The ultracontractivity of a non-symmetric Markovian semigroup and its applications

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1. Convergence of the transition probability

Killing at the boundary

- M: a compact connected Riemannia maniflod with a boundary ∂M.
- m: normalized Riemannian volume
- $\triangle$: the Laplace-Beltrami operator
- b: a vector field
- X_t, Y_t: diffusion processes generated by $\triangle$ and $\triangle + b$ respectively:

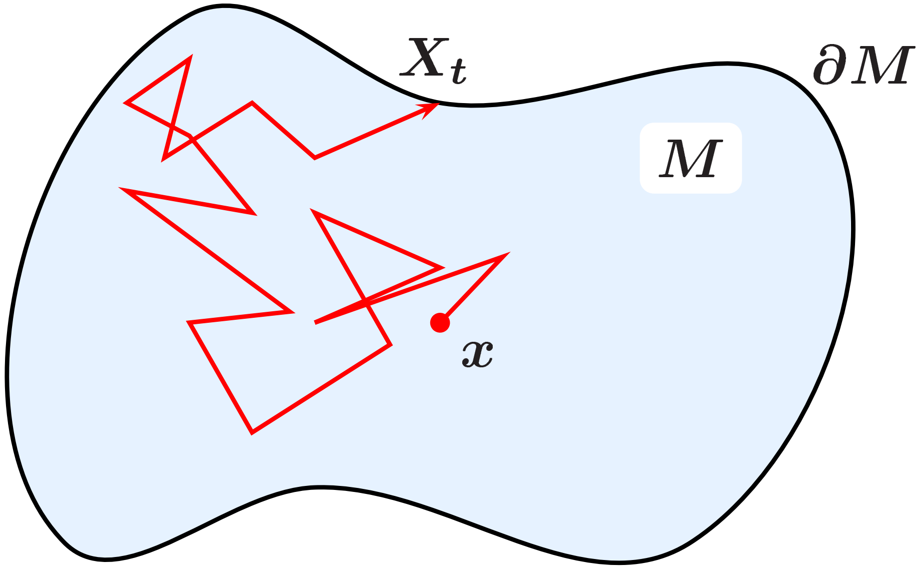
generator	fundamental solution
$\triangle$	$p(t,x,y)$
$\triangle + b$	$q(t,x,y)$

We assume $\mathrm{div}\, b = 0$ and we impose the Dirichlet boundary condition.

- Probablistic point of view

$$P_x(X_t \in dy) = p(t, x, y)m(dy)$$

(X_t) dies when it reaches the boundary.



- Differential equation point of view

  $u(t,x) = \int_M p(t,x,y) f(y) m(dy)$ satisfies the following differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} &= \triangle u \\ u(0,x) &= f(x) \\ u(t,x) &= 0, \quad x \in \partial M. \end{cases}$$

$$p(t,x,y) \to 0,$$

$$q(t,x,y) \to 0.$$

How fast?

$$\tilde{\lambda}_{1\to\infty} = -\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y\in M} p(t,x,y),$$

$$\lambda_{1\to\infty} = -\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y\in M} q(t,x,y).$$

Our aim is to show that

$$\tilde{\lambda}_{1\to\infty} \le \lambda_{1\to\infty}.$$

A Non-symmetric diffusion dies quicker than the symmetric diffusion.

Convergence to an invariant measure

- M: a compact connected Riemannia maniflod without boundary.
- X_t, Y_t: diffusion processes generated by $\triangle$ and $\triangle + b$ respectively:

generator	fundamental solution
$\triangle$	$p(t,x,y)$
$\triangle + b$	$q(t,x,y)$

We assume $\mathrm{div}\, b = 0$.

$$p(t,x,y) \to 1,$$
$$q(t,x,y) \to 1.$$

How fast?

$$\tilde{\gamma}_{1\to\infty} = -\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y\in M} |p(t,x,y) - 1|,$$
$$\gamma_{1\to\infty} = -\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y\in M} |q(t,x,y) - 1|.$$

Our aim is to show that

$$\tilde{\gamma}_{1\to\infty} \leq \gamma_{1\to\infty}.$$

A Non-symmetric diffusion converges to the invariant measure quicker than the symmetric diffusion.

2. Ultracontractivity

A semigroup $\{T_t\}$ is called ultracontractive if $T_t\colon L^1 \to L^\infty$ is bounded for all $t > 0$.

It is well-known that the following three conditions are equivalent for a symmetric Markovian semigroup. Let $\mu > 0$ be given.

(i) $\exists c_1 > 0, \forall f \in L^1$:

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0.$$

(ii) $\exists c_2 > 0, \forall f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$:

$$\|f\|_2^{2+4/\mu} \le c_2\, \mathcal{E}(f, f)\, \|f\|_1^{4/\mu}.$$

(iii) $\mu > 2, \exists c_3 > 0, \forall f \in \mathrm{Dom}(\mathcal{E})$:

$$\|f\|_{2\mu/(\mu-2)}^2 \le c_3\, \mathcal{E}(f, f).$$

We extend this result for non-symmetric Markovian semigroups.

Non-symmetric Markovian semigroups

We give a framework in generall Hilbert space scheme.

- H: a Hilbert space
- $\{T_t\}$: a contraction C_0 semigroup
- $\{T_t^*\}$: the dual semigroup
- $\mathfrak{A}$, $\mathfrak{A}^*$: the generators of $\{T_t\}$ and $\{T_t^*\}$

A natural bilinear form $\mathcal{E}$ is defined by

$$\mathcal{E}(u, v) = -(\mathfrak{A}u, v).$$

We do not assume the sector condition and so we can not use this bilinear form.

We introduce a symmetric bilinear form. For this, we assume the following condition:

$(\mathbf{A.1})$ $\mathbf{Dom}(\mathfrak{A}) \cap \mathbf{Dom}(\mathfrak{A}^*)$ is dense in $\mathbf{Dom}(\mathfrak{A})$ and $\mathbf{Dom}(\mathfrak{A}^*)$.

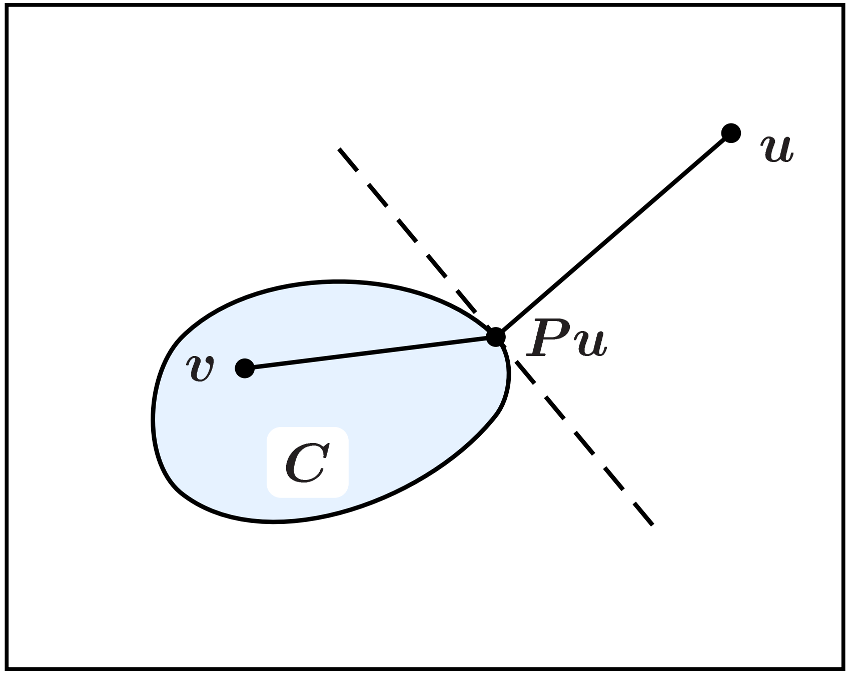
Under this condition, we define a symmetric bilinear form $\tilde{\mathcal{E}}$ by

$$\tilde{\mathcal{E}}(u,v) = -\frac{1}{2}\{(\mathfrak{A}u, v) + (u, \mathfrak{A}v)\}, \quad u, v \in \mathbf{Dom}(\mathfrak{A}) \cap \mathbf{Dom}(\mathfrak{A}^*).$$

Proposition 1. Under the condition (A.1), $\tilde{\mathcal{E}}$ is closable and its closure contains $\mathbf{Dom}(\mathfrak{A})$ and $\mathbf{Dom}(\mathfrak{A}^*)$.

Covex set preserving property

- C: a convex set of H.
- Pu: the shortest point from u to C



$$(u - Pu, v - Pu) \leq 0, \quad \forall v \in C.$$

Theorem 2. If $\{T_t\}$ and $\{T_t^*\}$ preserve a convex set C, then $Pu \in \mathrm{Dom}(\tilde{\mathcal{E}})$ for any $u \in \mathrm{Dom}(\tilde{\mathcal{E}})$ and we have

$$\tilde{\mathcal{E}}(Pu, u - Pu) \geq 0.$$

Markovian semigroup

- (M, m): a measure space
- $H = L^2(m)$: a Hilbert space
- $\{T_t\}$: a Markovian semigroup

We assume that $\{T_t^*\}$ is also a Markovian semigroup.

Under the assumption (A.1), we can define a symmetric bilinear form $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}$ is a Dirichlet form.

We have the following implications. For $\mu > 0$,

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0$$

$\Uparrow \quad \Downarrow$ under (1)

$$\|f\|_2^{2+4/\mu} \le c_2\, \tilde{\mathcal{E}}(f,f)\, \|f\|_1^{4/\mu}$$

$\Updownarrow$

$$\|f\|_{2\mu/(\mu-2)}^2 \le c_3\, \tilde{\mathcal{E}}(f,f) \quad (\mu > 2)$$

(1) $$(\mathfrak{A}^2 f, f)_2 + (\mathfrak{A} f, \mathfrak{A} f)_2 \ge 0.$$

(1) holds if $\mathfrak{A}$ is normal, i.e. $\mathfrak{A}\mathfrak{A}^* = \mathfrak{A}^*\mathfrak{A}$.

Moreover

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0,1]$$

$$\Uparrow \quad \Downarrow \text{ under } (2)$$

$$\|f\|_2^{2+4/\mu} \le c_2(\tilde{\mathcal{E}}(f,f) + \|f\|_2^2)\, \|f\|_1^{4/\mu}$$

$$\Updownarrow$$

$$\|f\|_{2\mu/(\mu-2)}^2 \le c_3(\tilde{\mathcal{E}}(f,f) + \|f\|_2^2) \quad (\mu > 2)$$

There there exists a constant $M > 0$ so that for all $f \in \mathbf{Dom}(\mathfrak{A}^2)$

$$((\mathfrak{A} - M)^2 f, f)_2 + ((\mathfrak{A} - M)f, (\mathfrak{A} - M)f)_2 \ge 0. \tag{2}$$

L^2 theory

We introduce three indices.

(3) $$\lambda_{\mathrm{P}} = \inf\left\{\frac{\tilde{\mathcal{E}}(f,f)}{\|f\|_2^2};\ f \neq 0\right\} \quad \text{i.e.,} \quad \lambda_{\mathrm{P}}\|f\|_2^2 \le \tilde{\mathcal{E}}(f,f).$$

(4) $$\lambda_{2\to 2} = -\lim_{t\to\infty}\frac{1}{t}\log\|T_t\|_{2\to 2}.$$

(5) $$\lambda_{\mathrm{B}} = \inf \Re(\sigma(-\mathfrak{A})).$$

(3) is equivalent to

(6) $$\|T_t f\|_2^2 \le e^{-2\lambda t}\|f\|_2^2, \quad \forall t > 0.$$

Theorem 3. We have the following inequalities:

$$\boldsymbol{\lambda}_{\mathbf{P}} \leq \boldsymbol{\lambda}_{\mathbf{2}\to\mathbf{2}} \leq \boldsymbol{\lambda}_{\mathbf{B}} \tag{7}$$

Theorem 4. If $\mathfrak{A}$ is norml, then we have

$$\boldsymbol{\lambda}_{\mathbf{P}} = \boldsymbol{\lambda}_{\mathbf{2}\to\mathbf{2}} = \boldsymbol{\lambda}_{\mathbf{B}}. \tag{8}$$

From these theorems, we have $\tilde{\boldsymbol{\lambda}}_{\mathbf{2}\to\mathbf{2}} \leq \boldsymbol{\lambda}_{\mathbf{2}\to\mathbf{2}}$.

Ultracontractivity

We introduce the following index:

$$\lambda_{1\to\infty} = -\varlimsup_{t\to\infty} \frac{1}{t} \log \|T_t\|_{1\to\infty}. \tag{9}$$

Theorem 5. Let $\mu > 0$ be given. Assume that there exists a constant $c_2 > 0$ such that

$$\|f\|_2^{2+(4/\mu)} \le c_2 \tilde{\mathcal{E}}(f,f)\|f\|_1^{4/\mu}, \quad \forall f \in \mathrm{Dom}(\tilde{\mathcal{E}}) \cap L^1. \tag{10}$$

Then we have $\lambda_{1\to\infty} = \lambda_{2\to 2}$. Therefore

$$\lambda_{1\to\infty} \ge \tilde{\lambda}_{1\to\infty}. \tag{11}$$

3. Dirichlet forms having invariant measure

We continue to assume the sector condition. In addition, we assume

- m is an invariant probability measure.

$$\int_M T_t f \, dm = \int_M f \, dm$$

- $T_t 1 = 1$ and $\mathfrak{A} 1 = 0$ and 1 is the unique eigenvalue.
- $m(f) = \int_M f(x) \, m(dx)$.

We have the following implications. For $\mu > 0$,

$$\|T_t f - m(f)\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0$$

$$\Uparrow \quad \Downarrow \text{ under } (12)$$

$$\|f - m(f)\|_2^{2+4/\mu} \le c_2\, \tilde{\mathcal{E}}(f, f)\, \|f - m(f)\|_1^{4/\mu}$$

$$\Updownarrow$$

$$\|f - m(f)\|_{2\mu/(\mu-2)}^2 \le c_3\, \tilde{\mathcal{E}}(f, f) \quad (\mu > 2)$$

$$(\mathfrak{A}^2 f, f)_2 + (\mathfrak{A} f, \mathfrak{A} f)_2 \ge 0. \tag{12}$$

Moreover

$$\|T_t f - m(f)\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0,1]$$

$$\Uparrow \quad \Downarrow \text{ under } (13)$$

$$\|f - m(f)\|_2^{2+4/\mu} \le c_2(\tilde{\mathcal{E}}(f,f) + \|f\|_2^2)\, \|f - m(f)\|_1^{4/\mu}$$

$$\Updownarrow$$

$$\|f - m(f)\|_{2\mu/(\mu-2)}^2 \le c_3(\tilde{\mathcal{E}}(f,f) + \|f\|_2^2) \quad (\mu > 2)$$

There there exists a constant $M > 0$ so that for all $f \in \mathbf{Dom}(\mathfrak{A}^2)$

$$((\mathfrak{A} - M)^2 f, f)_2 + ((\mathfrak{A} - M)f, (\mathfrak{A} - M)f)_2 \ge 0. \tag{13}$$

L^2 theory

We introduce the following three indices:

$$\gamma_{\mathrm{P}} = \inf\left\{ \frac{\tilde{\mathcal{E}}(f,f)}{\|f - m(f)\|_2^2};\ f \neq m(f) \right\} \text{ i.e., } \gamma_{\mathrm{P}}\|f - m(f)\|_2^2 \leq \tilde{\mathcal{E}}(f,f). \tag{14}$$

$$\gamma_{2\to 2} = -\lim \frac{1}{t} \log \|T_t - m\|_{2\to 2} \tag{15}$$

$$-\gamma_{\mathrm{SG}} = \sup \Re(\sigma(\mathfrak{A}) \setminus \{0\}). \tag{16}$$

γ_{P} is called a Poincaré constant. (14) is equivalent to

$$\|T_t f - m(f)\|_2^2 \leq e^{-2\lambda t}\|f - m(f)\|_2^2, \quad \forall t > 0. \tag{17}$$

We have the following theorems.

Theorem 6. We have the following inequalities:

$$\gamma_{\mathbf{P}} \leq \gamma_{\mathbf{2 \to 2}} \leq \gamma_{\mathbf{SG}}. \tag{18}$$

Theorem 7. If $\mathfrak{A}$ is normal, then we have

$$\gamma_{\mathbf{P}} = \gamma_{\mathbf{2 \to 2}} = \gamma_{\mathbf{SG}}. \tag{19}$$

From these theorem, we have

$$\tilde{\gamma}_{\mathbf{2 \to 2}} \leq \gamma_{\mathbf{2 \to 2}}.$$

Ultracontractivity

We introduce another indes $\gamma_{1\to\infty}$ as follows:

$$\gamma_{1\to\infty} = -\lim_{t\to\infty} \frac{1}{t} \log \|T_t - m\|_{1\to\infty} \tag{20}$$

Proposition 8. We have

$$\gamma_{1\to\infty} \leq \gamma_{2\to2}. \tag{21}$$

Moreover, if $\gamma_{1\to\infty} > -\infty$, then the identity holds.

Theorem 9. Let $\mu > 0$. Assume the following Nash inequality: there esists a constant $c_2 > 0$ such that

$$\|f - m(f)\|_2^{2+(4/\mu)} \leq c_2 \tilde{\mathcal{E}}(f,f) \|f - m(f)\|_1^{4/\mu}, \quad \forall f \in \mathbf{Dom}(\tilde{\mathcal{E}}) \cap L^1. \tag{22}$$

Then $\gamma_{1\to\infty} > 0$ and so $\gamma_{2\to 2} = \gamma_{1\to\infty}$. Therefore we have

$$\tilde{\gamma}_{1\to\infty} \leq \gamma_{1\to\infty}. \tag{23}$$

4. Compact Riemannian manifold with a boundary

- M: d-dimensional compact Riemannian manifold with a boundary ∂M.
- m: normalized Riemannian volume.
- The generator is given by

$$\mathfrak{A} = \triangle + b. \tag{24}$$

We assumet that $\mathbf{div}\, b \geq 0$ nad we impose the Dirichlet boundary condition:

$$f = 0 \quad \text{on } \partial M. \tag{25}$$

- The dual operator is

$$\mathfrak{A}^* = \triangle - \nabla_b - \operatorname{div} b. \tag{26}$$

- Associated symmetric form is

$$\tilde{\mathcal{E}}(u, v) = \int_M (\nabla u, \nabla v)\, dm + \frac{1}{2}\int_M uv \operatorname{div} b\, dm. \tag{27}$$

Theorem 10. We have

$$\tilde{\lambda}_{2\to 2} \le \lambda_{2\to 2}.$$

If $\mathfrak{A}$ is normal, then $\tilde{\lambda}_{2\to 2} = \lambda_{2\to 2}$.

Since M is compact, the following Nash inequality holds:

$$\|f\|_2^{2+(4/d)} \le c_2 \tilde{\mathcal{E}}(f, f) \|f\|_1^{4/d}.$$

Theorem 11. We have

$$\tilde{\lambda}_{1\to \infty} \le \lambda_{1\to \infty}.$$

If $\mathfrak{A}$ is normal, then $\tilde{\lambda}_{1\to \infty} = \lambda_{1\to \infty}$.

The semigroup T_t has a transition density $q(t, x, y)$ w.r.t. ν. $q(t, x, y)$ is C^∞ from the hypoellipticity. From the definition,

$$\lambda_{1\to\infty} = -\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y\in M} q(t, x, y).$$

Similary for $\tilde{\mathcal{E}}$, there exists a transition density $p(t, x, y)$ w.r.t. ν and

$$\tilde{\lambda}_{1\to\infty} = -\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y\in M} p(t, x, y).$$

We have

$$\tilde{\lambda}_{1\to\infty} \leq \lambda_{1\to\infty}.$$

Theorem 12. We have

$$\tilde{\boldsymbol{\lambda}}_{\mathbf{2\to 2}} \leq \boldsymbol{\lambda}_{\mathbf{2\to 2}} = \boldsymbol{\lambda}_{\mathbf{B}}. \tag{28}$$

If $\tilde{\boldsymbol{\lambda}}_{\mathbf{2\to 2}} = \boldsymbol{\lambda}_{\mathbf{2\to 2}}$, then $\mathfrak{A}$ has an eigenvalue $-\tilde{\boldsymbol{\lambda}}_{\mathbf{2\to 2}}$ and its eigenfunction coincides with the eigenfunction φ of $\frac{1}{2}(\mathfrak{A} + \mathfrak{A}^*)$ for the eigenvalue $-\tilde{\boldsymbol{\lambda}}_{\mathbf{2\to 2}}$. The vector fields b satisfies

$$\boldsymbol{b\varphi = -\frac{1}{2}(\mathrm{div}\, b)\varphi}. \tag{29}$$

Example: the unit disc

- $M = \{x \in \mathbb{R}^2; |x| \leq 1\}$
- $\operatorname{div} b = 0$.
- $r = (x_1^2 + x_2^2)^{1/2}$.

$$br \neq 0 \quad \Rightarrow \quad \tilde{\lambda}_{2\to 2} < \lambda_{2\to 2}$$

5. Compact Riemannian manifold without boundary

Let us return to the diffusion on a Riemannian manifold M generated by

$$\mathfrak{A}f = \triangle f + bf = \triangle f + (\nabla f, \omega_b).$$

If M is compact, then there exists an invariant probability measure.

- ν: an invariant probability measure: $\quad \nu = e^{-U}m$

We use the following notations

- ∇: the Levi-Civita covariant derivative
- ∇^*: the dual operator of ∇ w.r.t. m
- ∇^*_ν: the dual operator of ∇ w.r.t. ν
- ω_b: 1-form corresponding to b

We now change the reference measure to ν. So our Hilbert space changes to $L^2(\nu)$.

Set

$$\mathcal{G}_\nu = \{\mathfrak{A}\,;\ \mathfrak{A} \text{ has an invariant measure } \nu.\}$$

We set

$$\tilde{b} = \frac{1}{2}(\nabla U)^\sharp + b,$$
$$\omega_{\tilde{b}} = \frac{1}{2}\nabla U + \omega_b.$$

Theorem 13. $\mathfrak{A} \in \mathcal{G}_\nu$ if and ony if $\nabla_\nu^* \omega_{\tilde{b}} = 0$. In this case,

$$\mathfrak{A} f = -\nabla_\nu^* \nabla f + (\omega_{\tilde{b}}, \nabla f)$$

and

$$\mathfrak{A}_\nu^* f = -\nabla_\nu^* \nabla f - (\omega_{\tilde{b}}, \nabla f).$$

Further the associated symmetric Dirichlet form is given by

$$\tilde{\mathcal{E}}(f, h) = \int_M (\nabla f, \nabla h) d\nu.$$

T_t has a density $q(t,x,y)$ with respect to ν. Define

$$\gamma_{1\to\infty} = -\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y\in M} |q(t,x,y) - 1|.$$

Let $p(t,x,y)$ be a transition density for $\tilde{\mathcal{E}}$. Define

$$\tilde{\gamma}_{1\to\infty} = -\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y\in M} |p(t,x,y) - 1|.$$

Theorem 14. We have

$$\tilde{\gamma}_{1\to\infty} \le \gamma_{1\to\infty}.$$

The equality holds if $\mathfrak{A}$ is normal.

Recall that

$$\gamma_{\mathbf{SG}} = \inf\{\Re\eta;\ \eta \in \sigma(-\mathfrak{A}) \setminus \{0\}\}.$$

We have $\gamma_{1\to\infty} = \gamma_{\mathbf{SG}}$.

Theorem 15. If $\tilde{\gamma}_{1\to\infty} = \gamma_{1\to\infty}$, then $-\mathfrak{A}$ has an eigenvalue ξ so that $\Re\xi = \tilde{\gamma}_{1\to\infty}$ and its eigenfunctions is also an eigenfunction of $\nabla_\nu^*\nabla$ for an eigenvalue $\tilde{\gamma}_{1\to\infty}$.

Example: 2-dimensional torus

- $M = T^2$
- (x, y): the standard local coordinate
- $b = f(x)\dfrac{\partial}{\partial y} + g(y)\dfrac{\partial}{\partial x}$

Then

$$\begin{aligned} f = \text{constant},\ g = \text{constant} \quad &\Rightarrow \quad \tilde{\gamma}_{1\to\infty} = \gamma_{1\to\infty} \\ f = 0 \quad &\Rightarrow \quad \tilde{\gamma}_{1\to\infty} = \gamma_{1\to\infty} \\ f \neq \text{constant},\ g \neq \text{constant} \quad &\Rightarrow \quad \tilde{\gamma}_{1\to\infty} < \gamma_{1\to\infty}. \end{aligned}$$

Thanks a lot!