Semigoups that preserve a convex set in a Banach space

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Contents

1. Introduction

The following properties are well-discussed:

(1) Positivity preserving

(2) Markovian

(3) Excessive function

(4) Invariant set

Aim: We give a unified method to prove them.

Positivity preserving property

L^1 $$\int_{\{f<0\}} \mathfrak{A}f(x)d\mu(x) \geq 0$$

$$\uparrow \quad p \to 1$$

L^p $$\int \mathfrak{A}f(x)f_-^{p-1}(x)d\mu(x) \geq 0$$

$$\downarrow \quad p \to \infty$$

C_∞ $\mathfrak{A}f(x_0) \geq 0,\quad x_0:$ maximum point of f_-

2. Semigroups that preserve a convex set in a Banach space

- $\boldsymbol{B}$: Banach pace with a norm $\| \ \|$
- $\boldsymbol{B}^*$: the dual pace of $\boldsymbol{B}$
- $\boldsymbol{F(x) =: \{\varphi \in B^*; \langle x, \varphi\rangle = \|x\|^2\}}$ (conjugate mapping)
- $\boldsymbol{\{T_t\}}$: a $\boldsymbol{(C_0)}$-semigroup
- $\boldsymbol{\mathfrak{A}}$: the generator
- $\boldsymbol{\{G_\alpha\}}$: the resolvent
- $\boldsymbol{C}$: a convex set in $\boldsymbol{B}$

We are interested in the following property:

$$T_t C \subseteq C, \quad \forall t \geq 0,$$

i.e., T_t preserves the convex set C.

- $d(x, C) = \inf\{\|x - y\|; y \in C\}$
- $P(x) = \{y \in C; d(x, y) = d(x, C)\}$

We always assume that $P(x) \neq \emptyset$.

Theorem 1. $\gamma \in \mathbb{R}$ is fixed.

Suppose that $\forall \boldsymbol{x} \in \mathbf{Dom}(\boldsymbol{\mathfrak{A}})$, $\exists \boldsymbol{y} \in \boldsymbol{P}(\boldsymbol{x})$, $\underset{\sim}{\forall \boldsymbol{\varphi} \in \boldsymbol{F}(\boldsymbol{x}-\boldsymbol{y})}$:

$$\Re\langle \boldsymbol{\mathfrak{A}x}, \boldsymbol{\varphi}\rangle \leq \gamma \|\boldsymbol{x}-\boldsymbol{y}\|^{\mathbf{2}}, \tag{1}$$

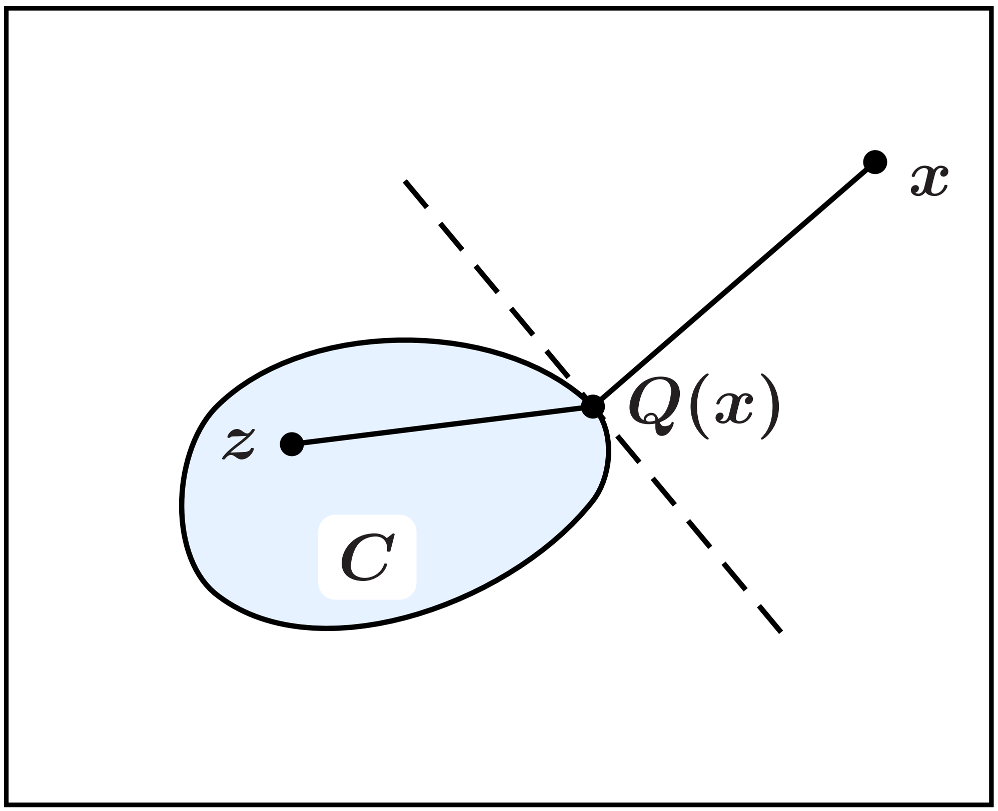
then the semigroup $\{\boldsymbol{T_t}\}$ preserves $\boldsymbol{C}$.

Conversely, if $\{\boldsymbol{T_t}\}$ preserves $\boldsymbol{C}$ and $\{\boldsymbol{e^{-\gamma t}T_t}\}$ is a contraction semigroup, then $\forall \boldsymbol{x} \in \mathbf{Dom}(\boldsymbol{\mathfrak{A}})$, $\forall \boldsymbol{y} \in \boldsymbol{P}(\boldsymbol{x})$, $\exists \boldsymbol{\varphi} \in \boldsymbol{F}(\boldsymbol{x}-\boldsymbol{y})$, so that (1) holds.

Good selection

$(\boldsymbol{Q}(\boldsymbol{x}), \boldsymbol{G}(\boldsymbol{x}))$: good selection

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \text{(i)} \quad \boldsymbol{Q}(\boldsymbol{x}) \in \boldsymbol{P}(\boldsymbol{x}), \quad \boldsymbol{G}(\boldsymbol{x}) \in \boldsymbol{F}(\boldsymbol{x} - \boldsymbol{Q}(\boldsymbol{x})) \\ \text{(ii)} \quad \forall \boldsymbol{z} \in \boldsymbol{C} : \ \Re\langle \boldsymbol{z} - \boldsymbol{Q}(\boldsymbol{x}), \boldsymbol{G}(\boldsymbol{x}) \rangle \le \boldsymbol{0} \end{cases}$$



Theorem 2. $\gamma \in \mathbb{R}$ is fixed.

Suppose that $\exists(\boldsymbol{Q(x)}, \boldsymbol{G(x)})$: good selection so that $\forall \boldsymbol{x} \in$ **Dom**$(\mathfrak{A})$:

$$\Re\langle \mathfrak{A}\boldsymbol{x}, \boldsymbol{G(x)}\rangle \le \gamma\|\boldsymbol{x} - \boldsymbol{Q(x)}\|^2, \tag{2}$$

then the semigroup $\{\boldsymbol{T_t}\}$ preserves $\boldsymbol{C}$.

Conversely, if $\{\boldsymbol{T_t}\}$ preserves $\boldsymbol{C}$ and $\{\boldsymbol{e^{-\gamma t}T_t}\}$ is a contraction semigroup, then for any good secection $(\boldsymbol{Q(x)}, \boldsymbol{G(x)})$ (if it exists) (2) holds.

Remark 1. In Hilbert space case, $\boldsymbol{P(x)}$ consists of one point and $\boldsymbol{F(x)} = \boldsymbol{x}$. In this case, the above theorem for $\boldsymbol{\gamma = 0}$ is proved by Brezis-Pazy (1970).

3. Examples

Positivity preserving property

$C = \{f;\ f \geq 0\}$

$Q(f) = f_+$

1. $C_\infty(E)$
   $G(f) = \|f_-\|_\infty\, \delta_{x_0}$, $\quad x_0$: maximaum point of f_-

$$\mathfrak{A} f(x_0) \geq \gamma f(x_0)$$

2. $L^p(d\mu) \quad (1 < p < \infty)$
   $G(f) = \|f_-\|_p^{2-p}\, f_-^{p-1}$

$$\int \mathfrak{A} f(x)\, f_-^{p-1}\, d\mu(x) \geq -\gamma \|f_-\|_p^p$$

3. $L^1(\mu)$

$G(f) = -\|f_-\|_1 \, \mathbf{1}_{\{f<0\}}$

$$\int_{\{f<0\}} \mathfrak{A} f(x)\, d\mu(x) \geq -\gamma \|f_-\|_1$$

Markovian peoperty

$C = \{f;\ f \leq 1\}$ (or $\{f;\ 0 \leq f \leq 1\}$), $Q(f) = f \wedge 1 = \min\{f, 1\}$

1. $C_\infty(E)$

$G(f) = \|(f-1)_+\|_\infty \, \delta_{x_0}$, $\quad x_0$: positive maximum point of f

$$\mathfrak{A} f(x_0) \leq 0$$

2. $\boldsymbol{L^p(d\mu)} \quad \boldsymbol{(1 < p < \infty)}$

$$\boldsymbol{G(f) = \|(f-1)_+\|_p^{2-p}\,(f-1)_+^{p-1}}$$

$$\boldsymbol{\int \mathfrak{A}f(x)\,(f-1)_+^{p-1}\,d\mu(x) \leq \gamma \|(f-1)_+\|_p^p}$$

3. $\boldsymbol{L^1(\mu)}$

$$\boldsymbol{G(f) = -\|(f-1)_+\|_1\,1_{\{f>1\}}}$$

$$\boldsymbol{\int_{\{f>1\}} \mathfrak{A}f(x)\,d\mu(x) \leq \gamma \|(f-1)_+\|_1}$$

L^1 contraction

The dual notion of the Markovian property is $\boldsymbol{L^1}$-contraction and positivity preserving. This time,

$$C = \{f;\ f \geq 0, \quad \int f\, d\mu = 1\}$$

$$Q(f) = (f - c)_+ \quad \text{with} \int (f - c)_+ d\mu = 1$$

$L^p(d\mu) \quad (1 < p < \infty)$

$$G(f) = \|f \wedge c\|_p^{2-p}\,(f \wedge c)^{p-1}$$

$$\int \mathfrak{A} f(x)\,(f \wedge c)^{p-1}\,d\mu(x) \leq \gamma \|f \wedge c\|_p^p$$

Excessive function

A non-negative function $\boldsymbol{u}$ is called excessive if

$$\boldsymbol{e^{-\alpha t} T_t u \leq u, \quad \forall t \geq 0.}$$

We do not need to assume that $\{T_t\}$ is Markovian. If we assume that $\{T_t\}$ is positivity preserving, then the above condition is equivalent to the invariance of the convex set $C = \{f;\ f \leq u\}$ under $\{e^{-\alpha t} T_t\}$.

So now

$$C = \{f;\ f \leq u\}, \quad Q(f) = f \wedge u = \min\{f, u\}$$

1. $C_\infty(E)$, $\quad G(f) = \|(f-u)_+\|_\infty \,\delta_{x_0}$, $\quad x_0$: positive maximum point of $f-u$

$$(\mathfrak{A}-\alpha)f(x_0) \le \gamma(f(x_0)-u(x_0))$$

2. $L^p(d\mu)$ $(1<p<\infty)$, $\quad G(f) = \|(f-u)_+\|_p^{2-p}\,(f-u)_+^{p-1}$

$$\int (\mathfrak{A}-\alpha)f(x)\,(f(x)-u(x))_+^{p-1}\,d\mu(x) \le \gamma\|(f-u)_+\|_p^p$$

3. $L^1(\mu)$, $\quad G(f) = -\|(f-u)_+\|_1\,\mathbf{1}_{\{f>u\}}$

$$\int_{\{f>u\}} (\mathfrak{A}-\alpha)f(x)\,d\mu(x) \le \gamma\|(f-u)_+\|_1$$

Invariant set

A set $\boldsymbol{K}$ is called invariant if

$$\mathbf{1}_{\boldsymbol{K}^c} \boldsymbol{T}_t \mathbf{1}_{\boldsymbol{K}} = \mathbf{0}, \quad \forall \boldsymbol{t} \geq \mathbf{0}.$$

So now

$\boldsymbol{C} = \{\boldsymbol{f};\, \mathbf{1}_{\boldsymbol{K}^c} \boldsymbol{f} = \mathbf{0}\}, \quad \boldsymbol{Q}(\boldsymbol{f}) = \mathbf{1}_{\boldsymbol{K}} \boldsymbol{f}$

1. $C_\infty(E)$, $\quad G(f) = \|1_{K^c} f\|_\infty \operatorname{sgn}(f(x_0))\delta_{x_0}$, $\quad x_0$: positive maximum $|f|$ in K^c.

$$\mathfrak{A}f(x_0) \operatorname{sgn}(f(x_0)) \leq \gamma |f(x_0)|$$

2. $L^p(d\mu)$ $(1 < p < \infty)$, $\quad G(f) = \|1_{K^c} f\|_p^{2-p} \, 1_{K^c} |f|^{p-1} \operatorname{sgn} f$

$$\int_{K^c} \mathfrak{A}f(x) \, |f(x)|^{p-1} \operatorname{sgn} f(x) \, d\mu(x) \leq \gamma \|1_{K^c} f\|_p^p$$

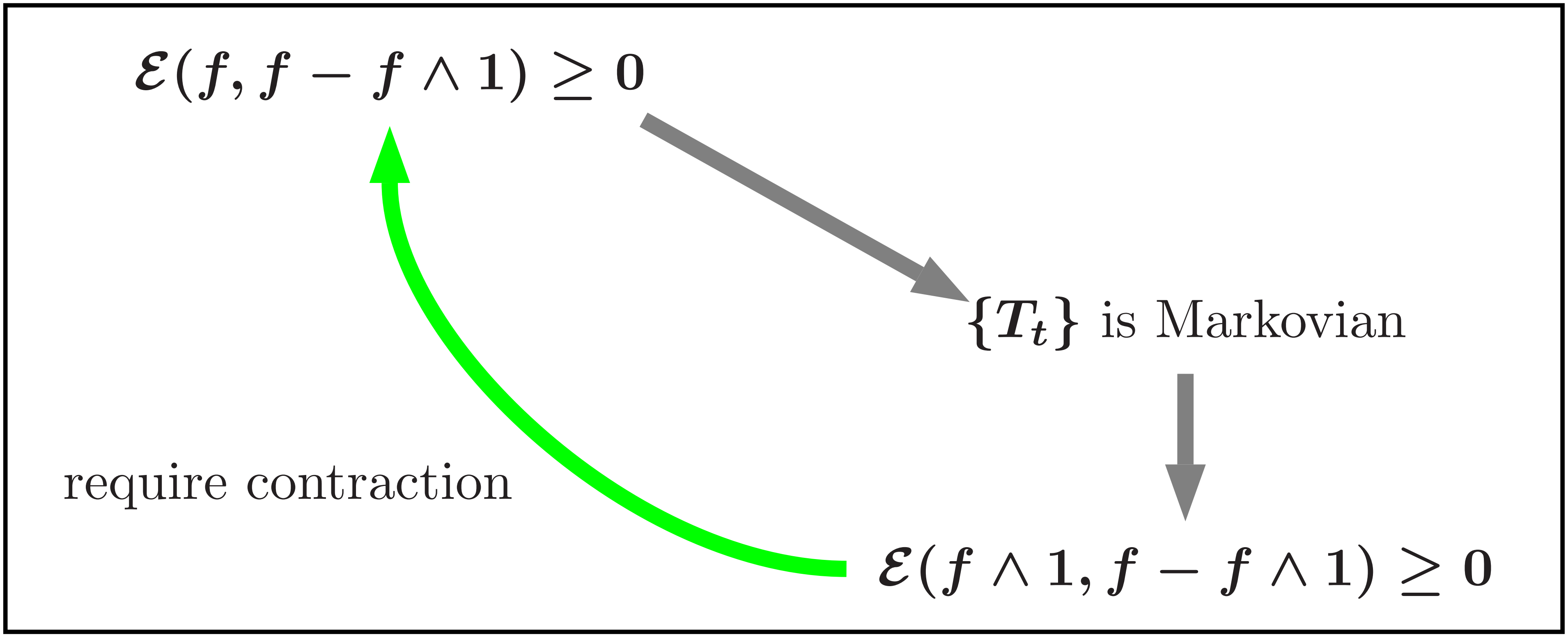
3. $L^1(\mu)$, $\quad G(f) = \|1_{K^c} f\|_1 \, 1_{K^c} \operatorname{sgn} f$

$$\int_{K^c} \mathfrak{A}f(x) \operatorname{sgn} f(x) \, d\mu(x) \leq \gamma \|1_{K^c} f\|_1$$

4. Hilber space case

We can give an conditions for preserving a convex set in terms of bilinear form. This was done by **Ouhabaz [1996]** for contraction semigroups. Our aim is to clarify when we need the contraction property or not.

Kown results



Ma-Röckner: Dirchelet forms

semi-Dirichlet form $\stackrel{\text{def}}{\Longleftrightarrow} \mathcal{E}(f + f \wedge 1, f - f \wedge 1) \geq 0$

Main results

$$\mathcal{E}(f, f - f \wedge 1) \geq 0$$

$$\Updownarrow$$

$\{T_t\}$ is Markovian contraction $\Longrightarrow$ $\{T_t\}$ is Markovian

$$\Updownarrow$$

$$\mathcal{E}(f \wedge 1, f - f \wedge 1) \geq 0$$

Main theorem

Theorem 3. We fix $\gamma \in \mathbb{R}$ and $\boldsymbol{\theta} \in [\mathbf{0}, \mathbf{1}]$. Let us consider the following condtitions:

(i) For any $\boldsymbol{x} \in \mathbf{Dom}(\boldsymbol{\mathcal{E}})$, $\boldsymbol{Px} \in \mathbf{Dom}(\boldsymbol{\mathcal{E}})$ and

$$\Re\mathcal{E}((1-\theta)x+\theta Px, x-Px) \geq -(1-\theta)\gamma|x-Px|^2.$$

(ii) $\{\boldsymbol{T_t}\}$ preserves $\boldsymbol{C}$.

(iii) $\boldsymbol{\mathcal{E}(P(x), x - P(x)) \geq 0}, \quad \forall \boldsymbol{x} \in \mathbf{Dom}(\boldsymbol{\mathcal{E}})$.

Then, the implications (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) holds.

If $\{\boldsymbol{e^{-\gamma t} T_t}\}$ is contractive, then the above three conditions are equivalent to each other.

If $\mathcal{E}$ is Hermitian, then the following condition (without the contraction property of $\{e^{-\gamma t}T_t\}$)

(iv) for any $x \in \mathbf{Dom}(\mathcal{E})$, $P(x) \in \mathbf{Dom}(\mathcal{E})$ and

$$\mathcal{E}(Px, Px) \leq \mathcal{E}(x, x) + \gamma|x - Px|^2, \quad \forall x \in \mathbf{Dom}(\mathcal{E})$$

deduces (ii). In addition, if we assume that $\{e^{-\gamma t}T_t\}$ is contractive, then all conditions (i) – (iv) are equiavlent to each other.

Positivity preserving property

Theorem 4. The following condtiotns are equiavlent to each other:

(i) $\{T_t\}$ preserves the positivity.

(ii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $|f| \in \mathbf{Dom}(\mathcal{E})$ and $\mathcal{E}(f_+, f_-) \leq 0$.

Further (i) or (ii) implies the following (iii):

(iii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $|f| \in \mathbf{Dom}(\mathcal{E})$ and $\mathcal{E}(|f|, |f|) \leq \mathcal{E}(f, f)$.

If, in addition, $\mathcal{E}$ is symmetric, then all conditions are equivalent to each other.

Theorem 5. We fix $\gamma \in \mathbb{R}$ and $\boldsymbol{\theta} \in [\mathbf{0}, \mathbf{1})$. The following tow conditions are equivalent to each other:

(i) $\{e^{-\gamma t} T_t\}$ is a positivity preserving contraction semigroup.

(ii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $|f| \in \mathbf{Dom}(\mathcal{E})$ and
$\mathcal{E}((1-\theta)f + \theta f_+, f - f_+) \geq -\gamma(1-\theta)\|f_-\|_2^2.$

If, in addition, $\mathcal{E}$ is symmetric, then the above condtions are equivalent to the following:

(iii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $|f| \in \mathbf{Dom}(\mathcal{E})$ and
$\mathcal{E}(f_+, f_+) \leq \mathcal{E}(f, f) + \gamma |f_-|^2.$

(iv) For any $f \in \mathbf{Dom}(\mathcal{E})$, $|f| \in \mathbf{Dom}(\mathcal{E})$ and
$0 \leq \mathcal{E}_\gamma(|f|, |f|) \leq \mathcal{E}_\gamma(f, f).$

Markovian property

Theorem 6. The following condtiotns are equiavlent to each other:

(i) $\{T_t\}$ is a Marvovian semigroup.

(ii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $f \wedge 1 \in \mathbf{Dom}(\mathcal{E})$ and $\mathcal{E}(f \wedge 1, f - f \wedge 1) \geq 0.$

Replacing $f \wedge 1$ with $f_+ \wedge 1$, we have the same result.

We may define that a bilinear form $\mathcal{E}$ is called semi-Dirichlet form if it satisfies the condition of (ii).

Theorem 7. We fix $\gamma \in \mathbb{R}$ and $\theta \in [0, 1)$. The following tow conditions are equivalent to each other:

(i) $\{T_t\}$ is a Markovian semigroup and $\{e^{-\gamma t}T_t\}$ is contractive.

(ii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $f \wedge 1 \in \mathbf{Dom}(\mathcal{E})$ and
$$\mathcal{E}((1-\theta)f + \theta(f \wedge 1), f - f \wedge 1) \geq -\gamma(1-\theta)\|f - f \wedge 1\|_2^2.$$

If, in addition, $\mathcal{E}$ is symmetric, (i) or (ii) is equivalent to the following:

(iv) For any $f \in \mathbf{Dom}(\mathcal{E})$, $f \wedge 1 \in \mathbf{Dom}(\mathcal{E})$ and
$$\mathcal{E}(f \wedge 1, f \wedge 1) \leq \mathcal{E}(f, f) + \gamma\|f - f \wedge 1\|_2^2.$$

Replacing $f \wedge 1$ with $f_+ \wedge 1$, we have the same result.

Excessive function

Theorem 8. We fix $\gamma \in \mathbb{R}$ and $\alpha \geq 0$. The following conditions are equivalent to each other:

(i) u is α-excessive and $\{T_t\}$ preserves the positivity.

(ii) For any $f \in \mathbf{Dom}(\mathcal{E})$, $f \wedge u \in \mathbf{Dom}(\mathcal{E})$ and $\mathcal{E}_\alpha(f \wedge u, f - f \wedge u) \geq 0$.

Theorem 9. We fix $\gamma \in \mathbb{R}$, $\boldsymbol{\alpha \geq 0}$ and $\boldsymbol{\theta \in [0, 1)}$. The following conditions are equivalent to each other:

(i) $\boldsymbol{u}$ is $\boldsymbol{\alpha}$-excessive and $\boldsymbol{\{e^{-(\alpha+\gamma)t}T_t\}}$ is a positivity preserving contraction semigroup.

(ii) For any $\boldsymbol{f \in \mathbf{Dom}(\mathcal{E})}$, $\boldsymbol{f \wedge u \in \mathbf{Dom}(\mathcal{E})}$ and
$\boldsymbol{\mathcal{E}_\alpha((1-\theta)f+\theta(f\wedge u), f-f\wedge u) \geq -\gamma(1-\theta)\|f-f\wedge u\|^2.}$

Invariant set

Theorem 10. The following conditions are equivalent to each other:

(i) $\boldsymbol{B}$ is invariant.

(ii) For any $\boldsymbol{f} \in \mathbf{Dom}(\boldsymbol{\mathcal{E}})$, $\mathbf{1}_{\boldsymbol{B}}\boldsymbol{f} \in \mathbf{Dom}(\boldsymbol{\mathcal{E}})$ and $\boldsymbol{\mathcal{E}}(\mathbf{1}_{\boldsymbol{B}}\boldsymbol{f}, \mathbf{1}_{\boldsymbol{B}^c}\boldsymbol{f}) \geq \mathbf{0}$.

(iii) For any $\boldsymbol{f} \in \mathbf{Dom}(\boldsymbol{\mathcal{E}})$, $\mathbf{1}_{\boldsymbol{B}}\boldsymbol{f} \in \mathbf{Dom}(\boldsymbol{\mathcal{E}})$ and $\boldsymbol{\mathcal{E}}(\mathbf{1}_{\boldsymbol{B}}\boldsymbol{f}, \mathbf{1}_{\boldsymbol{B}^c}\boldsymbol{f}) = \mathbf{0}$.

Thanks !