Non symmetric diffusions on a Riemannian manifold

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Contents

1. Non-symmetric Diffusion on a Riemannian manifold

- (M, g): d-dimensional connected complete Riemannian manifold.
- $m = \mathrm{vol}$: the Riemannian volume. b : a vector field on M.

We consider the following opetaror in $L^2(m)$:

(1)
$$\mathfrak{A} = \frac{1}{2}\triangle + \nabla_b.$$

The dual operator is

$$\mathfrak{A}^* = \frac{1}{2}\triangle - \nabla_b - \operatorname{div} b$$

and the symmetrization is

(2)
$$\frac{1}{2}(\mathfrak{A} + \mathfrak{A}^*) = \frac{1}{2}\triangle - \frac{1}{2}\operatorname{div} b$$

They are well-defined in $C_0^\infty(M)$.

The bilinear form $\mathcal{E}$ associated with $\mathfrak{A}$ is

$$\mathcal{E}(u,v) = -(\mathfrak{A}u, v) = \frac{1}{2}\int_M (\nabla u, \nabla v)\, dm - \int_M (\nabla_b u) v\, dm. \tag{3}$$

The symmetrization of this is

$$\tilde{\mathcal{E}}(u,v) = \frac{1}{2}\int_M (\nabla u, \nabla v)\, dm + \frac{1}{2}\int_M uv \operatorname{div} b\, dm. \tag{4}$$

This corresponds to the operator $\frac{1}{2}(\mathfrak{A} + \mathfrak{A}^*)$ in (2).

We are interested in when the semigroup associated to $\mathfrak{A}$ exists in L^2.

We impose the following condition to ensure that $-\mathfrak{A}$ is bounded from below.

(A.1) $\exists\, \gamma \in \mathbb{R} : \frac{1}{2} \operatorname{div} b \geq -\gamma.$

Under this condition, $\tilde{\mathcal{E}}$ is bounded from below and closable.

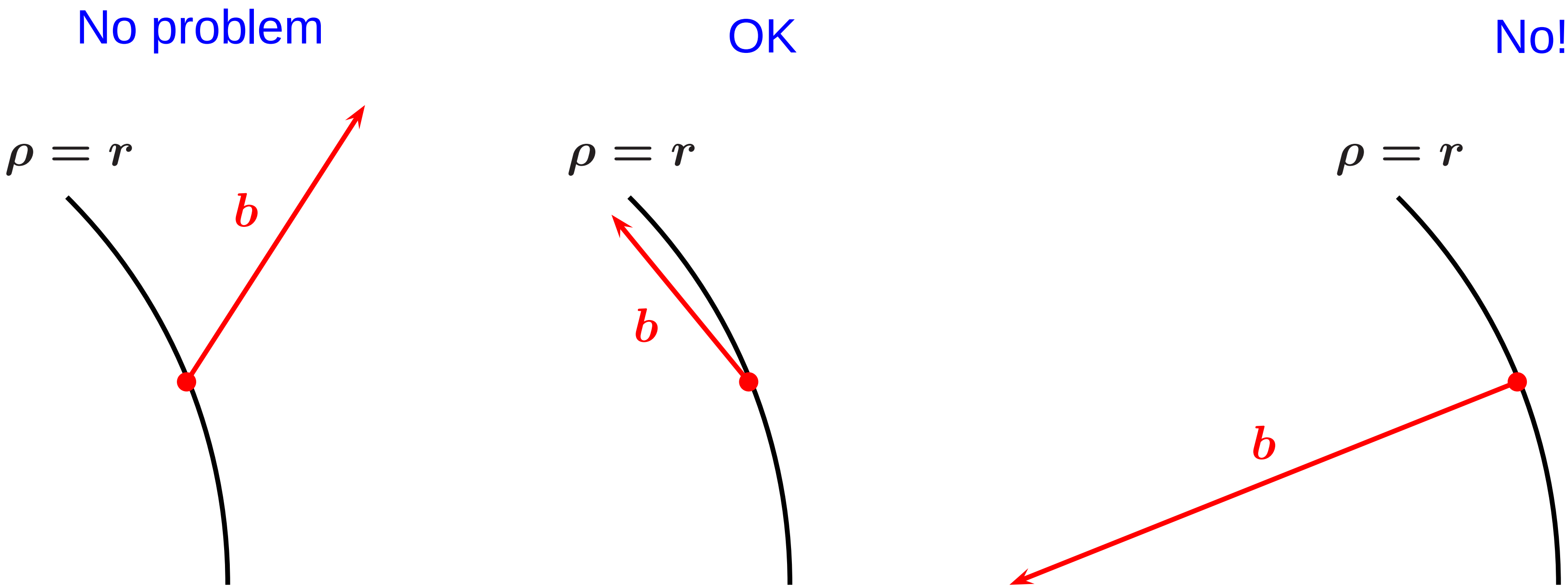
- d: the distance function
- $o \in M$: a fixed reference point
- $\rho(x) = d(o, x)$

We add the following condition for b :

(A.2) $\exists\, \kappa : [0, \infty) \to [0, 1]$ with $\int_0^\infty \kappa(x)\, dx = \infty$ so that

$$\kappa(\rho)\nabla_b \rho \geq -1.$$

- A typical example is $\kappa(x) = \frac{1}{x}$. $\quad \nabla_b \rho(x) \geq -\rho(x)$.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Theorem 1. Under the assumptions (A.1) and (A.2), the closure of $(\mathfrak{A}, C_0^\infty(M))$ generates a Markovian C_0-semigroup in $L^2(m)$.

We claim the following:

- the dissipativity: $((\mathfrak{A} - \gamma)u, u)_2 \le 0$.
- the maximality: $(\mathfrak{A} - \gamma - 1)(C_0^\infty(M))$ is dense in L^2.

In fact,

$$((\mathfrak{A} - \gamma)u, u)_2 = -\frac{1}{2}\int_M (|\nabla u|^2 + u^2 \operatorname{div} b)\, dm - \int_M \gamma u^2\, dm \le 0.$$

$$\begin{aligned} (\mathfrak{A} - \gamma - 1)^* u = 0 \quad &\Rightarrow \quad u \in C^\infty(M) \\ &\Rightarrow \quad (u, (\mathfrak{A} - \gamma - 1)(\chi_n u))_2 = 0 \\ &\Rightarrow \quad u = 0 \end{aligned}$$

The Markovian property is checked by the following criterion:

$$(\mathfrak{A}u, u - u \wedge 1)_2 \leq \gamma \|u - u \wedge 1\|_2^2. \tag{5}$$

Here $a \wedge b = \min\{a, b\}$.

We can also show the L^1-contraction property.

Proposition 2. Under the assumptions (A.1) and (A.2), $\{e^{-2t\gamma} T_t\}$ satisfies the L^1-contraction property.

We check the following criterion:

$$((\mathfrak{A} - 2\gamma)u, u_+ \wedge 1)_2 \leq -\gamma \|u_+ \wedge 1\|_2^2.$$

As for $\mathfrak{A}^*$

$$\mathfrak{A}^* = \frac{1}{2}\triangle - \nabla_b - \operatorname{div} b.$$

We need the following condition:

(A.2)* $\exists\, \kappa \colon [0,\infty) \to [0,1]$ with $\int_0^\infty \kappa(x)\,dx = \infty$ so that

$$\kappa(\rho)\nabla_b\rho \leq 1.$$

Theorem 3. Under the assumptions (A.1), (A.2)*, the closure of $(\mathfrak{A}^*, C_0^\infty(M))$ generates a C_0-semigroup in $L^2(m)$. It satisfies L^1-contraction property. If, in addition, $\operatorname{div} b \geq 0$, then the semigroup is Markovian.

2. Domain of the generator

If the Ricci curvature is bounded from below, then $\mathbf{Dom}(\triangle) = \mathbf{Dom}(\nabla^2)$. We can get similar result for $\mathfrak{A}$. To do so, we need the intertwining property. The following intertwining property is well known:

$$\nabla\triangle = \square_1\nabla.$$

Here $\square_1 = -(dd^* + d^*d)$ is the Hodge-Kodaira operator.

Now we define an operator $\vec{\mathfrak{A}}$ acting on 1-forms by

$$\vec{\mathfrak{A}}\theta = \frac{1}{2}\square_1\theta + \nabla_b\theta + \langle\nabla.b, \theta\rangle.$$

Then we have

$$\nabla\mathfrak{A} = \vec{\mathfrak{A}}\nabla.$$

As before, the bilinear form associated with the symmetrization of $\vec{\mathfrak{A}}$ is given by

$$\vec{\mathcal{E}}(\theta, \eta) = \frac{1}{2}(\nabla\theta, \nabla\eta)_2 + \int_M \{\frac{1}{2}\,\mathrm{Ric}(\theta, \eta) + \frac{1}{2}\,\mathrm{div}\, b(\theta, \eta) - (B\theta, \eta)\}\, dm.$$

where B is the symmetrization of ∇b : $B = \frac{1}{2}(\nabla b + (\nabla b)^*)$.

We have

$$(-\vec{\mathfrak{A}}\theta, \theta)_2 = \vec{\mathcal{E}}(\theta, \theta).$$

We impose the following condition so that $\vec{\mathcal{E}}$ is bounded from below.

(A.3) $\quad \mathrm{Ric}$ is bounded from below and $\exists\delta : \frac{1}{2}\,\mathrm{Ric} + \frac{1}{2}\,\mathrm{div}\, b - B \geq -\delta.$

Note that

$$\frac{1}{2}\|\nabla\theta\|_2^2 \leq \vec{\mathcal{E}}_\delta(\theta, \theta).$$

Theorem 4. Assume (A.1), (A.2), (A.2)* and (A.3). Then $u \in \mathrm{Dom}(\mathfrak{A})$ if and only if $u \in \mathrm{Dom}(\triangle)$ and $\nabla_b u \in L^2(m)$.

As for $\mathfrak{A}^*$

We have to handle $\mathrm{div}\, b$.

Define an operator $\vec{\mathfrak{D}}$ acting on 1-fomrs by

$$\vec{\mathfrak{D}}\theta = \frac{1}{2}\square_1\theta - \nabla_b\theta - \langle\nabla.b, \theta\rangle - \theta\,\mathrm{div}\,b.$$

The intertwining property holds as

$$\nabla\mathfrak{A}^* u = \vec{\mathfrak{D}}\nabla u - u\nabla\,\mathrm{div}\,b.$$

The bilinear form associated with the symmetrization of $\vec{\mathfrak{D}}$ is

$$\vec{\mathcal{E}}'(\theta, \eta) = \frac{1}{2}(\nabla\theta, \nabla\eta)_2 + \int_M \{\frac{1}{2}\,\mathrm{Ric}(\theta, \eta) + \frac{1}{2}(\theta, \eta)\,\mathrm{div}\,b + (B\theta, \eta)\}\,dm$$

We impose the following condition:

(A.4) Ric is bounded from below and $\exists\delta$: $\mathrm{Ric} + \frac{1}{2}\,\mathrm{div}\,b + B \geq -\delta'$ and $\frac{\nabla\,\mathrm{div}\,b}{\mathrm{div}\,b + 2\gamma + 2}$ is bounded.

Theorem 5. Assume (A.1), (A.2), (A.2)* and (A.4). Then $u \in \mathrm{Dom}(\mathfrak{A})$ if and only if $u \in \mathrm{Dom}(\triangle)$ and $\nabla_b u + \frac{1}{2}u\,\mathrm{div}\,b \in L^2$.

3. Convergence to the invariant measure

Le M be a compact connected Riemannia maniflod.

$$\tfrac{1}{2}\triangle \qquad\qquad p(t,x,y) \longrightarrow 1$$

$$\tfrac{1}{2}\triangle + b \quad (\mathrm{div}\, b = 0) \qquad\qquad q(t,x,y) \longrightarrow 1$$

How fast?

$$\lambda = -\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y\in M} |p(t,x,y) - 1|,$$

$$\gamma = -\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y\in M} |q(t,x,y) - 1|.$$

Our aim is to show that

$$\gamma \geq \lambda.$$

Dirichlet forms satisfying the sector condition

- (M, m): a measure space, $H = L^2(m)$: a Hilbert space
- $\mathcal{E}$: a Dirichlet form, $\tilde{\mathcal{E}}$: symmetrization of $\mathcal{E}$
- $\mathfrak{A}$: the generator
- $\{T_t\}$: a Markovian semigroup

We assume that $\mathcal{E}$ is non-negative definite and satisfies a weak sector condition:

$$|\mathcal{E}(f,g)| \leq K\mathcal{E}_1(f,f)^{1/2}\mathcal{E}_1(g,g)^{1/2}.$$

We also assumed that $\{T_t^*\}$ is a Markovian semigroup.

Ultracontractivity

Theorem 6. Let $\mu > 0$. We have the following equivalence:

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0,1]$$

$$\Updownarrow$$

$$\|f\|_2^{2+4/\mu} \le c_2(\tilde{\mathcal{E}}(f,f) + \|f\|_2^2)\, \|f\|_1^{4/\mu}$$

$$\Updownarrow$$

$$\|f\|_{2\mu/(\mu-2)}^2 \le c_3(\tilde{\mathcal{E}}(f,f) + \|f\|_2^2) \quad (\mu > 2)$$

Key estimate:

$$\tilde{\mathcal{E}}(T_s f, T_s f) \le C\{\tilde{\mathcal{E}}(f,f) + \|f\|_2^2\}$$

Dirichlet forms having invariant measure

We continue to assume the sector condition. In addition, we assume

- m is an invariant probability measure.

$$\int_M T_t f\, dm = \int_M f\, dm$$

- $T_t 1 = 1$ and $\mathfrak{A}1 = 0$.

The following inequality is called the Poincaré inequality

$$\|f - m(f)\|_2^2 \le \lambda^{-1} \tilde{\mathcal{E}}(f, f) \tag{6}$$

where

$$m(f) = \int_M f(x)\, m(dx).$$

This inequality is equivalent to

$$\|T_t f - m(f)\|_2^2 \le e^{-2\lambda t} \|f - m(f)\|_2^2.$$

Theorem 7. Let $\mu > 0$. We consider the following two conditions.

(i) There exists a constant c_1 so that for all $f \in L^1$

$$\|T_t f - m(f)\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0,1].$$

(ii) There exists a constant c_2 so that for all $f \in \mathrm{Dom}(\tilde{\mathcal{E}}) \cap L^1(m)$

$$\|f - m(f)\|_2^{2+4/\mu} \le c_2 \, \tilde{\mathcal{E}}(f,f) \, \|f\|_1^{4/\mu}.$$

Then, (i) & Poincaré inequality $\Leftrightarrow$ (ii).
Under the condition (ii), there exists a constant $c_4 > 0$ so that for all $f \in L^1$

$$\|T_t f - m(f)\|_\infty \le c_4 e^{-\lambda t} \|f\|_1, \quad \forall t \ge 1.$$

Here λ is a constant appears in the Poincaré inequality (6).

Proof.

$$\begin{aligned}\|T_t - m\|_{1\to\infty} &= \|(T_1 - m)(T_{t-2} - m)(T_1 - m)\|_{1\to\infty} \\ &\le \|T_1 - m\|_{2\to\infty}\, \|T_{t-2} - m\|_{2\to 2}\, \|T_1 - m\|_{1\to 2} \\ &\le \|T_1 - m\|_{2\to\infty}\, e^{-\lambda(t-2)}\, \|T_1 - m\|_{1\to 2}\end{aligned}$$

□

Let us investigate the convergense rete. Set $a_t = \|T_t - m\|_{1\to\infty}$ and define γ by

(7) $$\gamma = -\lim_{t\to\infty} \frac{1}{t} \log a_t.$$

Theorem 8. We have

$$\gamma \ge \lambda$$

and the equality holds if $\mathfrak{A}$ is normal. Here λ is the spectral gap (6).

Case that M is compact

Let us return to the diffusion on a Riemannian manifold M generated by

$$\mathfrak{A}f = \frac{1}{2}\triangle f + bf = \frac{1}{2}\triangle f + (\nabla f, \omega_b).$$

If M is compact, then there exists an invariant probability measure.

- ν: an invariant probability measure: $\nu = e^{-U}m$

We use the following notations

- ∇: the Levi-Civita covariant derivative
- ∇^*: the dual operator of ∇ w.r.t. m
- ∇^*_ν: the dual operator of ∇ w.r.t. ν
- ω_b: 1-form corresponding to b

We now change the reference measure to ν. So our Hilbert space changes to $L^2(\nu)$.

Set

$$\mathcal{G}_\nu = \{\mathfrak{A}\,;\ \mathfrak{A} \text{ has an invariant measure } \nu.\}$$

We set

$$\tilde{b} = \frac{1}{2}(\nabla U)^\sharp + b,$$
$$\omega_{\tilde{b}} = \frac{1}{2}\nabla U + \omega_b.$$

Theorem 9. $\mathfrak{A} \in \mathcal{G}_\nu$ if and ony if $\nabla_\nu^* \omega_{\tilde{b}} = 0$. In this case,

$$\mathfrak{A} f = -\frac{1}{2} \nabla_\nu^* \nabla f + (\omega_{\tilde{b}}, \nabla f)$$

and

$$\mathfrak{A}_\nu^* f = -\frac{1}{2} \nabla_\nu^* \nabla f - (\omega_{\tilde{b}}, \nabla f).$$

Further the associated symmetric Dirichlet form is given by

$$\tilde{\mathcal{E}}(f, h) = \frac{1}{2} \int_M (\nabla f, \nabla h) d\nu.$$

Normal operator

Theorem 10. $\mathfrak{A}$ is normal if and only if $\tilde{b}$ is a Killing field and $[\nabla U^\sharp, \tilde{b}] = 0$.

A vector field X is called a Killing field if $L_X g = 0$. It is known that X is a Killing field if and only if ∇X is skew-symmetric. This is also equivalent to

$$\operatorname{div} X = 0,$$

$$\nabla^* \nabla X + \operatorname{Ric}(X) = 0.$$

Recall

$$\mathfrak{A} = \frac{1}{2}\triangle_\nu + \nabla_{\tilde{b}},$$
$$\mathfrak{A}^* = \frac{1}{2}\triangle_\nu - \nabla_{\tilde{b}}.$$

Here

$$\triangle_\nu = -\nabla_\nu^*\nabla = -\nabla^*\nabla + \nabla U \cdot \nabla.$$

Then

$$\mathfrak{A}\mathfrak{A}^* - \mathfrak{A}^*\mathfrak{A} = [\nabla_{\tilde{b}}, \triangle_\nu].$$

Moreover

$$[\triangle_\nu, \nabla_{\tilde{b}}]f = 2(\nabla\omega_{\tilde{b}}, \nabla^2 f) + (-\nabla^*\nabla\omega_{\tilde{b}} + \mathrm{Ric}(\omega_{\tilde{b}}) + [\nabla U^\sharp, \tilde{b}]^\flat, \nabla f)$$

T_t has a density $p(t, x, y)$ with respect to ν. Define

$$\gamma = -\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y\in M} |p(t, x, y) - 1|.$$

Let λ be the spectral gap:

$$\lambda = \inf_{f\neq\nu(f)} \frac{\tilde{\mathcal{E}}(f, f)}{\|f - \nu(f)\|_\nu^2}$$

Theorem 11. We have

$$\gamma \geq \lambda.$$

The equality holds if $\mathfrak{A}$ is normal.

We can give a characterization of γ in terms of the spectrum:

$$\gamma = \inf\{\Re\eta;\ \eta \in \sigma(-\mathfrak{A})\}$$

Theorem 12. If $\gamma = \lambda$, then $-\mathfrak{A}$ has an eigenvalue ξ so that $\Re\xi = \lambda$ and its eigenfunctions is also an eigenfunction of $\frac{1}{2}\nabla_\nu^*\nabla$ for an eigenvalue λ.

Example: 2-dimensional torus

- $M = T^2$
- (x, y): the standard local coordinate
- $b = f(x)\dfrac{\partial}{\partial y} + g(y)\dfrac{\partial}{\partial x}$

Then

$$\begin{aligned} f = \text{constant},\ g = \text{constant} \quad &\Rightarrow \quad \gamma = \lambda \\ f = 0 \quad &\Rightarrow \quad \gamma = \lambda \\ f \neq \text{constant},\ g \neq \text{constant} \quad &\Rightarrow \quad \gamma > \lambda. \end{aligned}$$

Thanks a lot!