Non-symmetric diffusions on Riemannian manifolds*

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1 Non-symmetric diffusions on Riemannian manifolds

Let (M, g) be a complete Riemannian manifold. We deonte the Riemannian volume by $m =$ vol. We take a reference measure $\nu = e^{-U}d\text{vol}$. Here we assume that U is C^∞. We consider the following operator:

$$\mathfrak{A} = \frac{1}{2}\triangle + b - V. \tag{1}$$

Here $\triangle$ is the Laplace-Beltrami operator and b is a vector field on M. We regard it as an operator in $L^2(\nu)$. Denoting the covariant differentiation by ∇, we have $\triangle = -\nabla^*\nabla$. Here ∇^* is the dual operator of ∇ with respect to the Riemannian volume $d$vol. Our reference measure is ν and so we need to introduce the dual operator with respect to ν as follows:

$$\nabla_\nu^* = e^U \nabla^* e^{-U}.$$

Setting $\triangle_\nu = -\nabla_\nu^*\nabla$, $\mathfrak{A}$ can be expressed as

$$\mathfrak{A} = \frac{1}{2}\triangle_\nu + \nabla_{\tilde{b}} - V \tag{2}$$

where

$$\tilde{b} = \frac{1}{2}\nabla U^\sharp + b \tag{3}$$

The dual operator of $\mathfrak{A}$ is

$$\mathfrak{A}_\nu^* = \frac{1}{2}\triangle_\mu - \nabla_{\tilde{b}} - \operatorname{div}_\nu \tilde{b} - V.$$

We are interested in the semigroups generated by $\mathfrak{A}$ or $\mathfrak{A}^*$

We need the following assumptions. We take a point $o \in M$ and define $\rho(x) = d(o, x)$ where d is the Riemannian distance. We introduce the following conditions:

(B.1) $\frac{1}{2}\operatorname{div}_\nu \tilde{b} + V \geq 0$.

(B.2) $\nabla_{\tilde{b}}\rho/\rho$ is bounded from below for large ρ.

Theorem 1. *Under the conditions (B.1), (B.2), the closure of $(\mathfrak{A}, C_0^\infty(M))$ generates a C_0 semigroup in $L^2(\nu)$ and the semigroup is positivity preserving.*

We denote the associated semigroups by $\{T_t\}$.

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Theorem 2. *Assume (B.1), (B.2). The semigroup* $e^{-\alpha t}\{T_t\}$ *is Markovian if and only if* $V \geq -\alpha$.

Theorem 3. *Assume (B.1), (B.2). The semigroup* $e^{-\beta t}\{T_t\}$ *is* L^1*-contractive if and only if* $\operatorname{div}_\nu \tilde{b} + V \geq -\beta$.

We have the similar result for $\mathfrak{A}_\nu^*$. We need the following condition in place of (B.2).

(B.3) $\nabla_{\tilde{b}}\rho/\rho$ is bounded from above for large ρ.

Then we have the following:

Theorem 4. *Under the conditions (B.1), (B.3), the closure of* $(\mathfrak{A}_\nu^*, C_0^\infty(M))$ *generates a* C_0 *semigroup in* $L^2(\nu)$ *and the semigroup is positivity preserving.*

We denote the associated semigroups by $\{T_t^*\}$.

Theorem 5. *Assume (B.1), (B.3). The semigroup* $e^{-\beta t}\{T_t^*\}$ *is Markovian if and only if* $\operatorname{div}_\nu \tilde{b} + V \geq -\beta$.

Theorem 6. *Assume (B.1), (B.3). The semigroup* $e^{-\alpha t}\{T_t^*\}$ *is* L^1*-contractive if and only if* $V \geq -\alpha$.

2 Normal operators on Riemannian manifolds

As an application, we give an characterization of normal operators on Riemannian manifold.

We prepare a general theorem. Let H be a Hilbert space. Suppose we are given accretive operators A, B defined on $\mathscr{D}$. We assume that $\overline{A}$, $\overline{B}$ are m-dissipative.

Theorem 7. *Assume that* $A\mathscr{D} \subseteq \mathscr{D}$, $B\mathscr{D} \subseteq \mathscr{D}$ *and*

$$\begin{aligned} AB &= BA \quad \text{on } \mathscr{D}, \\ (Au, v) &= (u, Bv), \quad u, v \in \mathscr{D}. \end{aligned}$$

Then $\overline{A}$ *is normal and* $\overline{A}^* = \overline{B}$.

Now we return to the Riemannian manifold case. We consider the following operaotr in $H = L^2(\nu)$.

$$\mathfrak{A} = \frac{1}{2}\triangle_\nu + \nabla_b.$$

The dual operator is given by

$$\mathfrak{A}_\nu^* = \frac{1}{2}\triangle_\nu - \nabla_b - \operatorname{div}_\nu b.$$

Then we give a criterion for $\mathfrak{A} = \triangle_\nu + b$ being a normal operator as follows.

Theorem 8. *Assume that* $\operatorname{div}_\nu b$ *is bounded from below. Then* $\mathfrak{A}$ *is normal if and only if* b *is a Killing vector field and the following identies hold:*

$$\begin{aligned} (\frac{1}{2}\triangle_\nu + b)\operatorname{div}_\nu b &= 0, \\ [\nabla U, b] + \nabla \operatorname{div}_\nu b &= 0. \end{aligned}$$