EFFECTIVE BASE POINT FREE THEOREM FOR LOG CANONICAL PAIRS—KOLLÁR TYPE THEOREM

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Abstract. Kollár's effective base point free theorem for kawamata log terminal pairs is very important and was used in Hacon–McKernan's proof of pl flips. In this paper, we generalize Kollár's theorem for *log canonical* pairs.

1. Introduction

The main purpose of this paper is to show the power of the new cohomological technique introduced in [A]. The following theorem is the main theorem of this short note. It is a generalization of [K, 1.1 Theorem]. Kollár proved it only for kawamata log terminal pairs.

Theorem 1.1 (Effective base point free theorem)**.** *Let* (X, Δ) *be a* projective *log canonical pair with* $\dim X = n$. *Note that* Δ *is an effective* $\mathbb{Q}$*-divisor on* X. *Let* L *be a nef Cartier divisor on* X. *Assume that* $aL - (K_X + \Delta)$ *is nef and log big for some* $a \geq 0$. *Then there exists a positive integer* $m = m(n, a)$, *which only depends on* n *and* a, *such that* $|mL|$ *is base point free.*

For the relative statement, see Theorem 2.2.4 below.

Remark 1.2. We can take $m(n, a) = 2^{n+1}(n+1)!(\ulcorner a \urcorner + n)$ in Theorem 1.1.

By the results in [A], we can apply a modified version of X-method to log canonical pairs. More precisely, generalizations of Kollár's vanishing and torsion-free theorems to the context of embedded simple normal crossing pairs replace the Kawamata–Viehweg vanishing theorem in the world of log canonical pairs. For the details, see [F2]. Here, we generalize Kollár's arguments in [K] for log canonical pairs. This further illustrates the usefulness of our new cohomological package. For the benefit of the reader, we will explain the new vanishing and torsion-free theorems in the appendix (see Section 3). The starting point of

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our main theorem is the next theorem (see [A, Theorem 7.2]). For the proof, see [F2, Theorem 4.4]. Ambro's original statement is much more general than Theorem 1.3. Unfortunately, he gave no proofs in [A].

Theorem 1.3 (Base point free theorem for log canonical pairs)**.** *Let (X, Δ) be a log canonical pair and L a π-nef Cartier divisor on X, where $\pi : X \to V$ is a* projective *morphism. Assume that $aL-(K_X+\Delta)$ is π-nef and π-log big for some positive real number a. Then $\mathcal{O}_X(mL)$ is π-generated for all $m \gg 0$.*

The paper [F4] may help the reader to understand Theorem 1.3. In [F4], Theorem 1.3 is proved under the assumption that V is a point and $aL - (K_X + \Delta)$ is ample.

We summarize the contents of this paper. In Section 2, we prove Theorem 1.1. In Subsection 2.1, we give a slight generalization of Kollár's modified base point freeness method. We change Kollár's formulation so that we can apply our new cohomological technique. In Subsection 2.2, we use the modified base point freeness method to obtain Theorem 1.1. Here, we need Theorem 1.3. Section 3 is an appendix, where we quickly review our new vanishing and torsion-free theorems for the reader's convenience. The reader can find Angehrn–Siu type effective base point freeness and point separation for *log canonical* pairs in [F1].

Notation. We will work over the complex number field $\mathbb{C}$ throughout this paper.

Let r be a real number. The *integral part* $\llcorner r \lrcorner$ is the largest integer at most r and the *fractional part* $\{r\}$ is defined by $r - \llcorner r \lrcorner$. We put $\ulcorner r \urcorner = -\llcorner -r \lrcorner$ and call it the *round-up* of r.

Let X be a normal variety and B an effective $\mathbb{Q}$-divisor such that $K_X + B$ is $\mathbb{Q}$-Cartier. Then we can define the discrepancy $a(E, X, B) \in \mathbb{Q}$ for every prime divisor E over X. If $a(E, X, B) \geq -1$ (resp. > -1) for every E, then (X, B) is called *log canonical* (resp. *kawamata log terminal*). We sometimes abbreviate log canonical to *lc*.

Assume that (X, B) is log canonical. If E is a prime divisor over X such that $a(E, X, B) = -1$, then $c_X(E)$ is called a *log canonical center* (*lc center*, for short) of (X, B), where $c_X(E)$ is the closure of the image of E on X. A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on X is called *nef and log big* if L is nef and big and $L|_W$ is big for every lc center W of (X, B). The relative version of nef and log bigness can be defined similarly.

For a $\mathbb{Q}$-divisor $D = \sum_{i=1}^{r} d_i D_i$, where D_i is a prime divisor for every i and $D_i \neq D_j$ for $i \neq j$, we call D a *boundary* $\mathbb{Q}$-divisor if $0 \leq d_i \leq 1$ for every i. We denote by $\sim_{\mathbb{Q}}$ the $\mathbb{Q}$-linear equivalence of $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors.

We write $\mathrm{Bs}|D|$ the base locus of the linear system $|D|$.

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2. Effective base point free theorem

2.1. **Modified base point freeness method after Kollár.**

In this subsection, we slightly generalize Kollár's method in [K].

2.1.1. Let (X, Δ) be a log canonical pair and N a Cartier divisor on X. Let $g : X \to S$ be a proper surjective morphism onto a normal variety S with connected fibers. Let M be a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Assume that

$$N \sim_{\mathbb{Q}} K_X + \Delta + B + M, \tag{1}$$

where B is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $\mathrm{Supp} B$ contains no lc centers of (X, Δ) and that $B = g^*(B_S)$, where B_S is an effective ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on S. Let $X \setminus W$ be the largest open set such that $(X, \Delta + B)$ is lc. Assume that $W \neq \emptyset$, and let Z be an irreducible component of W such that $\dim g(Z)$ is maximal. We note that $g(W)$ is not equal to S since $B = g^*(B_S)$. Take a resolution $f : Y \to X$ such that the exceptional locus $\mathrm{Exc}(f)$ is a simple normal crossing divisor on Y, and put $h = g \circ f : Y \to S$. We can write

$$K_Y = f^*(K_X + \Delta) + \sum e_i E_i \text{ with } e_i \geq -1, \tag{2}$$

and

$$f^* B = \sum b_i E_i. \tag{3}$$

We can assume that $\mathrm{Supp}(f_*^{-1} B \cup f_*^{-1}\Delta \cup \sum E_i \cup h^{-1}(g(Z)))$ and $\mathrm{Supp}(h^{-1}(g(Z)))$ are simple normal crossing divisors. Let c be the largest real number such that $K_X + \Delta + cB$ is lc over the generic point of $g(Z)$. We note that

$$K_Y = f^*(K_X + \Delta + cB) + \sum (e_i - cb_i) E_i. \tag{4}$$

By the assumptions, we know $0 < c < 1$ and $c \in \mathbb{Q}$. If $cb_i - e_i < 0$, then E_i is f-exceptional. If $cb_i - e_i \geq 1$ and $g(Z)$ is a proper subset of $h(E_i)$, then $cb_i - e_i = 1$. We can write

$$f^* N \sim_{\mathbb{Q}} K_Y + f^* M + (1 - c) f^* B + \sum (cb_i - e_i) E_i \tag{5}$$

and

$$\sum \llcorner cb_i - e_i \lrcorner E_i = F + G_1 + G_2 - H, \tag{6}$$

where F, G_1, G_2, H are effective and without common irreducible components such that

- the h-image of every irreducible component of F is $g(Z)$,
- the h-image of every irreducible component of G_1 does not contain $g(Z)$,
- the h-image of every irreducible component of G_2 contains $g(Z)$ but does not coincide with $g(Z)$, and
- H is f-exceptional.

Note that $G_2 = \llcorner G_2 \lrcorner$ is a reduced simple normal crossing divisor on Y and that no lc center C of (Y, G_2) satisfies $h(C) \subset g(Z)$. Here, we used the fact that $\mathrm{Supp}(h^{-1}(g(Z)))$ and $\mathrm{Supp}(h^{-1}(g(Z)) \cup G_2)$ are simple normal crossing divisors on Y. We put $N' = f^*N + H - G_1$ and consider the short exact sequence

$$0 \to \mathcal{O}_Y(N' - F) \to \mathcal{O}_Y(N') \to \mathcal{O}_F(N') \to 0. \tag{7}$$

Note that

$$N' - F \sim_{\mathbb{Q}} K_Y + f^*M + (1-c)f^*B + \sum\{cb_i - e_i\}E_i + G_2.$$

So, the connecting homomorphism

$$h_*\mathcal{O}_F(N') \to R^1h_*\mathcal{O}_Y(N' - F) \tag{8}$$

is a zero map since $h(F) = g(Z)$ is a proper subset of S and every non-zero local section of $R^1h_*\mathcal{O}_Y(N' - F)$ contains $h(C)$ in its support, where C is some stratum of (Y, G_2). For the details, see Theorem 3.2, (a). Thus, we know that

$$0 \to h_*\mathcal{O}_Y(N' - F) \to h_*\mathcal{O}_Y(N') \to h_*\mathcal{O}_F(N') \to 0 \tag{9}$$

is exact. Moreover, by the vanishing theorem (see Theorem 3.2, (b)), we have

$$H^1(S, h_*\mathcal{O}_Y(N' - F)) = 0. \tag{10}$$

Therefore,

$$H^0(S, h_*\mathcal{O}_Y(N')) \to H^0(S, h_*\mathcal{O}_F(N')) \tag{11}$$

is surjective. It is easy to see that F is a reduced simple normal crossing divisor on Y. We note that no irreducible components of F appear in $\sum\{cb_i - e_i\}E_i$ and that

$$N'|_F \sim_{\mathbb{Q}} K_F + (f^*M + (1-c)f^*B)|_F + \sum\{cb_i - e_i\}E_i|_F + G_2|_F. \tag{12}$$

Thus, $h^i(S, h_*\mathcal{O}_F(N')) = 0$ for all $i > 0$ by the vanishing theorem (see Theorem 3.2, (b)). Thus, we obtain

$$h^0(F, \mathcal{O}_F(N')) = \chi(S, h_*\mathcal{O}_F(N')). \tag{13}$$

2.1.2. In our application, M will be a variable divisor of the form $M_j = M_0 + jL$, where M_0 is a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and $L = g^*L_S$ with an ample Cartier divisor L_S on S. Then we get that

$$h^0(F, \mathcal{O}_F(N_0' + jf^*L)) = \chi(S, h_*\mathcal{O}_F(N_0') \otimes \mathcal{O}_S(jL_S)) \tag{14}$$

is a polynomial in j for $j \geq 0$, where

$$N_0' = f^*N_0 + H - G_1 \tag{15}$$

and

$$N_0 \sim_{\mathbb{Q}} K_X + \Delta + B + M_0. \tag{16}$$

2.1.3. Assume that we establish $h^0(F, \mathcal{O}_F(N')) \neq 0$. By the above surjectivity (11), we can lift sections to $H^0(Y, \mathcal{O}_Y(f^*N + H - G_1))$. Since $F \not\subset \mathrm{Supp}G_1$, we get a section $s \in H^0(Y, \mathcal{O}_Y(f^*N + H))$ which is not identically zero along F. We know $H^0(Y, \mathcal{O}_Y(f^*N + H)) \simeq H^0(X, \mathcal{O}_X(N))$ because H is f-exceptional. Thus s descends to a section of $\mathcal{O}_X(N)$ which does not vanish along $Z = f(F)$.

2.2. **Proof of the main theorem.** The following lemma, which is the crucial technical result needed for Theorem 1.1, is essentially the same as [K, 2.2. Lemma].

Lemma 2.2.1. *Let $g : X \to S$ be a proper surjective morphism with connected fibers. Assume that X is projective, S is normal and (X, Δ) is lc for some effective $\mathbb{Q}$-divisor Δ. Let D_S^0 be an ample Cartier divisor on S and let $D_S \sim mD_S^0$ for some $m > 0$. We put $D^0 = g^*D_S^0$ and $D = g^*D_S$. Assume that $aD^0 - (K_X + \Delta)$ is nef and log big for some $a \geq 0$. Assume that $|D_S| \neq \emptyset$ and that $\mathrm{Bs}|D|$ contains no lc centers of (X, Δ), and let $Z_S \subset \mathrm{Bs}|D_S|$ be an irreducible component with minimal $k = \mathrm{codim}_S Z_S$. Then, with at most $\dim Z_S$ exceptions, Z_S is not contained in $\mathrm{Bs}|kD_S + (j + \ulcorner 2a \urcorner + 1)D_S^0|$ for $j \geq 0$.*

Proof. Pick general $B_i \in |D|$ and let

$$B = \frac{1}{2m}B_0 + B_1 + \cdots + B_k. \tag{17}$$

Then $B \sim_{\mathbb{Q}} \frac{1}{2}D^0 + kD$, $(X, \Delta + B)$ is lc outside $\mathrm{Bs}|D|$ and $(X, \Delta + B)$ is not lc at the generic points of $g^{-1}(Z_S)$. For the proof, see [K, (2.1.1) Claim]. We will apply the method in 2.1 with

$$N_j = kD + (j + \ulcorner 2a \urcorner + 1)D^0 \tag{18}$$

$$M_0 = \ulcorner 2a \urcorner D^0 - (K_X + \Delta) + \frac{1}{2}D^0, \text{and} \tag{19}$$

$$M_j = M_0 + jD^0. \tag{20}$$

We note that M_j is semi-ample for every $j \geq 0$ by Theorem 1.3 since M_j is nef and $M_j - (K_X + \Delta)$ is nef and log big. The crucial point is to show that

$$h^0(F, \mathcal{O}_F(N'_j)) = \chi(S, h_*\mathcal{O}_F(N'_j)) \tag{21}$$

is not identically zero, where

$$N'_j = f^*N_j + H - G_1 \tag{22}$$

for every j. Let $C \subset F$ be a general fiber of $F \to h(F) = Z_S$. Then

$$N'_0|_C = (h^*(kD_S + (\ulcorner 2a \urcorner + 1)D^0_S) + H - G_1)|_C = H|_C. \tag{23}$$

Hence $h_*\mathcal{O}_F(N'_0)$ is not the zero sheaf, and

$$H^0(F, \mathcal{O}_F(N'_j)) = H^0(S, h_*\mathcal{O}_F(N'_0) \otimes \mathcal{O}_S(jD^0_S)) \neq 0 \tag{24}$$

for $j \gg 1$. Therefore, $h^0(F, \mathcal{O}_F(N'_j))$ is a non-zero polynomial of degree $\dim Z_S$ in j for $j \geq 0$. Thus it can vanish for at most $\dim Z_S$ different values of j. This implies that

(25)

$$f(F) \not\subset \mathrm{Bs}|kD + (j + \ulcorner 2a \urcorner + 1)D^0| = g^{-1}\mathrm{Bs}|kD_S + (j + \ulcorner 2a \urcorner + 1)D^0_S|$$

by 2.1.3, with at most $\dim Z_S$ exceptions. Therefore, $Z_S = h(F) \not\subset \mathrm{Bs}|kD_S + (j + \ulcorner 2a \urcorner + 1)D^0_S|$. This is what we wanted. □

The next corollary is obvious by Lemma 2.2.1. For the proof, see [K, 2.3 Corollary].

Corollary 2.2.2. *Assume in addition that* $m \geq 2a + \dim S$ *and set* $k = \mathrm{codim}_S \mathrm{Bs}|D_S|$. *Then*

$$\dim \mathrm{Bs}|(2k+2)D_S| < \dim \mathrm{Bs}|D_S|. \tag{26}$$

Lemma 2.2.3. *We use the same notation as in* Theorem 1.1. *Then we can find an effective divisor* $D \in |2(\ulcorner a \urcorner + n)L|$ *such that* D *contains no lc centers of* (X, Δ).

Proof. Let C be an arbitrary lc center of (X, Δ). When (X, Δ) is kawamata log terminal, we put $C = X$. We consider the short exact sequence

$$0 \to \mathcal{I}_C \otimes \mathcal{O}_X(jL) \to \mathcal{O}_X(jL) \to \mathcal{O}_C(jL) \to 0, \tag{27}$$

where $\mathcal{I}_C$ is the defining ideal sheaf of C. By the vanishing theorem, $H^i(X, \mathcal{I}_C \otimes \mathcal{O}_X(jL)) = H^i(X, \mathcal{O}_X(jL)) = 0$ for all $i \geq 1$ and $j \geq a$ (see Theorem 3.3). Therefore, we have $H^i(C, \mathcal{O}_C(jL)) = 0$ for all $i \geq 1$ and $j \geq a$. Thus $h^0(C, \mathcal{O}_C(jL)) = \chi(C, \mathcal{O}_C(jL))$ is a non-zero polynomial in j since $|mL|$ is base point free for $m \gg 0$ (see Theorem 1.3). On the other hand, the map

$$H^0(X, \mathcal{O}_X(jL)) \to H^0(C, \mathcal{O}_C(jL)) \tag{28}$$

is surjective for $j \geq a$ since $H^1(X, \mathcal{I}_C \otimes \mathcal{O}_X(jL)) = 0$ for $j \geq a$ by the vanishing theorem (see Theorem 3.3). Thus, with at most $\dim C$ exceptions, $C \not\subset \mathrm{Bs}|(\ulcorner a \urcorner + j)L|$ for $j \geq 0$. Therefore, we can find an effective divisor $D \in |2(\ulcorner a \urcorner + n)L|$ such that D contains no lc centers. □

Proof of Theorem 1.1. By the base point free theorem for log canonical pairs (see Theorem 1.3), there exists a positive integer l such that $g = \Phi_{|lL|} : X \to S$ is a proper surjective morphism onto a normal variety with connected fibers such that $L \sim g^*L'$ for some ample Cartier divisor L' on S. By Lemma 2.2.3, we can find $D \in |2(\ulcorner a \urcorner + n)L|$ such that D contains no lc centers. Then Corollary 2.2.2 can be used repeatedly to lower the dimension of $\mathrm{Bs}|mL|$. This way we obtain that $|2^{n+1}(n+1)!(\ulcorner a \urcorner + n)L|$ is base point free. □

We close this section with the following theorem, which is the relative version of Theorem 1.1. We leave the proof for the reader's exercise. Of course, we need the relative version of Theorem 3.3 to check Theorem 2.2.4. See [A, Theorem 4.4] and [F2, Theorem 3.39].

Theorem 2.2.4. *Let (X, Δ) be a log canonical pair with* $\dim X = n$ *and $\pi : X \to V$ a* projective *surjective morphism. Note that Δ is an effective $\mathbb{Q}$-divisor on X. Let L be a π-nef Cartier divisor on X. Assume that $aL - (K_X + \Delta)$ is π-nef and π-log big for some $a \geq 0$. Then there exists a positive integer $m = m(n, a)$, which only depends on n and a, such that $\mathcal{O}_X(mL)$ is π-generated.*

3. Appendix: New cohomological package

In this appendix, we quickly review Ambro's formulation of Kollár's torsion-free and vanishing theorems.

3.1. Let Y be a simple normal crossing divisor on a smooth variety M, and let D be a boundary $\mathbb{Q}$-divisor on M such that $\mathrm{Supp}(D + Y)$ is simple normal crossing and D and Y have no common irreducible components. We put $B = D|_Y$ and consider the pair (Y, B). Let $\nu : Y^\nu \to Y$ be the normalization. We put $K_{Y^\nu} + \Theta = \nu^*(K_Y + B)$. A *stratum* of (Y, B) is an irreducible component of Y or the image of some lc center of (Y^ν, Θ). When Y is smooth and B is a boundary $\mathbb{Q}$-divisor on Y such that $\mathrm{Supp}B$ is simple normal crossing, we put $M = Y \times \mathbb{A}^1$

and $D = B \times \mathbb{A}^1$. Then $(Y, B) \simeq (Y \times \{0\}, B \times \{0\})$ satisfies the above conditions.

The following theorem is a special case of [A, Theorem 3.2].

Theorem 3.2. *Let (Y, B) be as above. Let $f : Y \to X$ be a proper morphism and L a Cartier divisor on Y.*

(a) Assume that $H \sim_{\mathbb{Q}} L - (K_Y + B)$ is f-semi-ample. Then every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the f-image of some strata of (Y, B).

(b) Let $\pi : X \to S$ be a proper morphism, and assume that $H \sim_{\mathbb{Q}} f^* H'$ for some π-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor H' on X. Then, $R^q f_* \mathcal{O}_Y(L)$ is π_*-acyclic, that is, $R^p \pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for all $p > 0$.

For the proof of Theorem 3.2, see [F2, Chapter 2]. By the above theorem, we can easily obtain the following theorem. For the details, see [A, Theorem 4.4] and [F2, Theorem 3.39].

Theorem 3.3. *Let (X, B) be an lc pair. Let C be an lc center of (X, B). We consider the short exact sequence*

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0,$$

where $\mathcal{I}_C$ is the defining ideal sheaf of C on X. Assume that X is projective. Let $\mathcal{L}$ be a line bundle on X such that $\mathcal{L} - (K_X + B)$ is ample. Then $H^q(X, \mathcal{L}) = 0$ and $H^q(X, \mathcal{I}_C \otimes \mathcal{L}) = 0$ for all $q > 0$. In particular, the restriction map $H^0(X, \mathcal{L}) \to H^0(C, \mathcal{L}|_C)$ is surjective.

A simple proof of Theorem 3.3 can be found in [F3] (cf. [F3, Theorem 4.1]). For a systematic treatment on this topic, we recommend the reader to see [F2].

References

[A] F. Ambro, Quasi-log varieties, Tr. Mat. Inst. Steklova **240** (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 220–239; translation in Proc. Steklov Inst. Math. 2003, no. 1 (240), 214–233

[F1] O. Fujino, Effective base point free theorem for log canonical pairs II—Angehrn–Siu type theorems—, preprint (2007).

[F2] O. Fujino, Introduction to the log minimal model program for log canonical pairs, preprint (2008).

[F3] O. Fujino, On injectivity, vanishing and torsion-free theorems for algebraic varieties, preprint (2009).

[F4] O. Fujino, Non-vanishing theorem for log canonical pairs, preprint (2009).

[K] J. Kollár, Effective base point freeness, Math. Ann. **296** (1993), 595–605.

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