On spectra of 1-dimensional diffusion operators

Ichiro Shigekawa (Kyoto University)

November 15, 2012, RIMS

Mathematical Quantum Field Theory and Related Topics

URL: http://www.math.kyoto-u.ac.jp/~ichiro/

Contents

1. Introduction

Hermite polynomials

Hermite polynomials are defined by

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \quad n = 0, 1, \ldots$$

These are eigenfunctions of the Ornstein-Uhlenbeck operator

$$\frac{d^2}{dx^2} - x\frac{d}{dx}.$$

We have

$$\frac{d}{dx} H_n(x) = H_{n-1}(x).$$

$$
\begin{array}{ccccc}
\text{eigenvalue} & & \xrightarrow{\frac{d}{dx}} & \mathbf{0} \\
\mathbf{0} & H_0(x) & \nearrow & H_0(x) \\
-1 & H_1(x) & \nearrow & H_1(x) \\
-2 & H_2(x) & \nearrow & H_2(x) \\
-3 & H_3(x) & \nearrow & H_3(x) \\
\vdots & \vdots & & \vdots
\end{array}
$$

In this talk, we give a general framework of this fact.

2. One dimensional diffusion processes

- $M = [0, \infty)$
- a, p: positive continuous functions on $(0, \infty)$

We consider the diffusion process generated by

(1) $$\mathfrak{A}u = \frac{1}{p}(apu')'.$$

This operator is regarded as a self-adjoint operator in $L^2(p)$. Here p denotes a measure $p(x)dx$ on $[0, \infty)$.

By formal calculation, the associated Dirichlet form is

(2) $$\mathcal{E}(u, v) = \int_0^\infty u'v'ap\,dx.$$

The speed measure and the scale function

- $dm = p(x)dx$: the speed measure.
- $s(x)$: the scale function.

$$m(x) = \int_0^x p(y)\,dy \tag{3}$$

$$s(x) = \int_0^x \frac{1}{a(y)p(y)}\,dy. \tag{4}$$

We assume that 0 is a regular boundary point (Feller's classification). So we have $m(0) = 0$, $s(0) = 0$. We also assume that ∞ is not a regular boundary point, i.e.,

$$m(\infty) + s(\infty) = \infty.$$

The precise domain of $\mathcal{E}$ is given by

$$\mathbf{Dom}(\mathcal{E}) = \{u \in L^2(p);\ u \text{ is absolutely continuous on } (0,\infty) \text{ and } u' \in L^2(ap)\}.$$

Proposition 1. If $u \in \mathbf{Dom}(\mathcal{E})$, then u is a.c. (absolutely continuous) on $[0,\infty)$, i.e., $u(0+)$ exists and u is a.c. on $[0,\infty)$ by defining $u(0) = u(0+)$. In this case, we have

$$|u(0)| \le \frac{1}{m(x)^{1/2}} \|u\|_{L^2(p)} + \mathcal{E}(u,u)^{1/2} s(x)^{1/2}. \tag{5}$$

Moreover, if $s(\infty) < \infty$ then $u(\infty)$ exists and if $s(\infty) < \infty$, $m(\infty) = \infty$ then $u(\infty) = 0$.

To show this, we use

$$|u(y) - u(x)| \leq \mathcal{E}(u,u)^{1/2}(s(y) - s(x))^{1/2}.$$

By Proposition 1, $u \mapsto u(0)$ is a continuous linear functional from $\mathbf{Dom}(\mathcal{E})$ to $\mathbb{R}$.

Now we define an operator $V\colon L^2(p) \to L^2(ap)$ by

$$Vu = u' \tag{6}$$

Here $\mathbf{Dom}(V) = \mathbf{Dom}(\mathcal{E})$.

If we impose the Dirichlet boundary condition at 0, we set $\mathbf{Dom}(V) = \mathbf{Dom}(\mathcal{E}) \cap \{u : u(0) = 0\}$.

Proposition 2. $V\colon L^2(p) \to L^2(ap)$ is a closed operator.

The dual operaotr V^*

We give a characterization of V^*.

Proposition 3. Take any $\theta \in L^2(ap)$. If $ap\theta$ is a.c. on $(0,\infty)$ and $\frac{(ap\theta)'}{p} \in L^2(p)$, then $ap\theta$ is a.c. on $[0,\infty)$, i.e., $ap\theta(0+)$ exists and $ap\theta$ is a.c. on $[0,\infty)$ by defining $ap\theta(0) = ap\theta(0+)$. We also have

$$|ap\theta(0+)| \leq \frac{\|\theta\|_{L^2(ap)}}{s(x)^{1/2}} + \|\frac{(ap\theta)'}{p}\|_{L^2(p)} m(x)^{1/2} \tag{7}$$

Moreoover, if $m(\infty) < \infty$ then $ap\theta(\infty)$ exists and if $m(\infty) < \infty$, $s(\infty) = \infty$ then $ap\theta(\infty) = 0$.

To show this, we use

$$|ap\theta(y) - ap\theta(x)| \leq \sqrt{\int_x^y \frac{(ap\theta)'^2}{p^2} p\, dt}(m(y) - m(x))^{1/2}.$$

Dense domain

We denote the set of all continuous functions on $[0, \infty)$ with compact support by C_0.

> **Proposition 4.** $\mathbf{Dom}(\mathcal{E}) \cap C_0$ is dense in $\mathbf{Dom}(\mathcal{E})$ and $\mathbf{Dom}(\mathcal{E}) \cap C_0 \cap \{u : u(0) = 0\}$ is dense in $\mathbf{Dom}(\mathcal{E}) \cap \{u : u(0) = 0\}$.

By using this, we have the following duality formula (integration by parts):

Proposition 5. For any $u \in \mathrm{Dom}(\mathcal{E})$ and any $\theta \in L^2(ap)$ satisfying $\frac{(ap\theta)'}{p} \in L^2(p)$, we have

$$\int_0^\infty u'\theta ap\,dt = -u(0)ap\theta(0+) - \int_0^\infty u(ap\theta)'\,dt \tag{8}$$

Further we have $uap\theta(\infty) = 0$.

Proposition 6. The dual opetator $V^*: L^2(ap) \to L^2(p)$ of $V: L^2(p) \to L^2(ap)$ is given by

(9) $$V^*\theta = -\frac{(ap\theta)'}{p}.$$

Here

(10)

$$\mathrm{Dom}(V^*) = \{\theta \in L^2(ap);\ \frac{(ap\theta)'}{p} \in L^2(p), ap\theta(0+) = 0\}$$

for the Neumann boundary condition and

(11) $$\mathrm{Dom}(V^*) = \{\theta \in L^2(ap);\ \frac{(ap\theta)'}{p} \in L^2(p)\}$$

for the Dirichlet boundary condition.

We define $\mathfrak{A} = -V^*V$. We can give a characterization of $\mathbf{Dom}(\mathfrak{A})$ as follows:

Theorem 7. We have that $u \in \mathbf{Dom}(\mathfrak{A})$ if and only if

1. u is a.c. on $(0, \infty)$ and $u' \in L^2(ap)$,
2. apu' is a.c. on $(0, \infty)$ and $\frac{(apu')'}{p} \in L^2(p)$,
3. $apu'(0+) = 0$.

If u satisfies these conditions, we have

$$\mathfrak{A}u = -V^*Vu = \frac{(apu')'}{p}$$

If the boundary condition is Dirichlet, the third condition is replaced by $u(0) = 0$.

Theorem 8. We have that $\theta \in \mathbf{Dom}(VV^*)$ if and only if

1. $ap\theta$ is a.c. on $(0,\infty)$ and $\frac{(ap\theta)'}{p} \in L^2(p)$,
2. $\frac{(ap\theta)'}{p}$ is a.c. on $(0,\infty)$ and $\left(\frac{(ap\theta)'}{p}\right)' \in L^2(ap)$,
3. $ap\theta(0+) = 0$.

In this case we have

$$\hat{\mathfrak{A}}\theta = -VV^*\theta = \left(\frac{(ap\theta)'}{p}\right)' \tag{12}$$

If the boundary condition is Dirichlet, the third condition $ap\theta(0+) = 0$ should be omitted.

If we assume that a and p are C^2 functions, we have

Corollary 9. We have

$$\mathfrak{A}u = -V^*Vu = au'' + bu', \tag{13}$$

$$\hat{\mathfrak{A}}\theta = -VV^*\theta = a\theta'' + (b + a')\theta' + b'\theta. \tag{14}$$

Here $b = a' + a(\log p)'$ $\quad$ ($b + a' = a' + a(\log ap)'$).

3. Super symmetry and the spectrum

The super symmetry is an efficient machinery to investigate the spectrum, which depends on the following well-known fact:

Proposition 10. Let T be a closed operator in a Hilbert space H. Then T^*T and TT^* have the same spectrum except for 0.

Let x be an eigenvector for a point spectrum λ of T^*T:

$$T^*Tx = \lambda x. \tag{15}$$

Then

$$(TT^*)Tx = T(T^*T)x = T\lambda x = \lambda Tx$$

which shows that Tx is an eigenvector for an eigenvalue λ of TT^*.

In the previous section, we took $T = V = \frac{d}{dx}$. So $\frac{d}{dx}$ give rise to a correspondence between eigenfunctions of the following operators:

$$\mathfrak{A}u = -V^*Vu = au'' + bu', \tag{16}$$

$$\hat{\mathfrak{A}}\theta = -VV^*\theta = a\theta'' + (a' + b)\theta' + b'\theta. \tag{17}$$

Here $b = a' + a(\log p)'$.

This can be seen from the following computation. Assume $au'' + bu' = \lambda u$. Then

$$a'u'' + a'u''' + b'u' + bu'' = \lambda u'.$$

Hence

$$a(u')'' + (a' + b)(u') + b'u' = \lambda u'.$$

Corollary 11. Assume that $b(x) \leq -c < 0$, then $-\mathfrak{A}$ has a spectral gap $\geq c$.

Hermite polynomials

We take $a = 1$, $p = e^{-x^2/2}$, $M = \mathbb{R}$. Then

$$b = a' + a(\log p)' = (-x^2/2)' = -x.$$

Hence

$$\mathfrak{A}u = -V^*Vu = u'' - xu'$$

$$\hat{\mathfrak{A}}\theta = -VV^*\theta = \theta'' - x\theta' - \theta.$$

$\mathfrak{A}$ is the Ornstein-Uhlenbeck operator. $u'' - xu'$ and $\theta'' - x\theta' - \theta$ have the same spectrum except for 0. This shows that the Ornstein-Uhlenbeck operator has eigenvalues $0, -1, -2, \ldots$. Eigenfunctions are Hermite polynomials

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}. \tag{18}$$

eigenvalue	$u'' - xu'$		$\theta'' - x\theta' - \theta$

$$\begin{array}{c} -1 \\ -2 \\ \vdots \\ -n \\ \vdots \end{array} \qquad \begin{pmatrix} H_1(x) \\ H_2(x) \\ \vdots \\ H_n(x) \\ \vdots \end{pmatrix} \quad \xrightarrow{\frac{d}{dx}} \quad \begin{pmatrix} H_0(x) \\ H_1(x) \\ \vdots \\ H_{n-1}(x) \\ \vdots \end{pmatrix}$$

Laguerre polynomials

We take $a = x$, $p = x^{\alpha-1}e^{-x}$, $M = [0, \infty)$. Then

$$b = a' + a(\log p)' = 1 + x((\alpha - 1)\log x - x)'$$
$$= 1 + x\left(\frac{\alpha - 1}{x} - 1\right) = \alpha - x.$$

Hence

$$\mathfrak{A}u = -V^*Vu = xu'' + (\alpha - x)u'$$
$$\hat{\mathfrak{A}}\theta = -VV^*\theta = x\theta'' + (\alpha + 1 - x)\theta' - \theta.$$

We call the operator $xu'' + (\alpha - x)u'$ as the Kummer operator.

Eigenvalues of the Kummer operator is $0, -1, -2, \ldots$.

Eigenfunctions are Laguerre polynomials:

$$L_n^{\alpha}(x) = e^x \frac{x^{-\alpha}}{n!} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha}), \quad n = 0, 1, 2, \ldots \tag{19}$$

We have

$$\frac{d}{dx} L_n^{\alpha-1}(x) = -L_{n-1}^{\alpha}(x).$$

eigenvalue $\quad xu'' + (\alpha - x)u' \qquad x\theta'' + (\alpha + 1 - x)\theta' - \theta$

$$\begin{array}{ccccc} -1 & \quad & \begin{pmatrix} L_1^{\alpha-1}(x) \\ L_2^{\alpha-1}(x) \\ \vdots \\ L_n^{\alpha-1}(x) \\ \vdots \end{pmatrix} & \xrightarrow{\frac{d}{dx}} & \begin{pmatrix} -L_0^{\alpha}(x) \\ -L_1^{\alpha}(x) \\ \vdots \\ -L_1^{\alpha}(x) \\ \vdots \end{pmatrix} \\ -2 & & & & \\ \vdots & & & & \\ -n & & & & \\ \vdots & & & & \end{array}$$

Laplacian

We take $a = 1$, $p = 1$, $M = \mathbb{R}$. Then $\mathfrak{A} = \dfrac{d^2}{dx^2}$.

Eigenfunctions are $e^{i\xi x}$. We have

$$\frac{d}{dx} e^{i\xi x} = i\xi e^{i\xi x}.$$

4. Logarithmic Sobolev inequality

For the Kummer operator, we have the following logarithmic Sobolev inequality.

Theorem 12. For $\mathfrak{A}u = xu'' + (\alpha - x)u'$, we have

$$\int_0^\infty u^2 \log(u^2/\|u\|_2^2)p(x)\,dx \le 4\mathcal{E}(u,u). \tag{20}$$

To show this, we use Bakry-Emery's Γ_2 criterion.

$\mathfrak{A}u = xu'' + (\alpha - x)u'$.

$$\Gamma(u,u) = \frac{1}{2}\{\mathfrak{A}(u^2) - 2u\mathfrak{A}u\} = xu'^2$$

$$\Gamma_2(u,u) = \frac{1}{2}\{\mathfrak{A}\Gamma(u,u) - 2\Gamma(\mathfrak{A}u,u)\}$$

$$= x^2(u'' + \frac{1}{2x}u')^2 + \frac{1}{2}(1 + \frac{2\alpha - 1}{2x})\Gamma(u,u).$$

So we have to assume that $\alpha \geq \frac{1}{2}$.

Thanks !