Semigoups that preserve a convex set in a Banach space

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August 5, 2011

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Contents

1. Semigroups

- B: a Banach space
- semigroup $\{T_t\}_{t\geq 0}$:
  (1) $T_0 = I$
  (2) $T_s T_t = T_{t+s}$
  (3) $t \mapsto T_t$ is strongly continuous.
- The generator $\mathfrak{A}$ is defined by

$$\mathfrak{A} f = \lim_{t \to 0} \frac{T_t f - f}{t}, \quad \text{if it exists.}$$

- G_α: the resolvent is is defined by

$$G_\alpha f = \int_0^\infty e^{-\alpha t} T_t f \, dt$$

- Markov process $((X_t), P_x)$ defines a semigroup

$$T_t f(x) = E_x[f(X_t)].$$

$\{T_t\}$ satisfies

$$0 \leq f \leq 1 \quad \Rightarrow \quad 0 \leq T_t f \leq 1.$$

This property is called a Markovian property.
$C = \{f;\, 0 \leq f \leq 1\}$. C is invariant under T_t.

- For any convex set C,

$$T_t f \in C,\ \forall f \in C \quad \Leftrightarrow \quad \alpha G_\alpha f \in C,\ \forall f \in C.$$

Many properties are characterized in this manner by changing a convex set C.

The following properties are well-discussed:

(1) Positivity preserving property

(2) Markovian property

(3) L^1 contraction

(4) Semigroup domination

(5) Excessive function

(6) Invariant set

Aim: We give a unified method to prove them.

Positivity preserving property

L^1
$$\int_{\{f<0\}} \mathfrak{A}f(x)\,d\mu(x) \geq 0$$

$$\Big\uparrow \quad p \to 1$$

L^p
$$\int \mathfrak{A}f(x)\,f_-^{p-1}(x)\,d\mu(x) \geq 0$$

$$\Big\downarrow \quad p \to \infty$$

C_∞ $\quad \mathfrak{A}f(x_0) \geq 0, \quad x_0:$ maximum point of f_-

2. Semigroups that preserve a convex set in a Banach space

- B : Banach pace with a norm $\|\ \|$
- B^* : the dual pace of B
- $F(x) =: \{\varphi \in B^*; \langle x, \varphi \rangle = \|x\|^2 = \|\varphi\|^2\}$ (conjugate mapping)
- $\{T_t\}$: a (C_0)-semigroup
- $\mathfrak{A}$: the generator
- $\{G_\alpha\}$: the resolvent
- C : a convex set in B

We are interested in the following property:

$$T_tC \subseteq C, \quad \forall t \geq 0,$$

i.e., T_t preserves the convex set C.

- $d(x, C) = \inf\{\|x - y\|; y \in C\}$
- $P(x) = \{y \in C; d(x, y) = d(x, C)\}$

We always assume that $P(x) \neq \emptyset$.

Theorem 1. $\gamma \in \mathbb{R}$ is fixed.

Suppose that $\forall \boldsymbol{x} \in \mathbf{Dom}(\mathfrak{A})$, $\exists \boldsymbol{y} \in \boldsymbol{P}(\boldsymbol{x})$, $\forall \varphi \in \boldsymbol{F}(\boldsymbol{x}-\boldsymbol{y})$:

$$\Re\langle \mathfrak{A}\boldsymbol{x}, \varphi\rangle \leq \gamma\|\boldsymbol{x}-\boldsymbol{y}\|^{2}, \tag{1}$$

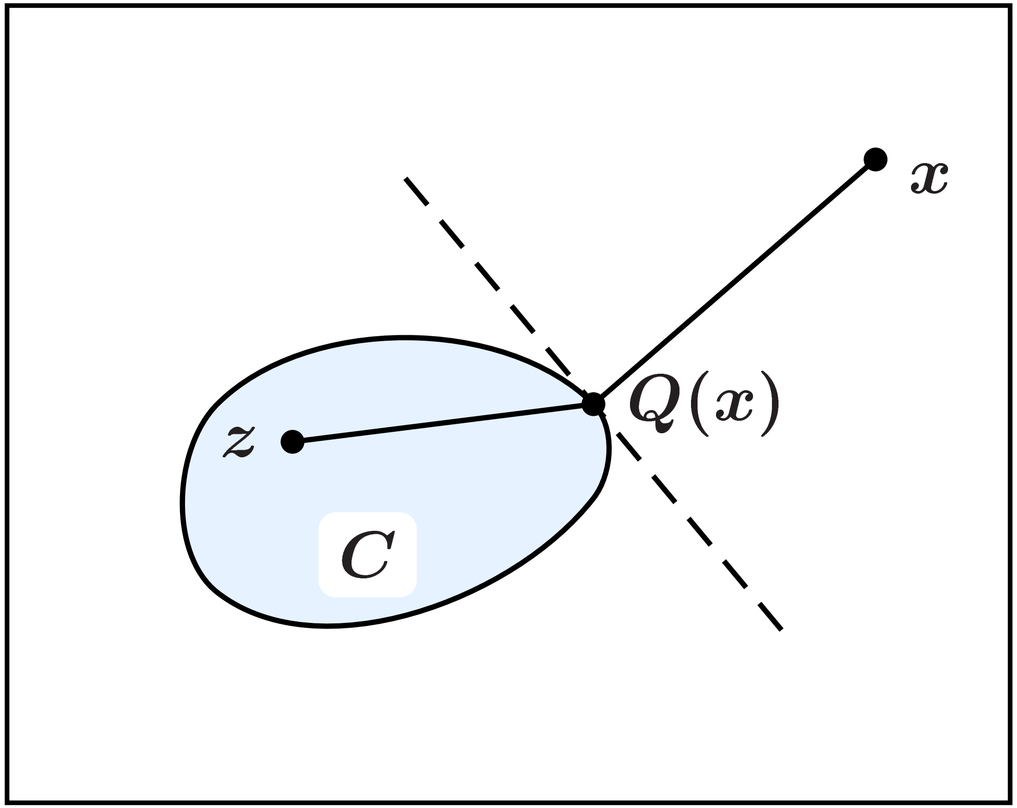
then the semigroup $\{\boldsymbol{T_t}\}$ preserves $\boldsymbol{C}$.

Conversely, if $\{\boldsymbol{T_t}\}$ preserves $\boldsymbol{C}$ and $\{\boldsymbol{e^{-\gamma t}T_t}\}$ is a contraction semigroup, then $\forall \boldsymbol{x} \in \mathbf{Dom}(\mathfrak{A})$, $\forall \boldsymbol{y} \in \boldsymbol{P}(\boldsymbol{x})$, $\exists \varphi \in \boldsymbol{F}(\boldsymbol{x}-\boldsymbol{y})$, so that (1) holds.

Good selection

$(Q(x), G(x))$: good selection

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \text{(i)} \quad Q(x) \in P(x), \quad G(x) \in F(x - Q(x)) \\ \text{(ii)} \quad \forall z \in C : \ \Re\langle z - Q(x), G(x)\rangle \leq 0 \end{cases}$$



Theorem 2. $\gamma \in \mathbb{R}$ is fixed.

Suppose that $\exists(\boldsymbol{Q}(\boldsymbol{x}), \boldsymbol{G}(\boldsymbol{x}))$: good selection so that $\forall \boldsymbol{x} \in \mathbf{Dom}(\boldsymbol{\mathfrak{A}})$:

$$\Re\langle \boldsymbol{\mathfrak{A}} \boldsymbol{x}, \boldsymbol{G}(\boldsymbol{x})\rangle \leq \gamma \|\boldsymbol{x} - \boldsymbol{Q}(\boldsymbol{x})\|^{\mathbf{2}}, \tag{2}$$

then the semigroup $\{\boldsymbol{T_t}\}$ preserves $\boldsymbol{C}$.

Conversely, if $\{\boldsymbol{T_t}\}$ preserves $\boldsymbol{C}$ and $\{\boldsymbol{e^{-\gamma t} T_t}\}$ is a contraction semigroup, then for any good secection $(\boldsymbol{Q}(\boldsymbol{x}), \boldsymbol{G}(\boldsymbol{x}))$ (if it exists) (2) holds.

Remark 1. In Hilbert space case, $\boldsymbol{P}(\boldsymbol{x})$ consists of one point and $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{x}$. In this case, the above theorem for $\gamma = \mathbf{0}$ is proved by Brezis-Pazy (1970).

Proof. Assuming (2), we will show that the resolvent preserves $\boldsymbol{C}$.

Take any $z \in C$ and set $x = \alpha G_\alpha z$. Then $\mathfrak{A}x = \alpha(x - z)$. Since $(Q(x), G(x))$ is a good selection, we have

$$\Re\langle z - Q(x), G(x)\rangle \leq 0. \tag{3}$$

Hence

$$\begin{aligned}
0 &\geq \Re\langle \mathfrak{A}x, G(x)\rangle - \gamma\|x - Q(x)\|^2 \\
&= \alpha\Re\langle x - z, G(x))\rangle - \gamma\|x - Q(x)\|^2 \\
&= \alpha\Re\langle x - Q(x) + Q(x) - z, G(x)\rangle - \gamma\|x - Q(x)\|^2 \\
&= (\alpha - \gamma)\|x - Q(x)\|^2 + \alpha\Re\langle Q(x) - z, G(x)\rangle \\
&\geq (\alpha - \gamma)\|x - Q(x)\|^2. \quad (\because (3))
\end{aligned}$$

We can take $\alpha > \gamma$ and so $x = Q(x) \in C$ follows. $\square$

3. Examples

Positivity preserving property

$C = \{f;\ f \geq 0\}$

$Q(f) = f_+$

1. $C_\infty(E)$
   $G(f) = \|f_-\|_\infty\, \delta_{x_0}$, x_0: maximaum point of f_-

$$\mathfrak{A} f(x_0) \geq \gamma f(x_0)$$

2. $L^p(d\mu) \quad (1 < p < \infty)$
   $G(f) = \|f_-\|_p^{2-p}\, f_-^{p-1}$

$$\int \mathfrak{A} f(x)\, f_-^{p-1}\, d\mu(x) \geq -\gamma \|f_-\|_p^p$$

3. $L^1(\mu)$

   $G(f) = -\|f_-\|_1 \, 1_{\{f<0\}}$

$$\int_{\{f<0\}} \mathfrak{A} f(x)\, d\mu(x) \geq -\gamma \|f_-\|_1$$

Markovian peoperty

$C = \{f;\ f \leq 1\}$ (or $\{f;\ 0 \leq f \leq 1\}$), $Q(f) = f \wedge 1 = \min\{f, 1\}$

1. $C_\infty(E)$

   $G(f) = \|(f-1)_+\|_\infty \, \delta_{x_0}$, $\quad x_0$: positive maximum point of f

$$\mathfrak{A} f(x_0) \leq 0$$

2. $L^p(d\mu) \quad (1 < p < \infty)$

$$G(f) = \|(f-1)_+\|_p^{2-p}\,(f-1)_+^{p-1}$$

$$\int \mathfrak{A}f(x)\,(f-1)_+^{p-1}\,d\mu(x) \le \gamma\|(f-1)_+\|_p^p$$

3. $L^1(\mu)$

$$G(f) = -\|(f-1)_+\|_1\,\mathbf{1}_{\{f>1\}}$$

$$\int_{\{f>1\}} \mathfrak{A}f(x)\,d\mu(x) \le \gamma\|(f-1)_+\|_1$$

L^1 contraction property

$C = \{f;\ f \geq 0,\ \int_M f\,d\mu \leq 1\}.$

$B = L^p(\mu)$

$Q(f) = f \wedge c$ where

$$\int_M (f-c)_+\,d\mu = 1.$$

$G(f) = \mathrm{sgn}(f)|f \wedge 1|^{p-1}.$

$$\int_M \mathfrak{A}f\,\mathrm{sgn}(f)|f \wedge 1|^{p-1}\,d\mu \leq \gamma\|f \wedge 1\|_p^p.$$

Excessive function

A non-negative function u is called excessive if

$$e^{-\alpha t} T_t u \leq u, \quad \forall t \geq 0.$$

We do not need to assume that $\{T_t\}$ is Markovian. If we assume that $\{T_t\}$ is positivity preserving, then the above condition is equivalent to the invariance of the convex set $C = \{f;\ f \leq u\}$ under $\{e^{-\alpha t} T_t\}$.

So now

$C = \{f;\ f \leq u\}, \quad Q(f) = f \wedge u = \min\{f, u\}$

1. $C_\infty(E)$, $\quad G(f) = \|(f-u)_+\|_\infty\, \delta_{x_0}$, $\quad x_0$: positive maximum point of $f-u$

$$(\mathfrak{A}-\alpha)f(x_0) \le \gamma(f(x_0)-u(x_0))$$

2. $L^p(d\mu)$ $(1<p<\infty)$, $\quad G(f) = \|(f-u)_+\|_p^{2-p}\,(f-u)_+^{p-1}$

$$\int (\mathfrak{A}-\alpha)f(x)\,(f(x)-u(x))_+^{p-1}\,d\mu(x) \le \gamma\|(f-u)_+\|_p^p$$

3. $L^1(\mu)$, $\quad G(f) = -\|(f-u)_+\|_1\, 1_{\{f>u\}}$

$$\int_{\{f>u\}} (\mathfrak{A}-\alpha)f(x)\,d\mu(x) \le \gamma\|(f-u)_+\|_1$$

Invariant set

A set $\boldsymbol{K}$ is called invariant if

$$\mathbf{1}_{K^c} T_t \mathbf{1}_K = 0, \quad \forall t \geq 0.$$

So now

$$C = \{f;\, \mathbf{1}_{K^c} f = 0\}, \quad Q(f) = \mathbf{1}_K f$$

1. $C_\infty(E)$, $\quad G(f) = \|1_{K^c} f\|_\infty \operatorname{sgn}(f(x_0))\delta_{x_0}$, $\quad x_0$: positive maximum $|f|$ in K^c.

$$\mathfrak{A} f(x_0) \operatorname{sgn}(f(x_0)) \leq \gamma |f(x_0)|$$

2. $L^p(d\mu)$ $(1 < p < \infty)$, $\quad G(f) = \|1_{K^c} f\|_p^{2-p} \, 1_{K^c} |f|^{p-1} \operatorname{sgn} f$

$$\int_{K^c} \mathfrak{A} f(x) \, |f(x)|^{p-1} \operatorname{sgn} f(x) \, d\mu(x) \leq \gamma \|1_{K^c} f\|_p^p$$

3. $L^1(\mu)$, $\quad G(f) = \|1_{K^c} f\|_1 \, 1_{K^c} \operatorname{sgn} f$

$$\int_{K^c} \mathfrak{A} f(x) \operatorname{sgn} f(x) \, d\mu(x) \leq \gamma \|1_{K^c} f\|_1$$

4. Hilber space case

We can give an conditions for preserving a convex set in terms of bilinear form. This was done by **Ouhabaz [1996]** for contraction semigroups. Our aim is to clarify when we need the contraction property or not.

- $\mathcal{E}(f, g) = -(\mathfrak{A} f, g)$.

Kown results

$$
\begin{tikzcd}
\mathcal{E}(f, f - f \wedge 1) \geq 0 \arrow[dr] & \\
 & \{T_t\} \text{ is Markovian} \arrow[d] \\
 & \mathcal{E}(f \wedge 1, f - f \wedge 1) \geq 0 \arrow[uul, bend left, "\text{require contraction}"]
\end{tikzcd}
$$

Ma-Röckner: Dirchelet forms

semi-Dirichlet form $\overset{\text{def}}{\Longleftrightarrow} \mathcal{E}(f + f \wedge 1, f - f \wedge 1) \geq 0$

Main results

$$\begin{tikzcd}
\mathcal{E}(f, f - f \wedge 1) \geq 0 \arrow[d, Leftrightarrow] & \\
\{T_t\} \text{ is Markovian contraction} \arrow[r, Rightarrow] & \{T_t\} \text{ is Markovian} \arrow[d, Leftrightarrow] \\
& \mathcal{E}(f \wedge 1, f - f \wedge 1) \geq 0
\end{tikzcd}$$

Main theorem

Theorem 3. We fix $\gamma \in \mathbb{R}$ and $\theta \in [0, 1]$. Let us consider the following condtitions:

(i) For any $x \in \mathbf{Dom}(\mathcal{E})$, $Px \in \mathbf{Dom}(\mathcal{E})$ and

$$\Re\mathcal{E}((1-\theta)x + \theta Px, x - Px) \geq -(1-\theta)\gamma|x - Px|^2.$$

(ii) $\{T_t\}$ preserves C.

(iii) $\mathcal{E}(P(x), x - P(x)) \geq 0, \quad \forall x \in \mathbf{Dom}(\mathcal{E}).$

Then, the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) holds.

If $\{e^{-\gamma t}T_t\}$ is contractive, then the above three conditions are equivalent to each other.

If $\mathcal{E}$ is Hermitian, then the following condition (without the contraction property of $\{e^{-\gamma t}T_t\}$)

(iv) for any $x \in \mathbf{Dom}(\mathcal{E})$, $P(x) \in \mathbf{Dom}(\mathcal{E})$ and

$$\mathcal{E}(Px, Px) \leq \mathcal{E}(x, x) + \gamma|x - Px|^2, \quad \forall x \in \mathbf{Dom}(\mathcal{E})$$

deduces (ii). In addition, if we assume that $\{e^{-\gamma t}T_t\}$ is contractive, then all conditions (i) – (iv) are equiavlent to each other.

Positivity preserving property

Theorem 4. The following condtiotns are equiavlent to each other:

(i) $\{T_t\}$ preserves the positivity.

(ii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $|f| \in \mathbf{Dom}(\mathcal{E})$ and $\mathcal{E}(f_+, f_-) \leq 0$.

Further (i) or (ii) implies the following (iii):

(iii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $|f| \in \mathbf{Dom}(\mathcal{E})$ and $\mathcal{E}(|f|, |f|) \leq \mathcal{E}(f, f)$.

If, in addition, $\mathcal{E}$ is symmetric, then all conditions are equivalent to each other.

Theorem 5. We fix $\gamma \in \mathbb{R}$ and $\boldsymbol{\theta} \in [\mathbf{0}, \mathbf{1})$. The following tow conditions are equivalent to each other:

(i) $\{e^{-\gamma t}T_t\}$ is a positivity preserving contraction semigroup.

(ii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $|f| \in \mathbf{Dom}(\mathcal{E})$ and
$\boldsymbol{\mathcal{E}((1-\theta)f+\theta f_+, f-f_+) \geq -\gamma(1-\theta)\|f_-\|_2^2}$.

If, in addition, $\mathcal{E}$ is symmetric, then the above condtions are equivalent to the following:

(iii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $|f| \in \mathbf{Dom}(\mathcal{E})$ and
$\boldsymbol{\mathcal{E}(f_+, f_+) \leq \mathcal{E}(f, f) + \gamma|f_-|^2}$.

(iv) For any $f \in \mathbf{Dom}(\mathcal{E})$, $|f| \in \mathbf{Dom}(\mathcal{E})$ and
$\boldsymbol{0 \leq \mathcal{E}_\gamma(|f|, |f|) \leq \mathcal{E}_\gamma(f, f)}$.

Markovian property

Theorem 6. The following condtiotns are equiavlent to each other:

(i) $\{T_t\}$ is a Marvovian semigroup.

(ii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $f \wedge 1 \in \mathbf{Dom}(\mathcal{E})$ and
$\mathcal{E}(f \wedge 1, f - f \wedge 1) \geq 0.$

Replacing $f \wedge 1$ with $f_+ \wedge 1$, we have the same result.

We may define that a bilinear form $\mathcal{E}$ is called semi-Dirichlet form if it satisfies the condition of (ii).

Theorem 7. We fix $\gamma \in \mathbb{R}$ and $\theta \in [0, 1)$. The following tow conditions are equivalent to each other:

(i) $\{T_t\}$ is a Markovian semigroup and $\{e^{-\gamma t} T_t\}$ is contractive.

(ii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $f \wedge 1 \in \mathbf{Dom}(\mathcal{E})$ and
$$\mathcal{E}((1-\theta)f + \theta(f \wedge 1), f - f \wedge 1) \geq -\gamma(1-\theta)\|f - f \wedge 1\|_2^2.$$

If, in addition, $\mathcal{E}$ is symmetric, (i) or (ii) is equivalent to the following:

(iv) For any $f \in \mathbf{Dom}(\mathcal{E})$, $f \wedge 1 \in \mathbf{Dom}(\mathcal{E})$ and
$$\mathcal{E}(f \wedge 1, f \wedge 1) \leq \mathcal{E}(f, f) + \gamma\|f - f \wedge 1\|_2^2.$$

Replacing $f \wedge 1$ with $f_+ \wedge 1$, we have the same result.

Excessive function

Theorem 8. We fix $\gamma \in \mathbb{R}$ and $\alpha \geq 0$. The following conditions are equivalent to each other:

(i) u is α-excessive and $\{T_t\}$ preserves the positivity.

(ii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $f \wedge u \in \mathbf{Dom}(\mathcal{E})$ and $\mathcal{E}_\alpha(f \wedge u, f - f \wedge u) \geq 0$.

Theorem 9. We fix $\gamma \in \mathbb{R}$, $\alpha \geq 0$ and $\theta \in [0,1)$. The following conditions are equivalent to each other:

(i) u is α-excessive and $\{e^{-(\alpha+\gamma)t}T_t\}$ is a positivity preserving contraction semigroup.

(ii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $f \wedge u \in \mathbf{Dom}(\mathcal{E})$ and $\mathcal{E}_\alpha((1-\theta)f+\theta(f\wedge u), f-f\wedge u) \geq -\gamma(1-\theta)\|f-f\wedge u\|^2.$

Invariant set

Theorem 10. The following conditions are equivalent to each other:

(i) B is invariant.

(ii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $1_B f \in \mathbf{Dom}(\mathcal{E})$ and $\boldsymbol{\mathcal{E}(1_B f, 1_{B^c} f) \geq 0}$.

(iii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $1_B f \in \mathbf{Dom}(\mathcal{E})$ and $\boldsymbol{\mathcal{E}(1_B f, 1_{B^c} f) = 0}$.

Thanks !