Uniqueness of Gibbs measures on $C(\mathbb{R} \longrightarrow \mathbb{R})$

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1. h-transformation

We consider the following operator on a domain $D \subseteq \mathbb{R}^d$:

$$L = \frac{1}{2}a_{ij}\partial_i\partial_j + b^i\partial_i + V.$$

We set $L_0 = L - V$. For positive function h, the **h-transform** of L is defined by

$$L^h f = \frac{1}{h}L(hf).$$

More explicitly,

$$L^h f = L_0 + a\frac{\nabla h}{h} \cdot \nabla + \frac{Lh}{h}.$$

If h is a harmonic function, i.e., $Lh = 0$, then L^h has no 0-th order term.

In the sequel, we assume that the semigroup generated by L has the transition measure: denoting $T_t = e^{tL}$

$$T_t f = \int_D p(t, x, dy) f(y).$$

Then, the transition measure of L^h is given by

$$p^h(t, x, dy) = \frac{1}{h(x)} p(t, x, dy) h(y).$$

2. Invariant function

$p(t, x, dy)$: a transition measure

φ is called a invariant function if

$$\varphi(x) = \int_D \varphi(y) p(t, x, dy), \quad \forall t \geq 0.$$

It is easy to see

$$\varphi \text{ is invariant} \iff h\text{-transform by } \varphi \text{ is conservative.}$$

principal eigenvalue

For any transition measure $p(t, x, dy)$ assiciated with L, there exist λ_c so that $L - \lambda$ is subcritical for $\lambda > \lambda_c$ and $L - \lambda$ is supercritical for $\lambda < \lambda_c$. Here

subcritical: Green measure exists

supercritical: no Green measure and no positive harmonic function

λ_c is called a (generalized) principal eigenvalue.

We will show that any 1-dimensional (minimal) diffusion process with $\lambda_c = 0$ has an invariant function.

3. One dimensional diffusion processes

$D = (l_-, l_+)$.

$\{(X_t), P_x\}$: a (minimal) diffusion on D (Dirichlet boundary condition)

$s(x)$: the sclae function

$dm(x)$: the speed measure (standard measure)

ζ : the explosion time

$\dfrac{d}{dm}\dfrac{d}{ds}$: the generator

Dirichlet form: $\mathcal{E}(f, g) = \int_D \frac{df}{ds}\frac{dg}{ds}ds$

From dm, we define a right continuous non-decreasing function m as

$$m(y) - m(x) = \int_{(x,y]} dm$$

Take any $a \in (l_-, l_+)$ and define

$$S(x) = \int_{(a,x]} \{m(y) - m(a)\} ds(y) = \int_{(a,x]} \{s(x) - s(u)\} dm(u),$$

$$M(x) = \int_{(a,x]} \{s(y) - s(a)\} dm(y) = \int_{(a,x]} \{m(x) - m(u)\} ds(u).$$

- $S(l_+) < \infty \Rightarrow l_+$ is called exit.
- $S(l_+) = \infty \Rightarrow l_+$ is called non-exit.
- $M(l_+) < \infty \Rightarrow l_+$ is called entrance.
- $M(l_+) = \infty \Rightarrow l_+$ is called non-entrance.

Feller's criterion:

$$(X_t) \text{ is conservative} \Leftrightarrow S(l_+) = \infty \text{ and } S(l_-) = \infty$$

h-transformation

Let v be a λ-harmonic function, i.e.,

$$\frac{d}{dm}\frac{d}{ds}v = \lambda v.$$

Define $d\hat{m} = v^2 dm$, $d\hat{s} = \frac{ds}{v^2}$. Then

$$\frac{1}{v}\left(\frac{d}{dm}\frac{d}{ds} - \lambda\right)(vf) = \frac{d}{d\hat{m}}\frac{d}{d\hat{s}}f. \tag{3.1}$$

$\frac{d}{d\hat{m}}\frac{d}{d\hat{s}}$ is the h-transform of $\frac{d}{dm}\frac{d}{ds} - \lambda$.

For 1-dimensional diffusions, we have $\boldsymbol{\lambda_c = \inf \sigma(-\frac{d}{dm}\frac{d}{ds})}$

Theorem 3.1. Let $(\boldsymbol{X_t})$ be a diffusion process on $\boldsymbol{D}$ with $\boldsymbol{\lambda_c = 0}$. Then there exist a invariant function.

	left	right	D	eigenvalue	h-transform
case 1	exit $\longleftarrow$	exit $\longrightarrow$	$(0, l)$	$\lambda_0 > 0$	$\varphi_0(x)$
case 2	exit $\longleftarrow$	non-exit $\not\longrightarrow$ $\not\longleftarrow$ non-entrance	$\begin{cases} (0, \infty) \\ (0, l) \end{cases}$	$\lambda_0 \geq 0$	$s(x) = x$
case 3	exit $\longleftarrow$	non-exit $\not\longrightarrow$ $\longleftarrow$ entrance	$(0, \infty)$	$\lambda_0 > 0$	$\varphi_0(x)$

4. Gibbs measure on $C(\mathbb{R} \longrightarrow \mathbb{R})$

We are given a potential function

- $V : \mathbb{R} \mapsto \mathbb{R}$: continuous and non-negative.

A Gibbs measure associated with V is formally expressed as

$$\mu(dx) \\ = Z^{-1} \exp\left\{ -\frac{1}{2} \int_{-\infty}^{\infty} |\dot{x}(t)|^2 \, dt - \int_{-\infty}^{\infty} V(x(t)) \, dt \right\} \prod_{t \in \mathbb{R}} dx(t).$$

Dobrushin-Lanford-Ruelle equation

Precise characterization of Gibbs measure is given as follows. For $I \subseteq \mathbb{R}$, we set $\mathcal{F}_I = \sigma\{x(t);\ t \in I\}$. Let $P_{s,x}^{t,y}$ be the pinned Brownian motion with $x(s) = x$ and $x(t) = y$. Then a probability measure μ is called a Gibbs measure if it satisfies

$$\mu(\,\cdot\,|\mathcal{F}_{[s,t]^c})(x(\cdot)) = Z^{-1}\exp\left\{-\int_s^t V(x(u))du\right\}P_{s,x}^{t,y} \otimes \delta_{x_{[s,t]^c}}.$$

Here Z is a normlizing constant.

We restrict ourselved to the following Gibbs measures:

$$\mathcal{G} = \{\mu \text{ satisfies DLR equation and the family } \{\mu \circ x(t)^{-1}\} \text{ is tight}\}.$$

Schrödinger operaotr

- $H = \frac{1}{2}\triangle - V$
- $\lambda_0 = \inf \sigma(-H)$: (generalized) principal eigenvalue
- $h = e^{-U}$: (generalized) eigenfunction for λ_0
- If $h \in L^2(\mathbb{R})$, then λ_0 is an eigenvalue.

$$Hh = -\lambda_0 h,$$
$$(H+\lambda_0)^h f = \frac{1}{h}(H+\lambda_0)(hf),$$
$$(H+\lambda_0)^h = \frac{1}{2}\triangle - \nabla U \cdot \nabla.$$

- $((H+\lambda_0)^h, L^2(h^2dx))$: a self-adjoint operaotr

- an associated Dirichlet form:

$$\mathcal{E}(f,g) = \frac{1}{2}\int_R (\nabla f, \nabla g) h^2 dx$$

U and V satisfy the following relation:

$$\triangle U - |\nabla U|^2 = 2(\lambda_0 - V).$$

The diffusion operaotr $\frac{1}{2}\triangle - \nabla U \cdot \nabla$ may generate a explosive diffusion. But we can assume that the associated diffusion is conservative by changing an eigenfunction if necessary (Theorem 3.1).

Transition measure associated with $\frac{1}{2}\triangle - V$ is given by

$$p(t,x,y) = E_{0,x}^{t,y}[\exp\{-\int_0^t V(x(s))\,ds\}]g(t,x,y)$$

where $E_{0,x}^{t,y}$ stands for the integral with respedt to the pinned Wiener measure $E_{0,x}^{t,y}$, and $g(t,x,y)$ is the Gauss kernel

$$g(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\Big\{-\frac{(y-x)^2}{2}\Big\}.$$

Set

$$q(t,x,y) = h(x)^{-1}p(t,x,y)e^{-t\lambda_0}h(y).$$

- $q(t,x,y)dy$: transition measure of the semigroup generated by $(H+\lambda_0)^h$
- h is an invariant function of $H+\lambda_0$

- $h^2\,dx$ is an invariant measure

Set

$$\hat{q}(t,x,y) = \frac{q(t,x,y)}{h(y)^2} = \frac{p(t,x,y)e^{-t\lambda_0}}{h(x)h(y)}$$

and

$$\nu(dx) = e^{-2U(x)}dx.$$

$\hat{q}(t,x,y)$ is a density function with respedt to ν.

When $\nu(\mathbb{R}) < \infty$, we assume that ν is normalized as $\nu(\mathbb{R}) = 1$.

Then the following is well-known:

- $\nu(\mathbb{R}) = 1 \implies \lim_{t\to\infty} \sup_{|x|,|y|\le R} |\hat{q}(t,x,y) - 1| = 0.$
- $\nu(\mathbb{R}) = \infty \implies \lim_{t\to\infty} \sup_{|x|,|y|\le R} |\hat{q}(t,x,y)| = 0.$

Theorem 4.1. If λ_0 is an eigenvalue then $\sharp(\mathcal{G}) = 1$, i.e., the uniqueness holds.

Sketch of proof

Under the assumption, we have $\nu(\mathbb{R}) = 1$.

- the law of $x(0)$ is ν.

$\because$ The density function of law of $x(0)$ is

$$\int_{\mathbb{R}^2} \frac{\hat{q}(t,x,z)h(z)^2\hat{q}(t,z,y)h(y)^2}{\hat{q}(2t,x,y)h(y)^2}\mu^{(x(-t),x(t))}(dx,dy)$$
$$= \int_{\mathbb{R}^2} \frac{\hat{q}(t,x,z)h(z)^2\hat{q}(t,z,y)}{\hat{q}(2t,x,y)}\mu^{(x(-t),x(t))}(dx,dy).$$

Non-existence

If $V = 0$, then $\sharp(\mathcal{G}) = 0$.

$$\frac{g(t,x,z)g(t,z,y)}{g(2t,x,y)}\mu^{(x(-t),x(t))}(dx,dy)$$

$$= \frac{1}{2\pi t}\exp\{-\frac{1}{2t}|z-x|^2 - \frac{1}{2t}|y-z|^2\}\sqrt{4\pi t}\exp\{\frac{1}{4t}|y-x|^2\}$$

$$\mu^{(x(-t),x(t))}(dx,dy)$$

$$= \frac{1}{\sqrt{\pi t}}\exp\{-\frac{1}{2t}|z-x|^2 - \frac{1}{2t}|y-z|^2 + \frac{1}{4t}|y-x|^2\}$$

$$\mu^{(x(-t),x(t))}(dx,dy)$$