Non symmetric diffusions on a Riemannian manifold

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Contents

1. Non-symmetric Diffusion on a Riemannian manifold

- (M, g): d-dimensional connected complete Riemannian manifold.
- $m = \mathrm{vol}$: the Riemannian volume. b : a vector field on M.

We consider the following opetaror in $L^2(m)$:

$$\mathfrak{A} = \frac{1}{2}\triangle + \nabla_b. \tag{1}$$

The dual operator is

$$\mathfrak{A}^* = \frac{1}{2}\triangle - \nabla_b - \operatorname{div} b$$

and the symmetrization is

$$\frac{1}{2}(\mathfrak{A} + \mathfrak{A}^*) = \frac{1}{2}\triangle - \frac{1}{2}\operatorname{div} b \tag{2}$$

They are well-defined in $C_0^\infty(M)$.

The bilinear form $\mathcal{E}$ associated with $\mathfrak{A}$ is

$$\mathcal{E}(u,v) = -(\mathfrak{A}u, v) = \frac{1}{2}\int_M (\nabla u, \nabla v)\, dm - \int_M (\nabla_b u) v\, dm. \tag{3}$$

The symmetrization of this is

$$\tilde{\mathcal{E}}(u,v) = \frac{1}{2}\int_M (\nabla u, \nabla v)\, dm + \frac{1}{2}\int_M uv \,\mathrm{div}\, b\, dm. \tag{4}$$

This coresspond the operator $\frac{1}{2}(\mathfrak{A} + \mathfrak{A}^*)$ in (2).

We are interested in when the semigroup associated to $\mathfrak{A}$ exists in L^2.

We impose the following condition to ensure that $-\mathfrak{A}$ is bounded from below.

(A.1) $\exists\, \gamma \in \mathbb{R} : \frac{1}{2}\,\mathrm{div}\, b \geq -\gamma$.

Under this condition, $\tilde{\mathcal{E}}$ is bounded from below and closable.

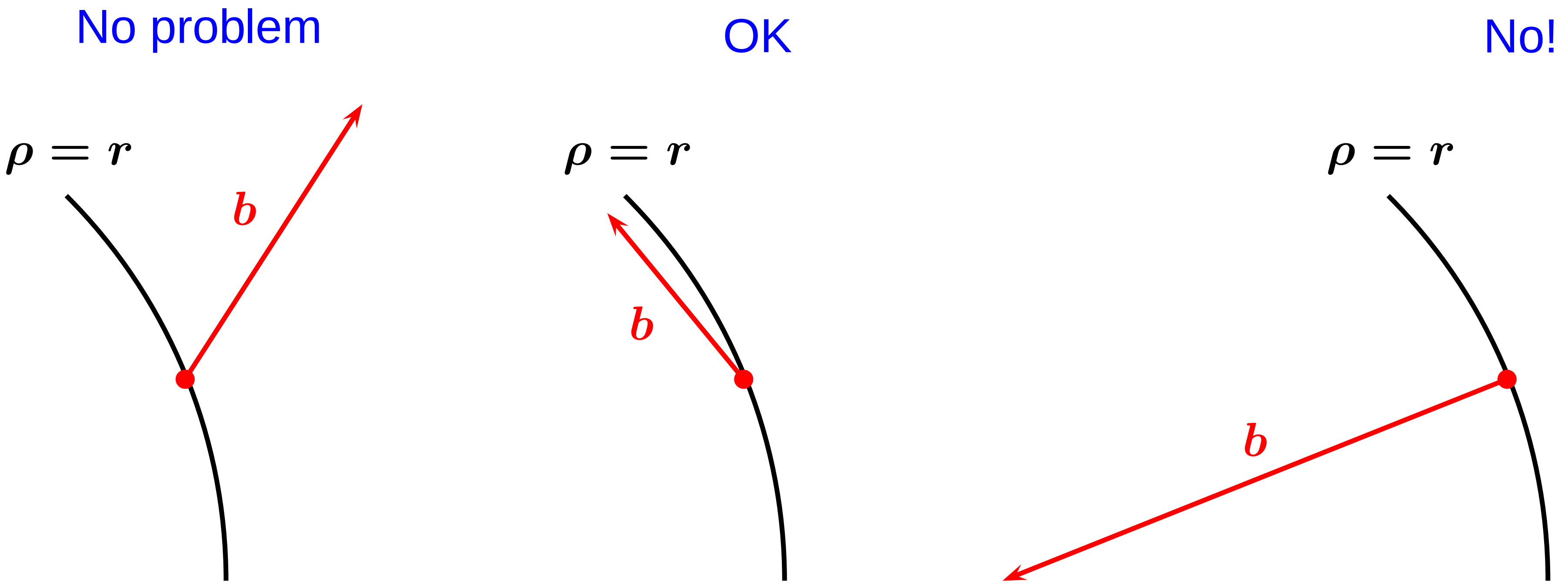
- d: the distance function
- $p \in M$
- $\rho(x) = d(p, x)$

We add the following condition for b :

(A.2) $\exists\, \kappa : [0, \infty) \to [0, 1]$ with $\int_0^\infty \kappa(x)\, dx = \infty$ so that

$$\kappa(\rho)\nabla_b \rho \geq -1.$$

- A typical example is $\kappa(x) = \frac{1}{x}$. $\nabla_b \rho(x) \geq -\rho(x)$.


No problem
$\rho = r$
b
OK
$\rho = r$
b
No!
$\rho = r$
b

Theorem 1. Under the assumptions (A.1) and (A.2), the closure of $(\mathfrak{A}, C_0^\infty(M))$ generates a Markovian C_0-semigroup in $L^2(m)$.

We claim the following:

- the dissipertivity: $((\mathfrak{A} - \gamma)u, u)_2 \leq 0$.
- the maximality: $(\mathfrak{A} - \gamma - 1)(C_0^\infty(M))$ is dense in L^2.

In fact,

$$((\mathfrak{A} - \gamma)u, u)_2 = -\frac{1}{2}\int_M (|\nabla u|^2 + u^2 \operatorname{div} b)\, dm - \int_M \gamma u^2\, dm \leq 0.$$

$$\begin{aligned} (\mathfrak{A} - \gamma - 1)^* u = 0 \quad &\Rightarrow \quad u \in C^\infty(M) \\ &\Rightarrow \quad (u, (\mathfrak{A} - \gamma - 1)(\chi_n u))_2 = 0 \\ &\Rightarrow \quad u = 0 \end{aligned}$$

The Markovian property is checked by the following criterion:

$$(\mathfrak{A}u, u - u \wedge 1)_2 \leq \gamma \|u - u \wedge 1\|_2^2. \tag{5}$$

Here $a \wedge b = \min\{a, b\}$.

We can also show the L^1-contraction property.

Proposition 2. Under the assumptions (A.1) and (A.2), $\{e^{-2t\gamma} T_t\}$ satisfies the L^1-contraction property.

We check the following criterion:

$$((\mathfrak{A} - 2\gamma)u, u_+ \wedge 1)_2 \leq -\gamma \|u_+ \wedge 1\|_2^2.$$

As for $\mathfrak{A}^*$

$$\mathfrak{A}^* = \frac{1}{2}\triangle - \nabla_b - \operatorname{div} b$$

We need the following condition:

(A.2)* $\exists\, \kappa : [0, \infty) \to [0, 1]$ with $\int_0^\infty \kappa(x)\, dx = \infty$ so that

$$\kappa(\rho)\nabla_b \rho \leq 1.$$

Theorem 3. Under the assumptions (A.1), (A.2)*, the closure of $(\mathfrak{A}^*, C_0^\infty(M))$ generates a C_0-semigroup in $L^2(m)$. It satisfies L^1-contraction property. If, in addition, $\operatorname{div} b \geq 0$, then the semigroup is Markovian.

2. Generator domain

If the Ricci curvature is bounded from below, then $\mathbf{Dom}(\triangle) = \mathbf{Dom}(\nabla^2)$. We can get similar result for $\mathfrak{A}$. To do so, we need the intertwining property. The following intertwining property is well known:

$$\nabla\triangle = \Box_1 \nabla.$$

Here $\Box_1 = -(dd^* + d^*d)$ is the Hodge-Kodaira operator.

Now we define an operator $\vec{\mathfrak{A}}$ acting on 1-forms by

$$\vec{\mathfrak{A}}\theta = \frac{1}{2}\Box_1\theta + \nabla_b\theta + \langle \nabla.b, \theta\rangle.$$

Then we have

$$\nabla\mathfrak{A} = \vec{\mathfrak{A}}\nabla.$$

As before, the bilinear form associated with the symmetrization of $\vec{\mathfrak{A}}$ is given by

$$\vec{\mathcal{E}}(\theta,\eta) = \frac{1}{2}(\nabla\theta, \nabla\eta)_2 + \int_M \{\frac{1}{2}\,\mathrm{Ric}(\theta,\eta) + \frac{1}{2}\,\mathrm{div}\, b(\theta,\eta) - (B\theta,\eta)\}\, dm.$$

where B is the symmetrization of ∇b : $B = \frac{1}{2}(\nabla b + (\nabla b)^*)$.

We have

$$(-\vec{\mathfrak{A}}\theta, \theta)_2 = \vec{\mathcal{E}}(\theta, \theta).$$

We impose the following condition so that $\vec{\mathcal{E}}$ is bounded from below.

(A.3) $\quad \mathrm{Ric}$ is bounded from below and $\exists \delta : \frac{1}{2}\,\mathrm{Ric} + \frac{1}{2}\,\mathrm{div}\, b - B \geq -\delta.$

Note that

$$\frac{1}{2}\|\nabla\theta\|_2^2 \leq \vec{\mathcal{E}}_\delta(\theta, \theta).$$

Theorem 4. Assume (A.1), (A.2), (A.2)* and (A.3). Then $u \in \mathbf{Dom}(\mathfrak{A})$ if and only if $u \in \mathbf{Dom}(\triangle)$ and $\nabla_b u \in L^2(m)$.

$$
\begin{aligned}
((\mathfrak{A}-\delta-1)u, \triangle u) &= -((\mathfrak{A}-\delta-1)u, \nabla^*\nabla u) \\
&= -(\nabla(\mathfrak{A}-\delta-1)u, \nabla u) \\
&= -((\vec{\mathfrak{A}}-\delta-1)\nabla u, \nabla u) \\
&= \vec{\mathcal{E}}_{\delta+1}(\nabla u, \nabla u).
\end{aligned}
$$

As for $\mathfrak{A}^*$

We have to handle $\operatorname{div} b$.

Define an operator $\vec{\mathfrak{D}}$ acting on 1-fomrs by

$$
\vec{\mathfrak{D}}\theta = \frac{1}{2}\Box_1\theta - \nabla_b\theta - \langle \nabla . b, \theta\rangle - \theta \operatorname{div} b.
$$

The intertwining property holds as

$$
\nabla\mathfrak{A}^* u = \vec{\mathfrak{D}}\nabla u - u\nabla \operatorname{div} b.
$$

The bilinear form associated with the symmetrization of $\vec{\mathfrak{D}}$ is

$$\vec{\mathcal{E}}'(\theta,\eta) = \frac{1}{2}(\nabla\theta,\nabla\eta)_2 + \int_M \{\frac{1}{2}\operatorname{Ric}(\theta,\eta) + \frac{1}{2}(\theta,\eta)\operatorname{div} b + (B\theta,\eta)\}\, dm$$

We impose the following condition:

(A.4) $\quad \operatorname{Ric}$ is bounded from below and $\exists\delta$: $\operatorname{Ric} + \frac{1}{2}\operatorname{div} b + B \geq -\delta'$ and $\frac{\nabla \operatorname{div} b}{\operatorname{div} b + 2\gamma + 2}$ is bounded.

Theorem 5. Assume (A.1), (A.2), (A.2)* and (A.4). Then $u \in \operatorname{Dom}(\mathfrak{A})$ if and only if $u \in \operatorname{Dom}(\triangle)$ and $\nabla_b u + \frac{1}{2} u \operatorname{div} b \in L^2$.

$$
\begin{aligned}
&((\mathfrak{A}^* - \delta' - 1)u, \triangle u)_2 \\
&\qquad = -((\mathfrak{A}^* - \delta' - 1)u, \nabla^* \nabla u)_2 \\
&\qquad = -(\nabla(\mathfrak{A}^* - \delta' - 1)u, \nabla u)_2 \\
&\qquad = -((\vec{\mathfrak{D}} - \delta' - 1)\nabla u, \nabla u)_2 + (u\nabla \operatorname{div} b, \nabla u)_2 \\
&\qquad = \vec{\mathcal{E}}'_{\delta'+1}(\nabla u, \nabla u) + (u\nabla \operatorname{div} b, \nabla u)_2.
\end{aligned}
$$

$\frac{\nabla \operatorname{div} b}{\operatorname{div} b + 2\gamma + 2}$ is bounded

3. L^p semigroup

So far, we have considered in L^2 setting. What about in L^p case? $(1 < p < \infty)$

Theorem 6. Under the assumptions (A.1) and (A.2), the closure of $(\mathfrak{A}, C_0^\infty(M))$ generate a C_0-semigroup in L^p. The semigroup satisfies Markovian property. Further $\{e^{-2t\gamma} T_t\}$ satisfies the L^1-contraction property.

We set $\gamma_p = \frac{p}{2}\gamma$.

We claim the following:

- the dissipertivity: $\displaystyle\int_M (\mathfrak{A} - \gamma_p) u \,\mathrm{sgn}(u)|u|^{p-1}\, dm \leq 0$.
- the maximality: $(\mathfrak{A} - \gamma - 1)(C_0^\infty(M))$ is dense in L^p.

To see

$$\int_M \triangle u \ \mathrm{sgn}(u)|u|^{p-1}\, dm \leq 0, \tag{6}$$

define φ_ε ($\varepsilon > 0$) by

$$\varphi_\varepsilon(t) = t(t^2 + \varepsilon)^{(p/2)-1}.$$

$\varphi'_\varepsilon(t) \geq 0$. Hence

$$\int_M \triangle u\, \varphi_\varepsilon(u)\, dm = -\int_M \nabla u\, \varphi'_\varepsilon(u) \nabla u\, dm = -\int_M \varphi'_\varepsilon(u)|\nabla u|^2\, dm \leq 0.$$

Letting $\varepsilon \to 0$, we have (6).

As for $\nabla_b u$, set $\varphi(t) = |t|^p$. Then $\varphi'(t) = p\,\mathrm{sgn}(t)|t|^{p-1}$.

$$\nabla_b(|u|^p) = \nabla_b \varphi(u) = p\,\mathrm{sgn}(u)|u|^{p-1}\nabla_b u.$$

Hence

$$\int_M \nabla_b u\,\mathrm{sgn}(u)|u|^{p-1}\,dm = \frac{1}{p}\int_M \nabla_b(|u|^p)\,dm = -\frac{1}{p}\int_M |u|^p\,\mathrm{div}\,b\,dm.$$

So

$$\begin{aligned}\int_M (\mathfrak{A} - \gamma_p)u\,\mathrm{sgn}(u)|u|^{p-1}\,dm \\ = \int_M \triangle u\,\mathrm{sgn}(u)|u|^{p-1} - \int_M (\frac{1}{p}\,\mathrm{div}\,b + \gamma_p)|u|^p\,dm \leq 0.\end{aligned}$$

- the Markovian property:

$$\int_M \mathfrak{A}u\,(u-1)_+^{p-1}\,dm \le \frac{2\gamma}{p}\|(u-1)_+\|_p^p$$

- the L^1-contraction property:

$$\int_M (\mathfrak{A}-2\gamma)u\,(u_+\wedge 1)^{p-1}\,dm \le 2\gamma(\frac{1}{p}-1)\|u_+\wedge 1\|_p^p.$$

4. Ultracontractivity

A semigroup $\{T_t\}$ is called ultracontractive if $T_t \colon L^1 \to L^\infty$ is bounded for all $t > 0$.

It is well-known that the following three conditions are equivalent for a Markovian symmetric semigroup.

Let $\mu > 0$ be given.

(i) $\exists c_1 > 0$, $\forall t > 0$, $\forall f \in L^1$:

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1.$$

(ii) $\exists c_2 > 0$, $\forall f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$:

$$\|f\|_2^{2+4/\mu} \le c_2\, \mathcal{E}(f,f)\, \|f\|_1^{4/\mu}.$$

(iii) $\mu > 2$, $\exists c_3 > 0$, $\forall f \in \mathrm{Dom}(\mathcal{E})$:

$$\|f\|_{2\mu/(\mu-2)}^2 \le c_3\, \mathcal{E}(f,f).$$

We assume that $\operatorname{div} b \geq 0$ and (A.2) and (A.2)*. Then the following inplication holds:

$$\|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1$$

$$\Uparrow$$

$$\|f\|_2^{2+4/\mu} \leq c_2 \tilde{\mathcal{E}}(f,f) \, \|f\|_1^{4/\mu}$$

$$\Uparrow$$

$$\|f\|_{2\mu/(\mu-2)}^2 \leq c_3 \tilde{\mathcal{E}}(f,f)$$

To show this, set $u(t) = \|T_t f\|_2^2$. Then

$$-\frac{du}{dt} = -2(\mathfrak{A} T_t f, T_t f) \geq 2\|T_t f\|_2^{2+4/\mu} / (c_2 \|T_t f\|_1^{4/\mu}) \geq 2u^{1+2/\mu} / (c_2 \|f\|_1^{4/\mu}).$$

Hence

$$\frac{d}{dt}(u^{-2/\mu}) \geq \frac{4}{c_2 \mu \|f\|_1^{4/\mu}}.$$

The rest is the same as the symmetric case.

Theorem 7. Assume that $\mathbf{div}\, b \geq 0$ and (A.2), (A.2)*. If there exists $c_2 > 0$ so that

$$\|f\|_2^{2+4/\mu} \leq c_2\, \tilde{\mathcal{E}}(f, f)\, \|f\|_1^{4/\mu}$$

then, there exists $c_1 > 0$ so that

$$\|T_t f\|_\infty \leq c_1 t^{-\mu/2} \|f\|_1.$$

Remark 1. Under the above condition, we have

$$\frac{1}{2} \int_M |\nabla u|^2 \, dm \leq \tilde{\mathcal{E}}(u, u).$$

Thanks a lot!