Non-symmetric diffusions on Riemannian manifolds and the ultracontractivity

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1. Introduction

Let (X_t) be a diffusion on a compact Riemannian manifold M generated by $\frac{1}{2}\triangle + b$. Its has a transition probability density $p(t,x,y)$. We can see that $p(t,x,y)$ converges to an invariant measure $\nu(dx) = \rho(x)\mathrm{vol}(dx)$.

We are interested in the convergence rrate γ:

$$\gamma = -\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y} |p(t,x,y) - \rho(x)|.$$

We give a lower bound of γ.

Our main tool is the ultracontractivity of the semigorup.

Ultracontractivity

A semigroup $\{T_t\}$ is called ultracontractive if $T_t\colon L^1 \to L^\infty$ is bounded for all $t > 0$.

It is well-known that the following three conditions are equivalent for a symmetric Markovian semigroup. Let $\mu > 0$ be given.

(i) $\exists c_1 > 0$, $\forall f \in L^1$:

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0.$$

(ii) $\exists c_2 > 0$, $\forall f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$:

$$\|f\|_2^{2+4/\mu} \le c_2\, \mathcal{E}(f,f)\, \|f\|_1^{4/\mu}.$$

(iii) $\mu > 2$, $\exists c_3 > 0$, $\forall f \in \mathrm{Dom}(\mathcal{E})$:

$$\|f\|_{2\mu/(\mu-2)}^2 \le c_3\, \mathcal{E}(f,f).$$

We extend this result for non-symmetric Markovian semigroups.

2. Non-symmetric Markovian semigroups

We give a framework in generall Hilbert space scheme.

- H: a Hilbert space
- $\{T_t\}$: a contraction C_0 semigroup
- $\{T_t^*\}$: the dual semigroup
- $\mathfrak{A}$, $\mathfrak{A}^*$: the generators of $\{T_t\}$ and $\{T_t^*\}$

A natural bilinear form $\mathcal{E}$ is defined by

$$\mathcal{E}(u, v) = -(\mathfrak{A}u, v).$$

We do not assume the sector condition and so we can not use this bilinear form.

We introduce a symmetric bilinear form. For this, we assume the following condition:

$(\mathbf{A.1})$ $\mathbf{Dom}(\mathfrak{A}) \cap \mathbf{Dom}(\mathfrak{A}^*)$ is dense in $\mathbf{Dom}(\mathfrak{A})$ and $\mathbf{Dom}(\mathfrak{A}^*)$.

Under this condition, we define a symmetric bilinear form $\tilde{\mathcal{E}}$ by

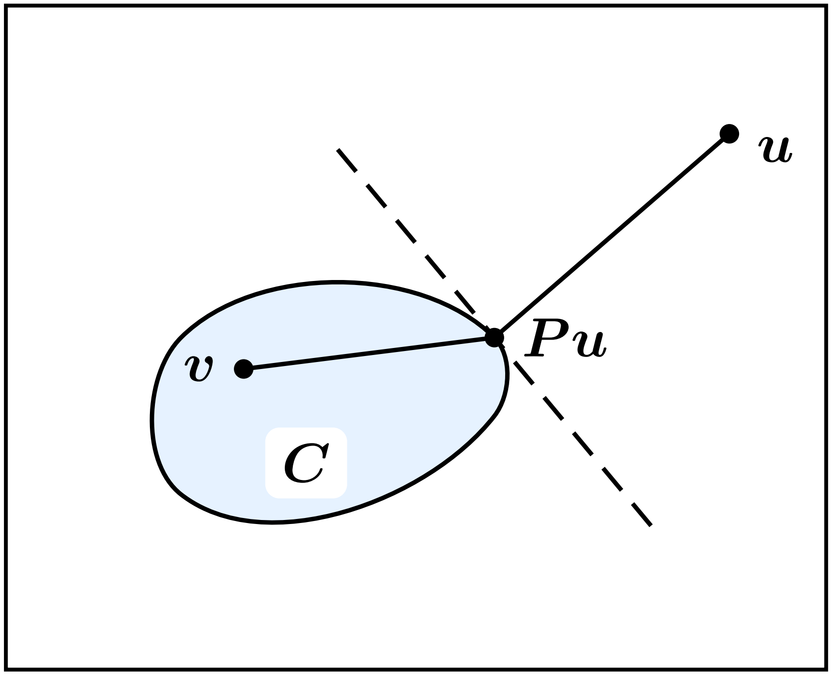
$$\tilde{\mathcal{E}}(u,v) = -\frac{1}{2}\{(\mathfrak{A}u, v) + (u, \mathfrak{A}v)\}, \quad u, v \in \mathbf{Dom}(\mathfrak{A}) \cap \mathbf{Dom}(\mathfrak{A}^*).$$

Proposition 1. Under the condition (A.1), $\tilde{\mathcal{E}}$ is closable and its closure contains $\mathbf{Dom}(\mathfrak{A})$ and $\mathbf{Dom}(\mathfrak{A}^*)$.

Covex set preserving property

- C: a convex set of H.
- Pu: the shortest point from u to C

$$(u - Pu, v - Pu) \le 0, \quad \forall v \in C.$$



Theorem 2. If $\{T_t\}$ and $\{T_t^*\}$ preserve a convex set C, then $Pu \in \mathrm{Dom}(\tilde{\mathcal{E}})$ for any $u \in \mathrm{Dom}(\tilde{\mathcal{E}})$ and we have

$$\tilde{\mathcal{E}}(Pu, u - Pu) \ge 0.$$

Markovian semigroup

- (M, m): a measure space
- $H = L^2(m)$: a Hilbert space
- $\{T_t\}$: a Markovian semigroup

We assume that $\{T_t^*\}$ is also a Markovian semigroup.

Under the assumption (A.1), we can define a symmetric bilinear form $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}$ is a Dirichlet form.

We have the following implications.

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0$$

$$\Uparrow \quad \Downarrow \text{ under } (1)$$

$$\|f\|_2^{2+4/\mu} \le c_2 \, \tilde{\mathcal{E}}(f,f) \, \|f\|_1^{4/\mu}$$

$$\Uparrow$$

$$\|f\|_{2\mu/(\mu-2)}^2 \le c_3 \, \tilde{\mathcal{E}}(f,f) \quad (\mu > 2)$$

(1) $$(\mathfrak{A}^2 f, f)_2 + (\mathfrak{A} f, \mathfrak{A} f)_2 \ge 0.$$

(1) holds if $\mathfrak{A}$ is normal, i.e. $\mathfrak{A}\mathfrak{A}^* = \mathfrak{A}^*\mathfrak{A}$.

Moreover

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0,1]$$

$$\Uparrow \quad \Downarrow \text{ under } (2)$$

$$\|f\|_2^{2+4/\mu} \le c_2(\tilde{\mathcal{E}}(f,f) + \|f\|_2^2)\, \|f\|_1^{4/\mu}$$

$$\Uparrow$$

$$\|f\|_{2\mu/(\mu-2)}^2 \le c_3(\tilde{\mathcal{E}}(f,f) + \|f\|_2^2) \quad (\mu > 2)$$

There there exists a constant $M > 0$ so that for all $f \in \mathbf{Dom}(\mathfrak{A}^2)$

$$((\mathfrak{A} - M)^2 f, f)_2 + ((\mathfrak{A} - M)f, (\mathfrak{A} - M)f)_2 \ge 0. \tag{2}$$

3. Dirichlet forms satisfying the sector condition

From now on, we assume the sector condition for the Dirichlet form $\mathcal{E}$.

In this case, we have

$$\|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0,1]$$

$$\Updownarrow$$

$$\|f\|_2^{2+4/\mu} \leq c_2(\tilde{\mathcal{E}}(f,f) + \|f\|_2^2)\,\|f\|_1^{4/\mu}$$

$$\Uparrow$$

$$\|f\|_{2\mu/(\mu-2)}^2 \leq c_3(\tilde{\mathcal{E}}(f,f) + \|f\|_2^2) \quad (\mu > 2)$$

Key estimate:

$$\tilde{\mathcal{E}}(T_s f, T_s f) \leq C\{\tilde{\mathcal{E}}(f,f) + \|f\|_2^2\}$$

Theorem 3. $\mu > 2$. Suppose that there exists a constant c_1 so that for any $f \in L^1$

$$\|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0, 1].$$

Then, for any $\tilde{\mu} > \mu$, there exists a constant $c_3 > 0$ so that for all $f \in \mathrm{Dom}(\tilde{\mathcal{E}})$

$$\|f\|^2_{2\tilde{\mu}/(\tilde{\mu}-2)} \leq c_3(\tilde{\mathcal{E}}(f, f) + \|f\|^2_2)$$

Key estimate: for $s < \frac{1}{2}$,

$$\|(1 - \mathfrak{A})^s f\|^2_2 \leq C(\tilde{\mathcal{E}}(f, f) + \|f\|^2_2).$$

4. Dirichlet forms having invariant measure

We continue to assume the sector condition. In addition, we assume

- m is an invariant probability measure.

$$\int_M T_t f \, dm = \int_M f \, dm$$

- $T_t 1 = 1$ and $\mathfrak{A} 1 = 0$.

The following inequality is called the Poincaré inequality

$$\|f - m(f)\|_2^2 \leq \lambda^{-1} \tilde{\mathcal{E}}(f, f) \tag{3}$$

where

$$m(f) = \int_M f(x) \, m(dx).$$

This inequality is equivalent to

$$\|T_t f - m(f)\|_2^2 \leq e^{-2\lambda t} \|f - m(f)\|_2^2.$$

Theorem 4. $\mu > 0$. We consider the following two conditions.

(i) There exists a constant c_1 so that for all $f \in L^1$

$$\|T_t f - m(f)\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t \in (0, 1].$$

(ii) There exists a constant c_2 so that for all $f \in \mathrm{Dom}(\tilde{\mathcal{E}}) \cap L^1(m)$

$$\|f - m(f)\|_2^{2+4/\mu} \le c_2 \, \tilde{\mathcal{E}}(f, f) \, \|f\|_1^{4/\mu}.$$

Then, (ii) is equivalent to (i) with the Poincaré inequality.
Under the condition (ii), there exists a constant $c_4 > 0$ so that for all $f \in L^1$

$$\|T_t f - m(f)\|_\infty \le c_4 e^{-\lambda t} \|f\|_1, \quad \forall t \ge 1.$$

Here λ is a constant appears in the Poincaré inequality (3).

Proof.

$$\begin{aligned} \|T_t - m\|_{1\to\infty} &= \|(T_1 - m)(T_{t-2} - m)(T_1 - m)\|_{1\to\infty} \\ &\le \|T_1 - m\|_{2\to\infty} \, \|T_{t-2} - m\|_{2\to 2} \, \|T_1 - m\|_{1\to 2} \\ &\le \|T_1 - m\|_{2\to\infty} \, e^{-\lambda(t-2)} \, \|T_1 - m\|_{1\to 2} \end{aligned}$$ □

Let us investigate the convergense rete. Set $a_t = \|T_t - m\|_{1\to\infty}$ and define γ by

$$\gamma = -\lim_{t\to\infty} \frac{1}{t} \log a_t. \tag{4}$$

Theorem 5. We have

$$\gamma \ge \lambda$$

and the equality holds if $\mathfrak{A}$ is normal. Here λ is the spectral gap (3).

Theorem 6. $\mu > 2$. Assume that there exists a constant c_1 so that

$$\|T_t f - m(f)\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0$$

and the Poincaré inequality holds.
Then, for any $\tilde{\mu} > \mu$, there exists a constant $c_3 > 0$ so that for all $f \in \mathrm{Dom}(\tilde{\mathcal{E}})$

$$\|f - m(f)\|^2_{2\tilde{\mu}/(\tilde{\mu}-2)} \le c_3 \tilde{\mathcal{E}}(f, f).$$

5. Non-symmetric diffusions on Riemannian manifolds

- (M, g): a complete connected Riemannian manifold
- $m = \mathrm{vol}$: the Riemannian volume
- b: a smooth vector field

We consider a diffusion generated by

$$\mathfrak{A} = \frac{1}{2}\triangle + b.$$

We regard it as an operator in $L^2(m)$.

The dual operator is

$$\mathfrak{A}^* = \frac{1}{2}\triangle - b - \mathrm{div}\, b.$$

Associated symmetric bilinear form $\tilde{\mathcal{E}}$ is

$$\tilde{\mathcal{E}}(u, v) = \frac{1}{2}\int_M (\nabla u, \nabla v)\, dm + \frac{1}{2}\int_M uv\, \mathrm{div}\, b\, dm.$$

We have to show the existence of associated semigroups.

- $o \in M$: any fixed point
- d: the Riemannian distance
- $\rho(x) = d(o, x)$

We assume the following conditions:

(A.2) $\quad \operatorname{div} b \geq 0$.

(A.3) There exists a non-increasing function $\kappa : [0, \infty) \to [0, \infty)$ with $\int_0^\infty \kappa(x)\, dx = \infty$ so that $|\nabla_b \rho| \leq \frac{1}{\kappa(\rho)}$.

Typical example of κ is $\kappa(x) = \frac{1}{cx}$.

Theorem 7. Under the conditions (A.2), (A.3), The closure of $(\mathfrak{A}, C_0^\infty(M))$ generates a C_0 semigroup in $L^2(m)$ and the semigroup is Markovian.
The same is true for $(\mathfrak{A}^*, C_0^\infty(M))$.

We denote the associated semigroups by $\{T_t\}$ and $\{T_t^*\}$.

Theorem 8. Assume (A.2), (A.3) and that there exists a constant c_2 so that for all $f \in \mathrm{Dom}(\tilde{\mathcal{E}}) \cap L^1(m)$

$$\|f\|_2^{2+4/\mu} \leq c_2\, \tilde{\mathcal{E}}(f, f)\, \|f\|_1^{4/\mu}.$$

Then, there exists a constant c_1 so that for all $f \in L^1$

$$\|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1, \quad \forall t > 0. \tag{5}$$

Remark 1. Under the condition (A.2), we have

$$\frac{1}{2} \int_M |\nabla u|^2\, dm \leq \tilde{\mathcal{E}}(u, u).$$

If the Brownian motion satisfies (5), then the diffusion satisfies (5).

Case that M is compact

If M is compact, then there exists an invariant probability measure.

- ν: an invariant probability measure
- $\nu = e^{-U} m$

We use the following notations

- ∇: the covariant derivative
- ∇^*: the dual operator of ∇ w.r.t. m
- ∇^*_ν: the dual operator of ∇ w.r.t. ν
- ω_b: 1-form corresponding to b

$$\mathfrak{A} f = \frac{1}{2}\triangle f + bf = \frac{1}{2}\triangle f + (\nabla f, \omega_b)$$

We now change the reference measure to ν. So our Hilbert space changes to $L^2(\nu)$.

Set

$$\mathcal{G}_\nu = \{\mathfrak{A}\,;\ \mathfrak{A} \text{ has an invariant measure } \nu.\}$$

We set

$$\tilde{b} = \frac{1}{2}(\nabla U)^\sharp + b,$$
$$\omega_{\tilde{b}} = \frac{1}{2}\nabla U + \omega_b.$$

Theorem 9. $\mathfrak{A} \in \mathcal{G}_\nu$ if and ony if $\boldsymbol{\nabla}_\nu^* \omega_{\tilde{b}} = \mathbf{0}$. In this case,

$$\mathfrak{A} f = -\frac{1}{2} \nabla_\nu^* \nabla f + (\omega_{\tilde{b}}, \nabla f)$$

and

$$\mathfrak{A}_\nu^* f = -\frac{1}{2} \nabla_\nu^* \nabla f - (\omega_{\tilde{b}}, \nabla f).$$

Further the associated symmetric Dirichlet form is given by

$$\tilde{\mathcal{E}}(f, h) = \frac{1}{2} \int_M (\nabla f, \nabla h) d\nu.$$

Normal operator

Theorem 10. $\mathfrak{A}$ is normal if and only if $\tilde{b}$ is a Killing field and $[\nabla U^\sharp, \tilde{b}] = 0$.

A vector field X is called a Killing field if $L_X g = 0$. It is known that X is a Killing field if and only if ∇X is skew-symmetric. This is also equivalent to

$$\operatorname{div} X = 0,$$

$$\nabla^* \nabla X + \operatorname{Ric}(X) = 0.$$

Recall

$$\mathfrak{A} = \frac{1}{2}\triangle_\nu + \nabla_{\tilde{b}},$$
$$\mathfrak{A}^* = \frac{1}{2}\triangle_\nu - \nabla_{\tilde{b}}.$$

Here

$$\triangle_\nu = -\nabla_\nu^*\nabla = \nabla^*\nabla + \nabla U \cdot \nabla.$$

Then

$$\mathfrak{A}\mathfrak{A}^* - \mathfrak{A}^*\mathfrak{A} = [\nabla_{\tilde{b}}, \triangle_\nu].$$

Moreover

$$[\triangle_\nu, \nabla_{\tilde{b}}]f = 2(\nabla\omega_{\tilde{b}}, \nabla^2 f) + (-\nabla^*\nabla\omega_{\tilde{b}} + \mathrm{Ric}(\omega_{\tilde{b}}) + [\nabla U^\sharp, \tilde{b}]^\flat, \nabla f)$$

T_t has a density $p(t,x,y)$ with respect to ν. Define

$$\gamma = -\lim_{t\to\infty} \frac{1}{t} \log \sup_{x,y\in M} |p(t,x,y) - 1|.$$

Let λ be the spectral gap:

$$\|f - \nu(f)\|_\nu^2 \le \lambda^{-1} \tilde{\mathcal{E}}(f,f)$$

Theorem 11. We have

$$\gamma \ge \lambda.$$

The equality holds if $\mathfrak{A}$ is normal.

Thank you!