Non-symmetric diffusions on a Riemannian manifold

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Geometry and Probability in Yamagata

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Contents

1. Non-symmetric diffusions on a Riemannian manifold

- (M, g): d-dimensional connected complete Riemannian manifold.
- vol : the Riemannian volume.
- $d\nu = e^{-U} d\mathrm{vol}$: a reference measure
- b : a vector field on M.
- V : a potential funciton on M.
- $\triangle$: the Laplace-Beltrami operator.

We consider the following opetaror in $L^2(\nu)$:

(1)
$$\mathfrak{A} = \frac{1}{2}\triangle + \nabla_b - V.$$

We need to change the expression of $\mathfrak{A}$.

- ∇: the covariant differentiation
- $\triangle = -\nabla^* \nabla$
- ∇^*: the dual operator of ∇ with respect to $\mathbf{vol}$.

The dual operator of ∇ with respect to ν is given by

$$\nabla_\nu^* = e^U \nabla^* e^{-U}$$

Then

$$\triangle u = -\nabla_\nu^* \nabla u + (\nabla U, \nabla u).$$

So we set

$$\tilde{b} = \frac{1}{2} \nabla U^\sharp + b. \tag{2}$$

Then

$$\mathfrak{A} = -\frac{1}{2}\nabla_{\nu}^{*}\nabla + \nabla_{\tilde{b}} - V$$

The dual opetrator of $\mathfrak{A}$ with respect to ν is given by

$$\mathfrak{A}_{\nu}^{*} = -\frac{1}{2}\nabla_{\nu}^{*}\nabla - \nabla_{\tilde{b}} - \mathrm{div}_{\nu}\,\tilde{b} - V. \tag{3}$$

Here

$$\mathrm{div}_{\nu}\,X = e^{U}\,\mathrm{div}(e^{-U}X) = \mathrm{div}\,X - XU.$$

They are well-defined in $C_0^{\infty}(M)$.

The bilinear form associated with $\mathfrak{A}$ is

$$\begin{aligned}\mathcal{E}(u,v) &= -(\mathfrak{A}u,v)_2\\ &= \frac{1}{2}\int_M (\nabla u, \nabla v)\,d\nu - \int_M (\nabla_{\tilde{b}} u) v\,d\nu + \int_M V uv\,d\nu.\end{aligned}$$

We denote the symmetrization of $\mathcal{E}$ by $\tilde{\mathcal{E}}$:

$$\tilde{\mathcal{E}}(u,v) = \frac{1}{2}\int_M (\nabla u, \nabla v)\,d\nu + \frac{1}{2}\int_M (\mathrm{div}_\nu\, \tilde{b}) uv\,d\nu + \int_M V uv\,d\nu.$$

$\tilde{\mathcal{E}}$ is associated with $\frac{1}{2}\{\mathfrak{A} + \mathfrak{A}^*_\nu\}$.

We are interested in when the semigroup associated to $\mathfrak{A}$ exists in L^2.

We impose the following condition to ensure that $-\mathfrak{A}$ is bounded from below.

(B.1) $\exists\, \gamma \in \mathbb{R}: \quad \frac{1}{2}\,\mathrm{div}_\nu\, \tilde{b} + V \geq -\gamma.$

Under this condition, $\tilde{\mathcal{E}}$ is bounded from below and closable.

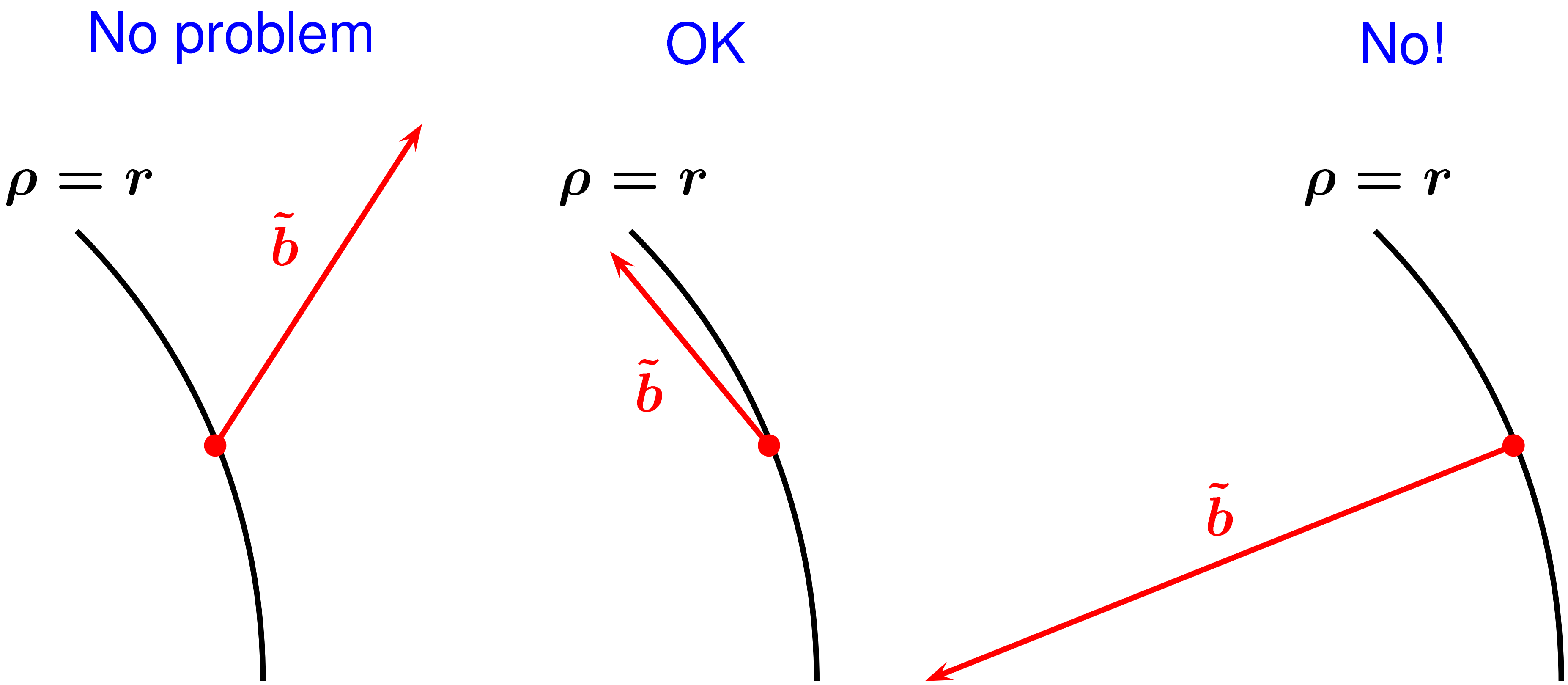
- d: the distance function
- $o \in M$: a fixed reference point
- $\rho(x) = d(o, x)$

We add the following condition for $\tilde{b}$:

(B.2) $\exists\, \kappa : [0, \infty) \to [0, 1]$ with $\int_0^\infty \kappa(x)\, dx = \infty$ so that

$$\kappa(\rho) \nabla_{\tilde{b}} \rho \geq -1.$$

A typical example is $\kappa(x) = \frac{1}{x}$. $\quad \nabla_{\tilde{b}} \rho(x) \geq -\rho(x)$.


No problem
OK
No!
$\rho = r$
$\rho = r$
$\rho = r$
$\tilde{b}$
$\tilde{b}$
$\tilde{b}$

Theorem 1. Under the assumptions (B.1) and (B.2), the closure of $(\mathfrak{A}, C_0^\infty(M))$ generates a positivity preserving C_0-semigroup in $L^2(m)$.

We claim the following:

- the dissipativity: $((\mathfrak{A} - \gamma)u, u)_2 \leq 0$.
- the maximality: $(\mathfrak{A} - \gamma - 1)(C_0^\infty(M))$ is dense in L^2.

In fact,

$$((\mathfrak{A} - \gamma)u, u)_2 = -\frac{1}{2}\int_M (|\nabla u|^2 + u^2 \operatorname{div} b)\, dm - \int_M (V + \gamma)u^2\, dm$$

$$\begin{aligned}(\mathfrak{A}-\gamma-1)^*u=0 \quad &\Rightarrow \quad u\in C^\infty(M)\\ &\Rightarrow \quad (u,(\mathfrak{A}-\gamma-1)(\chi_n^2 u))_2=0\\ &\Rightarrow \quad u=0\end{aligned}$$

The positivity preserving property is checked by the following criterion:

(4) $$(\mathfrak{A}u, u_+)_2 \le \gamma \|u_+\|_2^2.$$

Assume the following Sobolev inequality: there exist $p>2$ and $C>0$ so that

$$\|u\|_p^2 \le C(\|\nabla u\|_2^2 + \|u\|_2^2).$$

Then the condition (B.1) can be relaxed as follows:

(B.1)′ $\exists\, \gamma \in \mathbb{R} : (\frac{1}{2}\,\mathrm{div}_\nu\, \tilde{b} + V + \gamma)_- \in L^{p/(p-2)}(\nu).$

Markovian property

The semigroup generated by $\mathfrak{A}$ is denoted by $\{T_t\}$. We can also give a criterion for the Markovian property of $\{T_t\}$.

Proposition 2. Under the assumptions (B.1) and (B.2), $\{e^{-\alpha t}T_t\}$ is Markovian if and only if $V + \alpha \geq 0$.

To show this, we use the following characterization: $\{e^{-\alpha t}T_t\}$ is Markovian if and only if

$$((\mathfrak{A} - \alpha)u, (u-1)_+)_2 \leq (\gamma - \alpha)\|(u-1)_+\|_2^2$$

L^1 contraction property

Similarly, we have

Proposition 3. Under the assumptions (B.1) and (B.2), $\{e^{-\beta t}T_t\}$ has the L^1 contraction property if and only if $\mathbf{div}_\nu \tilde{b} + V \geq -\beta$.

To show this, we use the following characterization: $\{e^{-\beta t}T_t\}$ has L^1 contraction property if and only if

$$((\mathfrak{A} - \beta)u, u_+ \wedge 1)_2 \leq (\gamma - \beta)\|u_+ \wedge 1\|_2^2$$

As for $\mathfrak{A}_\nu^*$

$$\mathfrak{A}^* = -\frac{1}{2}\nabla_\nu^*\nabla - \nabla_{\tilde{b}} - \mathrm{div}_\nu\, \tilde{b}.$$

We need the following condition:

(B.2)* $\exists\, \kappa : [0, \infty) \to [0, 1]$ with $\int_0^\infty \kappa(x)\, dx = \infty$ so that

$$\kappa(\rho)\nabla_{\tilde{b}}\rho \leq 1.$$

Theorem 4. Under the assumptions (B.1), (B.2)*, the closure of $(\mathfrak{A}_\nu^*, C_0^\infty(M))$ generates a positivity preserving C_0-semigroup in $L^2(m)$. We denote the associated semigroup by $\{T_t^*\}$.
$\{e^{-\alpha t}T_t\}$ is Markovian if and only if $\mathrm{div}_\nu\, \tilde{b} + V \geq -\beta$. Further $\{e^{-\beta t}T_t\}$ has the L^1 contraction property if and only if $V + \alpha \geq 0$.

2. Criterion for normal operators

normal operator

An operator A in a Hilbert space H is called normal if $AA^* = A^*A$.

- A, B: dissipative operators on $\mathcal{D}$
- Assume that $\overline{A}$, $\overline{B}$ are m-dissipative

Theorem 5. Assume that $A\mathcal{D} \subseteq \mathcal{D}$, $B\mathcal{D} \subseteq \mathcal{D}$ and

$$AB = BA \quad \text{on } \mathcal{D},$$

$$(Au, v) = (u, Bv), \quad u, v \in \mathcal{D}.$$

Then $\overline{A}$ is normal and $\overline{A}^* = \overline{B}$.

Examples on a Riemannian manifold

- M: a complete Riemannian manifold
- $\mathbf{vol}$: the Riemannian volume
- $\nu = e^{-U} d\mathbf{vol}$.

Define an operator on $H = L^2(\nu)$ by

$$\mathfrak{A} = \frac{1}{2}\triangle_\nu + \nabla_b$$

where $\triangle_\nu = -\nabla_\nu^* \nabla$. Then

$$\mathfrak{A}_\nu^* = \frac{1}{2}\triangle_\nu - \nabla_b - \mathrm{div}_\nu\, b.$$

Here div_ν denotes the divergence with respect to ν.

Theorem 6. Let b a Killing vector field and assume that $\operatorname{div}_\nu b$ is bounded from below. Then the closures of $\mathfrak{A}$ and $\mathfrak{A}_\nu^*$ are m-dissipative.

We give a criterion for $\mathfrak{A} = \frac{1}{2}\triangle_\nu + \nabla_b$ being a normal operator.

Theorem 7. Assume that $\operatorname{div}_\nu b$ is bounded from below. Then $\mathfrak{A}$ is normal if and only if b is a Killing vector field and the following identies hold:

$$(\frac{1}{2}\triangle_\nu + \nabla_b)\operatorname{div}_\nu b = 0,$$
$$[(\nabla U)^\sharp, b] + (\nabla \operatorname{div}_\nu b)^\sharp = 0.$$

If M is a compact manifold, the above theorem simplified as follows:

Theorem 8. $\mathfrak{A}$ is normal if and only if b is a Killing vector field and the following identies hold:

$$\mathbf{div}_\nu \, b = 0,$$

$$[(\nabla U)^\sharp, b] = 0.$$

3. Examples of normal operators

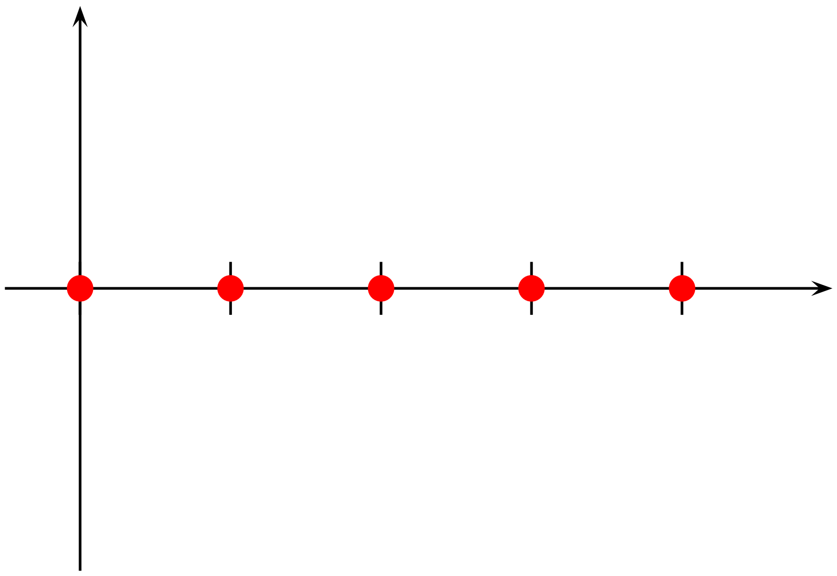
Ornstein-Uhlenbeck operator with rotation

- $M = \mathbb{R}^2$
- $\nu = \frac{1}{2\pi} e^{-(x^2+y^2)/2} dx dy$
- $b = c(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})$

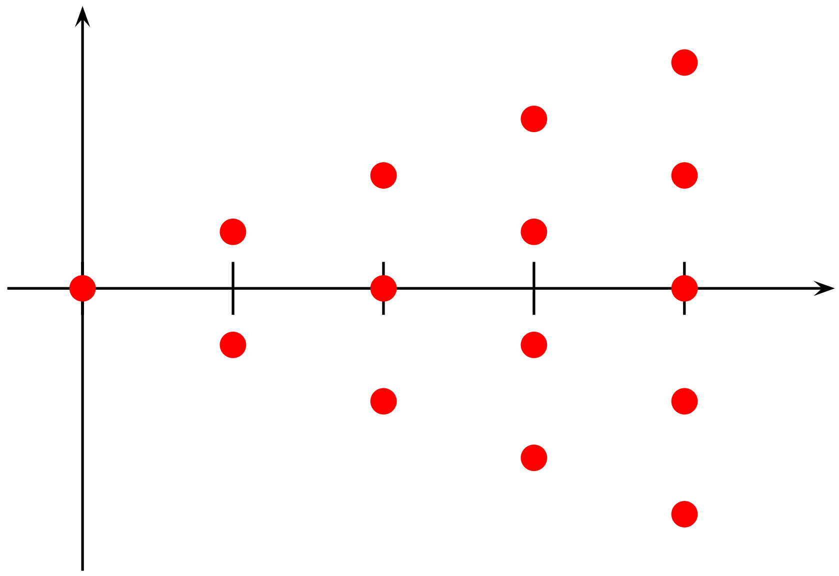
Then $\mathfrak{A} = -\nabla_\nu^* \nabla + \nabla_b$ is a normal operator in $L^2(\nu)$.

Theorem 9. The spectrum of $\mathfrak{A}$ is

$$\{(p+q) - (p-q)ci\}_{p,q=0}^{\infty} \tag{5}$$



the spectrum of $\nabla_\nu^* \nabla$



the spectrum of $-\mathfrak{A}$

One-dimensional Brownian motion with drift

We consider an operator $\mathfrak{A} = \frac{d^2}{dx^2} - c\frac{d}{dx}$ on $L^2(\mathbb{R}, \nu)$. Here ν is a measure defined by

$$\nu(dx) = e^{-cx}dx. \tag{6}$$

Then $\mathfrak{A}$ is a self-adjoint operator with

$$(\mathfrak{A}f, g) = -\int_{\mathbb{R}} f'(x)g'(x)\,\nu(dx).$$

To investigate the spectrum of $\mathfrak{A}$, we use the following isometric map $I\colon L^2(\nu) \longrightarrow L^2(dx)$:

$$If(x) = e^{-cx/2}f(x).$$

We have

$$I \circ \mathfrak{A} \circ I^{-1} = \frac{d^2}{dx^2} - \frac{c^2}{4},$$

i.e., the following diagram is commutative:

$$\begin{array}{ccc} L^2(\nu) & \xrightarrow{\mathfrak{A}} & L^2(\nu) \\ I \big\downarrow & & \big\downarrow I \\ L^2(dx) & \xrightarrow{\frac{d^2}{dx^2} - \frac{c^2}{4}} & L^2(dx) \end{array}$$

Hence the spectrum $-\mathfrak{A}$ is

$$\sigma(-\mathfrak{A}) = [\frac{c^2}{4}, \infty). \tag{7}$$

We now consider an perturbation of $\mathfrak{A}$. Let b be an vector field defined by

$$b = k\frac{d}{dx}.$$

We consider an operator of the form $\mathfrak{A} + b$. We are interested in how the spectrum changes. b is clearly an Killing vector field. The divergence of b with respect to ν

$$\operatorname{div}_\nu b = -ck$$

and so it satisfies

$$(\mathfrak{A} + b) \operatorname{div}_\nu b = 0,$$
$$[(\nabla U)^\sharp, b] + \nabla \operatorname{div}_\nu b = 0.$$

Here $U(x) = cx$. By Theorem 7, $\mathfrak{A} + b$ is a normal operator. Under the transformation of I, we have

$$I \circ (\mathfrak{A} + b) \circ I^{-1} = \frac{d^2}{dx^2} + k\frac{d}{dx} - \frac{c(c-2k)}{4}.$$

It is enough to get the spectrum of $\frac{d^2}{dx^2} + k\frac{d}{dx}$. Recall the Fourier transform as

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x}\, dx.$$

This gives an isometry from $L^2(dx)$ onto $L^2(d\xi)$. Note that

$$\int_{\mathbb{R}} (\frac{d^2}{dx^2} + k\frac{d}{dx}) f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}} (-\xi^2 + ik\xi)\hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi$$

which means that

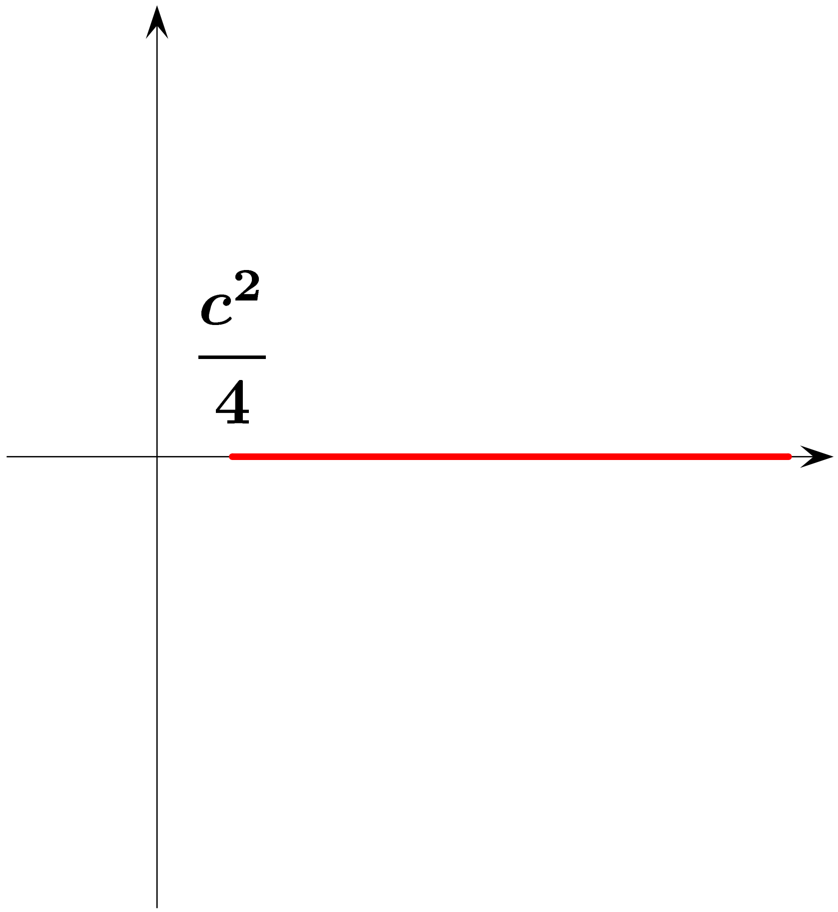
$$\sigma(\frac{d^2}{dx^2} + k\frac{d}{dx}) = \{-\xi^2 + ik\xi; \xi \in \mathbb{R}\}.$$

Theorem 10. We have

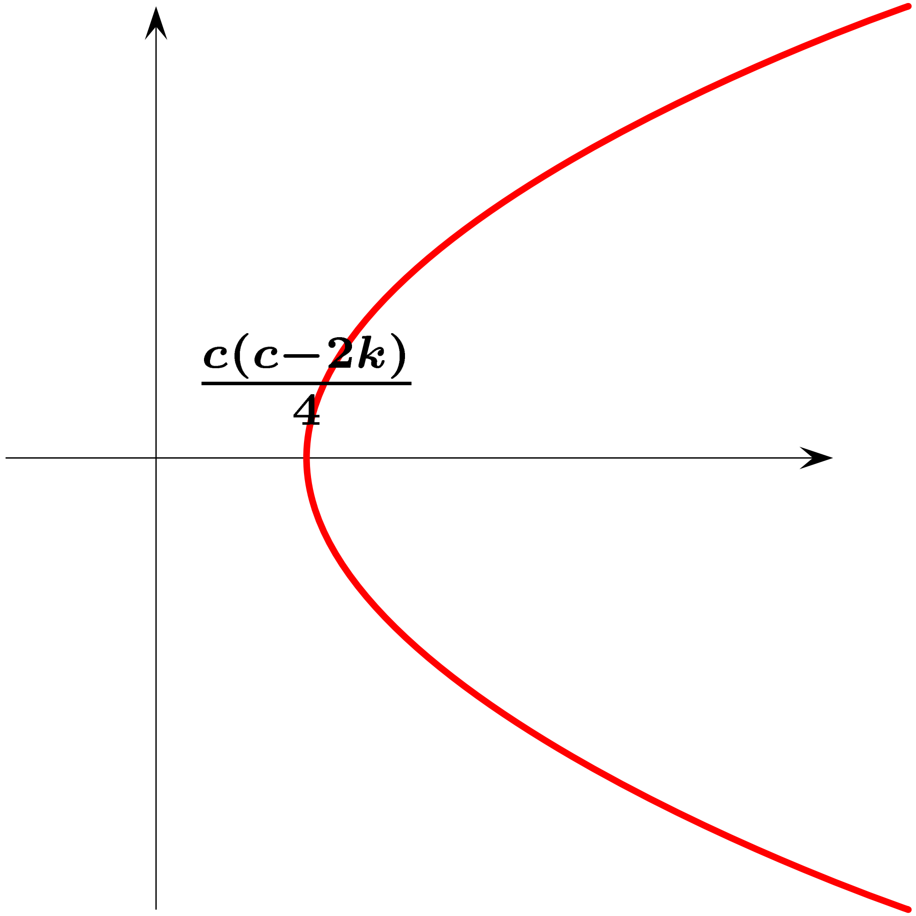
$$\sigma(-\mathfrak{A}) = [\frac{c^2}{4}, \infty)$$

and

$$\sigma(-\mathfrak{A} - b) = \{\frac{c(c-k)}{2} + \xi^2 + ik\xi;\ \xi \in \mathbb{R}\}.$$



$-\mathfrak{A}$



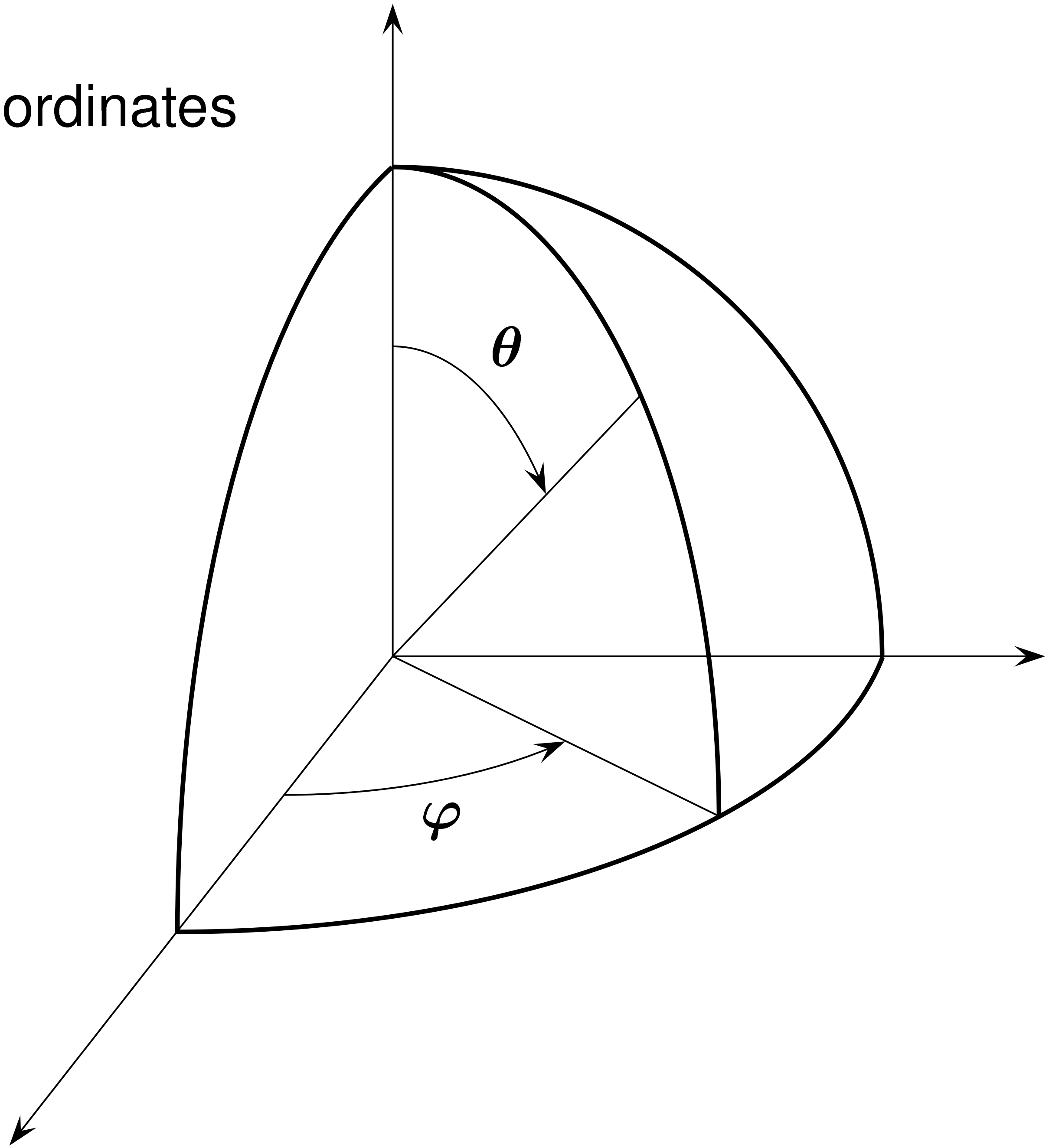
$-\mathfrak{A} - k\frac{d}{dx}$

Normal operator on S^2

The Laplace-Berltrami operaotr on S^2 is given as follows;

$$\triangle = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

Here, we take the polar coordinates as follows polar coordinates



Eigenvalues are $n(n+1)$, $n = 0, 1, 2, \ldots$.

Corresponding eigenfunctions are given as follows:

- Legendre polynomials

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} (1 - x^2)^n.$$

- ODE of Legendre polynomials

$$(1 - x^2) P_n'' - 2x P_n' = -n(n+1) P_n.$$

- Associated Legendre functions of the first kind

$$P_n^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x).$$

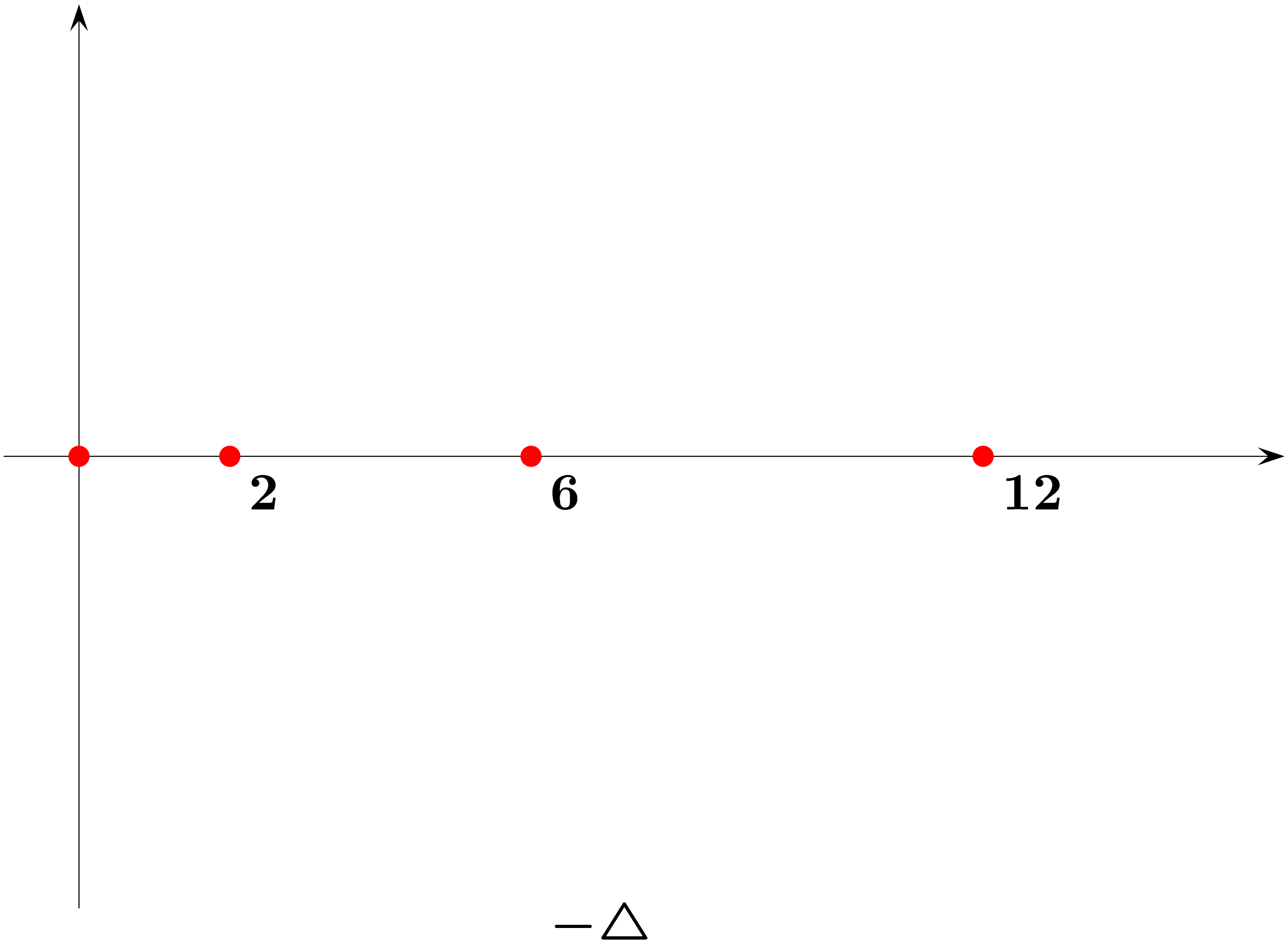
- ODE of associated Legendre functions

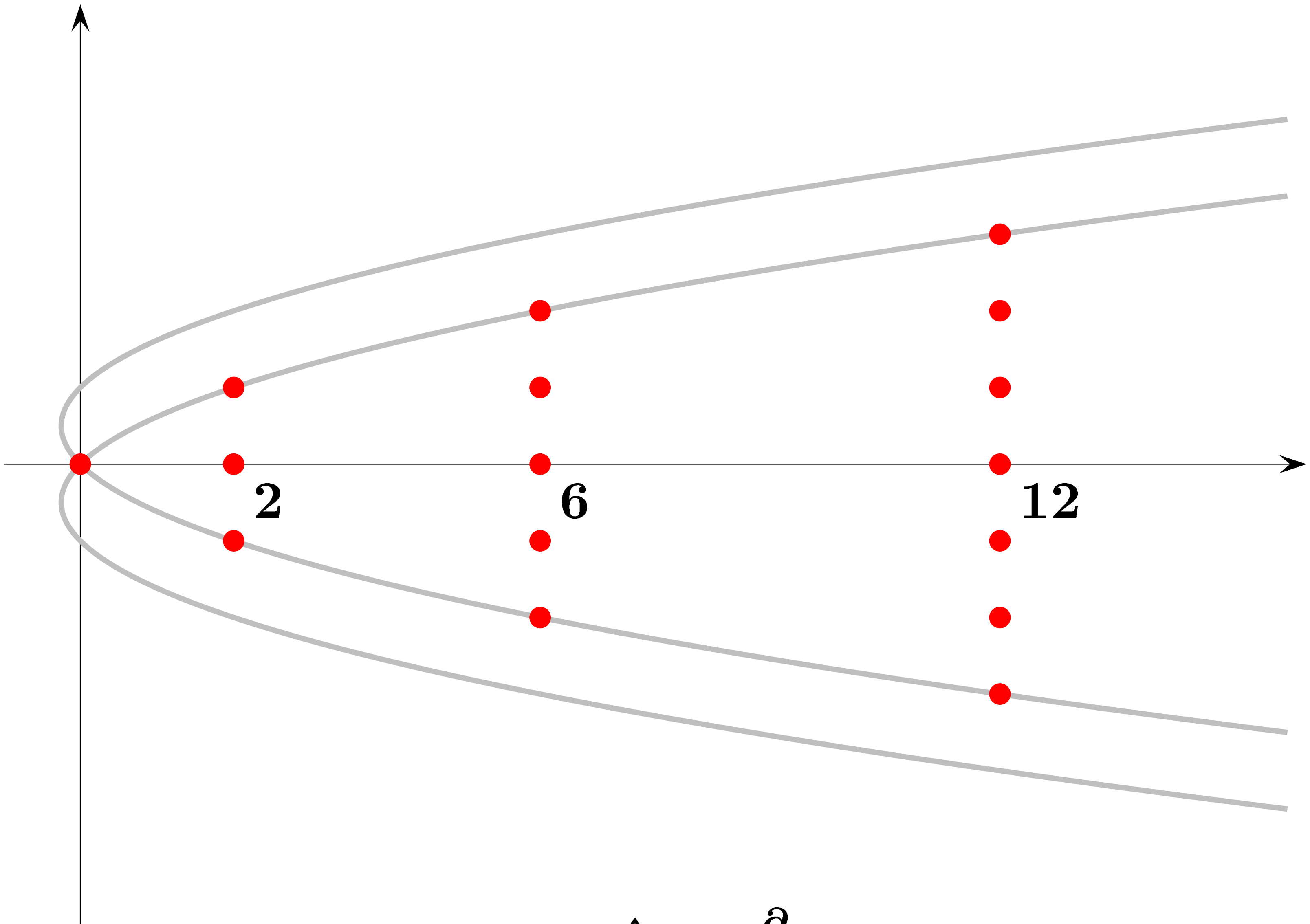
$$(1-x^2)\frac{d^2}{dx^2}P_n^m(x) - 2x\frac{d}{dx}P_n^m(x) + \left[n(n+1) - \frac{m^2}{1-x^2}\right]P_n^m(x) = 0.$$

Now eigenfunctions for the eigenvalue $-n(n+1)$ are

$$P_n^m(\cos\theta)e^{im\varphi}, \quad P_n^m(\cos\theta)e^{-im\varphi},$$

$$n = 0, 1, \ldots, \; m = 0, 1, \ldots, n.$$


2
6
12
$-\triangle$



$$-\triangle - \frac{\partial}{\partial \varphi}$$

Thanks !