{d}{dm} = -\frac{d}{ds}^*$ | $\frac{d}{ds} : L^2(dm) \to L^2(ds)$ |
| Stein | $\left(a\frac{d}{dx} + b\right)\frac{d}{dx}$ | $a\frac{d}{dx} + b = -\frac{d}{dx}^*$ | $\frac{d}{dx} : L^2(\rho) \to L^2(a\rho)$ |

By the Feller's duality and the Stein's duality, we can get the following pairings:

| | |
|---|---|
| Feller's pair | $\frac{d}{dm}\frac{d}{ds} \longleftrightarrow \frac{d}{ds}\frac{d}{dm}$ |
| Stein's pair | $(a\frac{d}{dx}+b)\frac{d}{dx} \longleftrightarrow \frac{d}{dx}(a\frac{d}{dx}+b)$ |

These pairs have a feature that they have the same spectrum except for $0$. This is called a supersymmetry. Further we have

- If $f$ is an eigenfunction, then so are $f'$, $\frac{d}{ds}f$.
- If $\theta$ is an eigenfunction, then so are $a\theta' + b\theta$, $\frac{d}{dm}\theta$.

# Classification of the diffusions

We can classify the diffusions according to the form of $a$.
Based on the degree of $a$, we have the following six cases.

(I) $a = 1$ on $(-\infty, \infty)$

(II) $a = x$ on $(0, \infty)$

(III-1) $a = x^2$ on $(0, \infty)$

(III-2-a) $a = x(1 - x)$ on $(0, 1)$

(III-2-b) $a = x(x + 1)$ on $(0, \infty)$

(III-3) $a = x^2 + 1$ on $(-\infty, \infty)$

To sum up, we have

| | complete family | incomplete family | | special function |
|---|---|---|---|---|
| $\alpha$-family | $a = 1$ | | | |
| $\beta$-family | $a = x$ | $a = x^2$ | | ${}_0F_1,\ {}_1F_1$ |
| $\gamma$-family | $a = x(1 - x)$ | $a = x(1 + x)$ | $a = 1 + x^2$ | ${}_2F_1$ |

Further, associated speed measures are given as follows:

| | complete family | incomplete family | |
|---|---|---|---|
| $\alpha$-family | $e^{\beta x^2/2}$ | | |
| $\beta$-family | $x^\alpha e^{\beta x}$ | $x^\alpha e^{\beta/x}$ | |
| $\gamma$-family | $x^\alpha(1 - x)^\beta$ | $x^\alpha(1 + x)^\beta$ | $(1 + x^2)^\alpha \exp\{2\beta \arctan x\}$ |

# Doob's $h$-transformation

To observe the spectrum, Doob's $h$-transformation is an important tool. It gives a unitary equivalence with other operator. It's based on a harmonic function. We give a typical example.

**Theorem 2**

*Setting $\varphi = \rho^{-1}$, we have*

$$\mathfrak{A}\varphi = (a'' - b')\varphi. \tag{5}$$

From our assumption, $a'' - b'$ is a constant. Hence $\varphi$ is a harmonic function.

Using this fact, we can show the following.

**Theorem 3**

*The following operator*

$$\tilde{\mathfrak{A}} = a\frac{d^2}{dx^2} + (2a' - b)\frac{d}{dx} + (a'' - b') \tag{6}$$

*in a Hilbert space $L^2(\rho^{-1}\,dx)$ has the same spectrum as $\mathfrak{A}$.*

In fact, we have the following commutative diagram.

$$\begin{array}{ccc} L^2(I;\rho\,dx) & \xrightarrow{\mathfrak{A}} & L^2(I;\rho\,dx) \\ J\uparrow & & \uparrow J \\ L^2(I;\rho^{-1}\,dx) & \xrightarrow{\tilde{\mathfrak{A}}} & L^2(I;\rho^{-1}\,dx) \end{array} \tag{7}$$

# Black-Scholes family

# (III-1) $a = x^2, I = (0, \infty)$.

Our generator is of the form

$$\mathfrak{A} = x^2 \frac{d}{dx^2} + (\alpha x - \beta)\frac{d}{dx}. \tag{8}$$

When $\beta = 0$, the associated diffusion is the Black-Scholes model. So we call this class as Black-Scholes family.

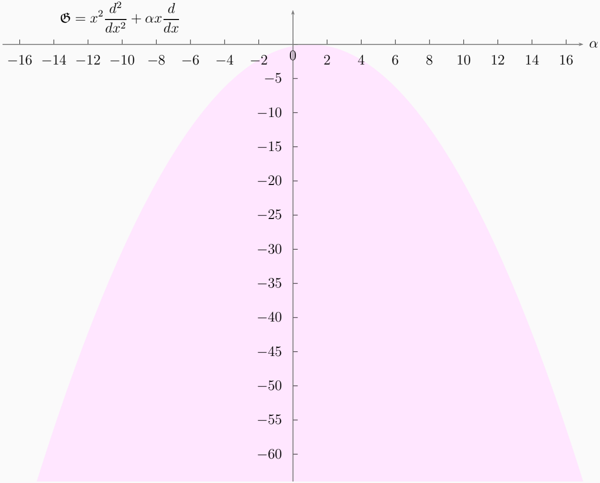
**Case $\beta = 0$**

The generator is of the form:

$$\mathfrak{A} = x^2 \frac{d}{dx^2} + \alpha x \frac{d}{dx}. \tag{9}$$

The spectrum is given by

$$\sigma(\mathfrak{A}) = (-\infty, -\frac{1}{4}(\alpha - 1)^2] \qquad (10)$$



## Case $\beta = -1$

The generator is of the form:

$$\mathfrak{A} = x^2 \frac{d}{dx^2} + (\alpha x + 1)\frac{d}{dx}. \tag{11}$$

For $n = 0, 1, 2, \ldots$, define $\lambda_n(\alpha)$ by

$$\lambda_n(\alpha) = n(n - 1 + \alpha). \tag{12}$$

Then the spectrum is given by

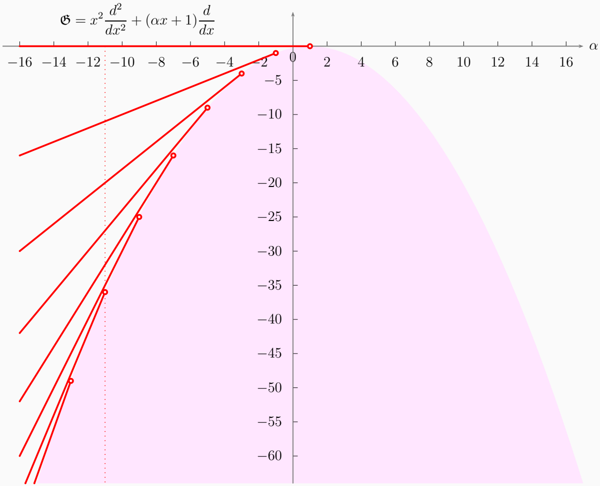
$$\sigma_{\text{ess}}(\mathfrak{A}) = (-\infty, -\frac{1}{4}(\alpha - 1)^2]$$

$$\sigma_{\text{p}}(\mathfrak{A}) = \{\lambda_n(\alpha);\ 0 \leq n < \frac{1-\alpha}{2}\}.$$

The associated eigenfunction is given by

$$P_n^{(\alpha)}(x) = x^n L_n^{(1-2n-\alpha)}\left(\frac{1}{x}\right). \tag{13}$$

Here, $L_n^{(1-2n-\alpha)}$ is a Laguerre polynomial.



$\mathfrak{G} = x^2 \frac{d^2}{dx^2} + (\alpha x + 1)\frac{d}{dx}$
α
−16 −14 −12 −10 −8 −6 −4 −2 0 2 4 6 8 10 12 14 16
−5 −10 −15 −20 −25 −30 −35 −40 −45 −50 −55 −60

## Case $\beta = 1$

The generator is of the form:

$$\mathfrak{A} = x^2 \frac{d}{dx^2} + (\alpha x - 1)\frac{d}{dx}. \tag{14}$$

For $n = 1, 2, \ldots,$ define $\xi_n(\alpha)$ by

$$\xi_n(\alpha) = n(n + 1 - \alpha) \tag{15}$$

Then the spectrum is given as

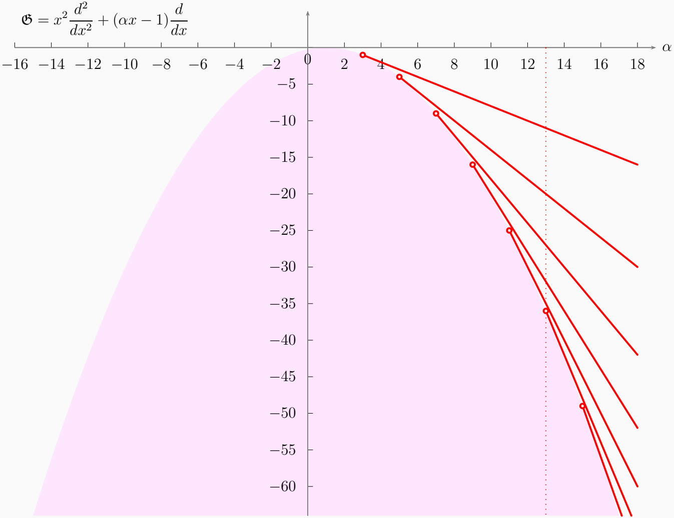
$$\sigma_{\mathrm{ess}}(\mathfrak{A}) = (-\infty, -\frac{1}{4}(\alpha - 1)^2]$$

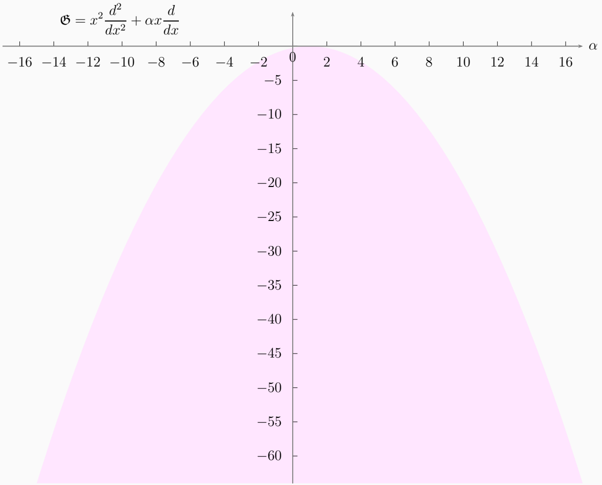
$$\sigma_{\mathrm{p}}(\mathfrak{A}) = \{\xi_n(\alpha);\ 1 \le n < \frac{\alpha - 1}{2}\}.$$

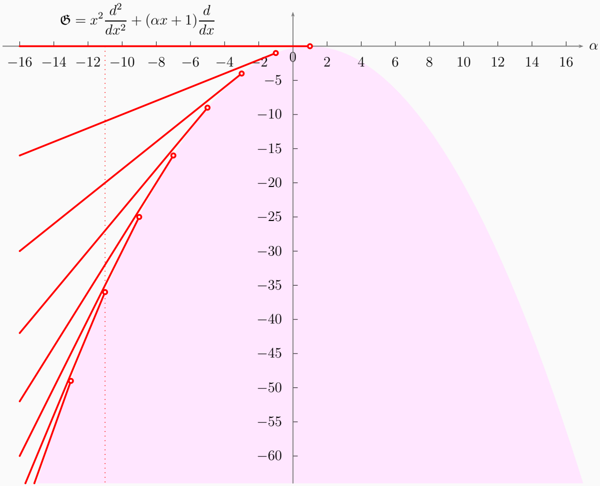
The associated eigenfunction is given by

$$x^{-\alpha+2}e^{-1/x}P_{n-1}^{(4-\alpha)}(x) = x^{n-\alpha+1}e^{-1/x}L_{n-1}^{(\alpha-2n-1)}\left(\frac{1}{x}\right).$$

Here $L_{n-1}^{(\alpha-2n-1)}$ is a Laguerre polynomial.


$\mathfrak{G} = x^2 \frac{d^2}{dx^2} + (\alpha x - 1)\frac{d}{dx}$
$\alpha$
−16 −14 −12 −10 −8 −6 −4 −2 0 2 4 6 8 10 12 14 16 18
−5 −10 −15 −20 −25 −30 −35 −40 −45 −50 −55 −60


$\mathfrak{G} = x^2 \frac{d^2}{dx^2} + \alpha x \frac{d}{dx}$
$\alpha$
−16 −14 −12 −10 −8 −6 −4 −2 0 2 4 6 8 10 12 14 16
−5 −10 −15 −20 −25 −30 −35 −40 −45 −50 −55 −60


$\mathfrak{G} = x^2 \frac{d^2}{dx^2} + (\alpha x + 1)\frac{d}{dx}$
α
−16 −14 −12 −10 −8 −6 −4 −2 0 2 4 6 8 10 12 14 16
−5 −10 −15 −20 −25 −30 −35 −40 −45 −50 −55 −60

# Jacobi family

## (III-2-a) $a = x(1-x)$, $I = (0,1)$

Our generator is of the form

$$\mathfrak{A} = x(1-x)\frac{d^2}{dx^2} + ((\alpha+1)(1-x) - (\beta+1))x)\frac{d}{dx}. \qquad (16)$$

We call this family as Jacobi family since eigenfunctions are Jacobi polynomials.

**Case $\alpha > -1, \beta > -1$.**

For $n = 0, 1, 2, \dots$ define $\lambda_n$ by

$$\lambda_n(\alpha,\beta) = -n(n+\alpha+\beta+1). \qquad (17)$$

The spectrum is given by

$$\sigma(\mathfrak{A}) = \{\lambda_n(\alpha,\beta);\ n = 0, 1, 2, \dots\}. \qquad (18)$$

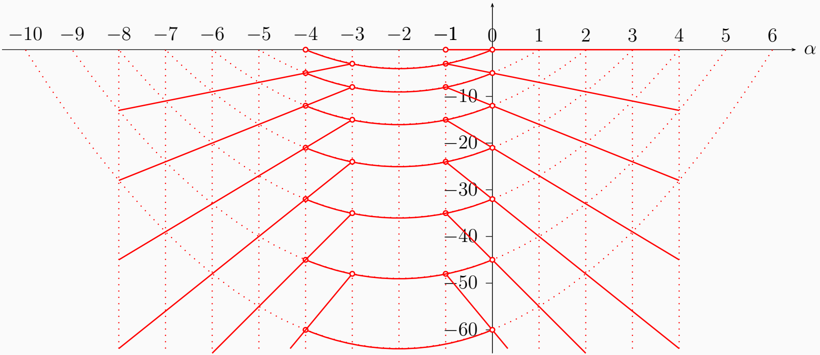
The associated eigenfunction is given by

$$K(\alpha, \beta, n; x) = {}_2F_1(-n, \alpha + \beta + n + 1; \alpha + 1; x)$$

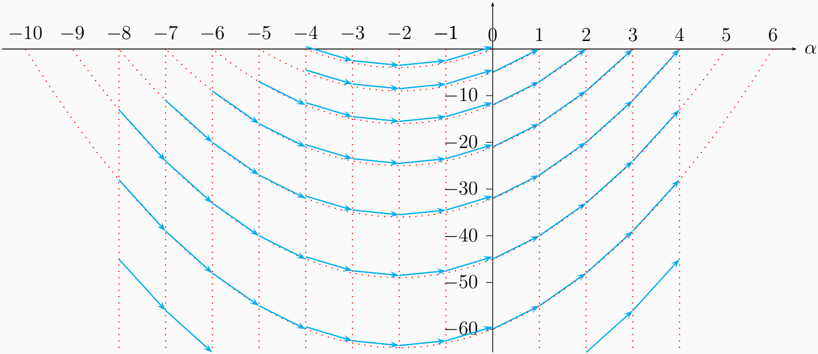
Here ${}_2F_1$ is a hypergeometric function. Since $n$ is an integer, $K(\alpha, \beta, n; x)$ is a polynomial. In this case we have a complete basis of polynomials.

Other cases can be obtained similarly.

To sum up, we have the following picture of spectra. Here we choose $\alpha$ as a parameter and restrict to the case $\boldsymbol{\beta = \alpha + 3}$.



The Stein's correspondence of differentiation is shown as



# Fisher family

# (III-2-b) $a = x(1+x)$, $I = [0, \infty)$

The generator is given by

$$\mathfrak{A} = x(1+x)\frac{d^2}{dx^2} + ((\alpha+1)(1+x) + (\beta+1))x)\frac{d}{dx}. \quad (19)$$

We call this family as Fisher family since speed mesures are of Fisher distribution.

**Case $\alpha > -1$**

The condition $\alpha > -1$ corresponds to that the boundary $0$ is entrance.

For $n = 0, 1, 2, \dots$, define

$$\lambda_n(\alpha,\beta) = \left(n - \frac{|\beta| + \beta}{2}\right)\left(n + \alpha - \frac{|\beta| - \beta}{2} + 1\right) = \begin{cases} (n-\beta)(n+\alpha+1), \\ n(n+\alpha+\beta+1), \end{cases}$$

Then

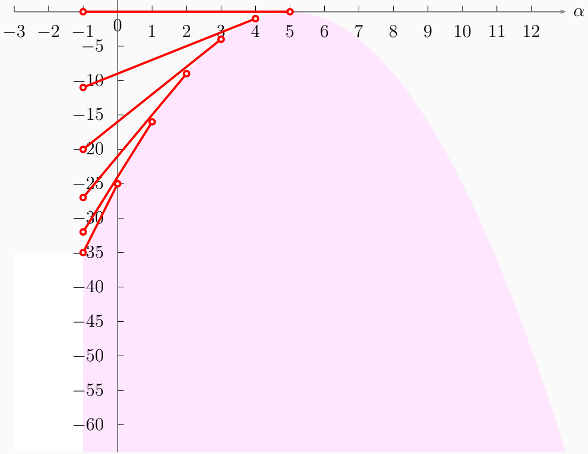
**Theorem 4**

*The spectrum of $\mathfrak{A}$ is given as*

$$\sigma_{ess}(\mathfrak{A}) = \left(-\infty, -\frac{(\alpha+\beta+1)^2}{4}\right]$$

$$\sigma_p(\mathfrak{A}) = \{\lambda_n(\alpha,\beta);\ 0 \le n < \left[\frac{-\alpha+|\beta|-1}{2}\right]\}$$

$$\mathfrak{G} = x(1+x)\frac{d^2}{dx^2} + ((\alpha+1)(1+x)+(\beta+1)x)\frac{d}{dx}, \qquad \beta = \alpha - 11$$



## Case $\alpha < 0$

The condition $\alpha < 0$ corresponds to that the boundary $0$ is exit.

For $n = 1, 2, \dots$, define

$$\xi_n(\alpha, \beta) = \Big(n - \frac{|\beta| - \beta}{2}\Big)\Big(n - \alpha - \frac{|\beta| + \beta}{2} + 1\Big)$$
$$= \begin{cases} n(n - \alpha - \beta - 1), & \beta \geq 0, \\ (n + \beta)(n - \alpha - 1), & \beta \leq 0. \end{cases}$$

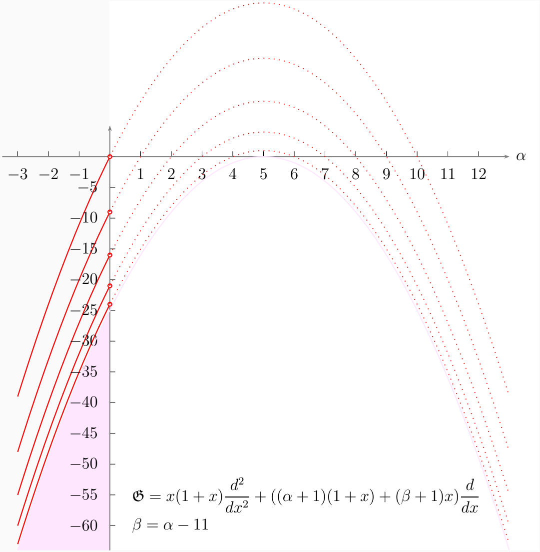
Then

**Theorem 5**

*The spectrum of $\mathfrak{A}$ is given as*

$$\sigma_{ess}(\mathfrak{A}) = \Big(-\infty, -\frac{(\alpha+\beta+1)^2}{4}\Big]$$

$$\sigma_p(\mathfrak{A}) = \{\xi_n(\alpha,\beta);\ 1 \le n < \Big[\frac{\alpha+|\beta|+1}{2}\Big]\}.$$


α
−3 −2 −1 1 2 3 4 5 6 7 8 9 10 11 12
−5 −10 −15 −20 −25 −30 −35 −40 −45 −50 −55 −60
$\mathfrak{G} = x(1+x)\frac{d^2}{dx^2} + ((\alpha+1)(1+x) + (\beta+1)x)\frac{d}{dx}$
$\beta = \alpha - 11$

# Student family

# (III-3) $a = 1 + x^2$, $I = (-\infty, \infty)$

The generator is given by

$$\mathfrak{A} = (1 + x^2)\frac{d^2}{dx^2} + (2(\alpha + 1)x + 2\beta)\frac{d}{dx}. \qquad (20)$$

We call this family as Student family since speed mesures are of student's $t$-distribution when $\beta = 0$.

**Theorem 6**

*The spectrum of $\mathfrak{A}$ is as follows: For the essential spectrum,*

$$\sigma_{ess}(\mathfrak{A}) = \left(-\infty, -(\alpha + \frac{1}{2})^2\right]. \tag{21}$$

*For the point spectrum, in the case $\alpha < -\frac{1}{2}$, it consists of*

$$\lambda_n(\alpha) = n(n + 2\alpha + 1), \quad 0 \leq n < -\alpha - \frac{1}{2} \tag{22}$$

*and in the case of $\alpha > \frac{1}{2}$, it consists of*

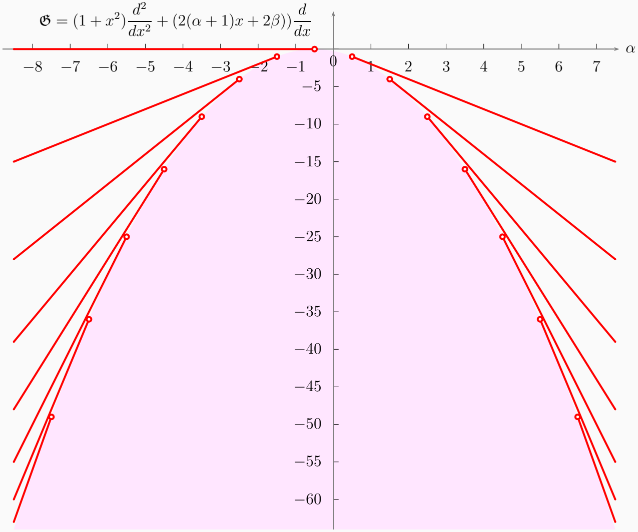
$$\xi_n(\alpha) = n(n - 2\alpha - 1), \quad 1 \leq n < \alpha + \frac{1}{2}. \tag{23}$$

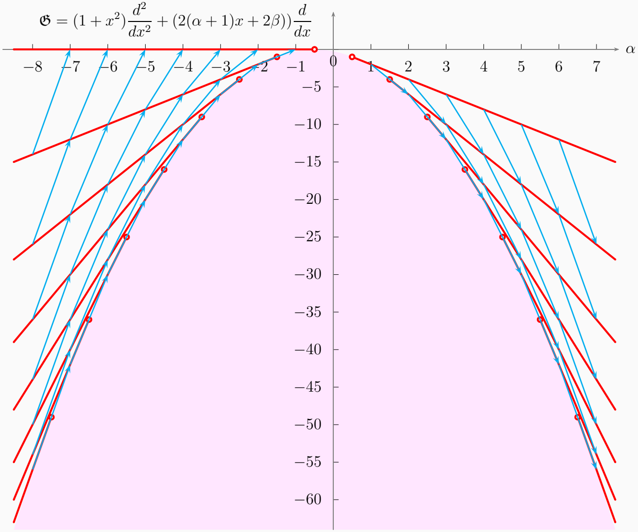
*There is no point spectrum when $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$.*

The associated eigenfunction is given by

$$x \mapsto K(\alpha + i\beta, \alpha - i\beta, n, \frac{1 - ix}{2}).$$

# We draw a picture for a fixed $\boldsymbol{\beta}$.





$\mathfrak{G} = (1+x^2)\frac{d^2}{dx^2} + (2(\alpha+1)x + 2\beta)\frac{d}{dx}$
$\alpha$
−8
−7
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6
7
−5
−10
−15
−20
−25
−30
−35
−40
−45
−50
−55
−60

Thank you very much