Application of non-commutative Gröbner basis to calculations of E_2 terms of the Adams spectral sequence

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Outline

- I am interested in the application of Gröbner basis to the study of algebraic topology.
- The theory of Gröbner bases for polynomial rings was developed by Bruno Buchberger in 1965.
- One can view it as a multivariate, non-linear generalization of:
  - the Euclidean algorithm for computation of greatest common divisors,
  - Gaussian elimination for linear systems, and
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- For example, I studied the ring structrure of the cohomology of the Oriented Grassmann manifolds using Gröbner basis. I obtaind the exact values of the cup-length for an infinite family of the Oriented Grassmann manifolds.
  - $\mathrm{cup}(\widetilde{G}_{n,3}) = n+1$ for $n = 2^{m+1} - 4, m \geq 1$.
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Steenrod algebra, $\mathcal{A}_2$, which is generated by operations ($i \geq 0$)

$$\mathrm{Sq}^i \colon H^*(\quad;\mathbb{F}_2) \to H^{*+i}(\quad;\mathbb{F}_2),$$

- $\mathrm{Sq}^0 =$ the identity homomorphism.
- If $x \in H^n(X;\mathbb{F}_2)$, then $\mathrm{Sq}^n x = x^2$.
- $x, y \in H^*(X;\mathbb{F}_2)$,
  $\mathrm{Sq}^k(x \cup y) = \sum_{i=0}^k \mathrm{Sq}^i x \cup \mathrm{Sq}^{k-i} y$, the Cartan formula.
- The following relations hold among the generators: if $0 < a < 2b$

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{i=0}^{[a/2]} \binom{b-1-i}{a-2i} \mathrm{Sq}^{a+b-i} \mathrm{Sq}^i$$

These relations are called the Adem relations.

Outline

Problem

For topological space X and Y, compute the groups of morphisms

$$\{Y, X\}_k := \lim_{n\to\infty} [\Sigma^{n+k} Y, \Sigma^n X].$$

If $Y = S^0$, then $\{Y, X\}_k = \pi_k^S(X)$; the stable k-th homotopy group of X.

Outline

Theorem (Adams spectarl sequence)

There is a spectral sequence, converging to $_{(2)}\{Y, X\}_*$, *with* E_2*-terms given by*

$$E_2^{s,t} \cong \mathrm{Ext}_{\mathcal{A}_2}^{s,t}(H^*(X;\mathbb{F}_2), H^*(Y;\mathbb{F}_2)) \tag{1}$$

and differentials d_r *of bi-degree* $(r, r-1)$.

E_2-terms of Adams spectral sequence are given by $\mathcal{A}_2$-free resolution of $\mathcal{A}_2$-module $M = H^*(X; \mathbb{F}_2)$

$$\cdots \xrightarrow{d_{i+1}} R_i \xrightarrow{d_i} R_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} R_0 \xrightarrow{d_0} M \to 0$$

where R_i is free $\mathcal{A}_2$-module ($\oplus \mathcal{A}_2$) and $\mathcal{A}_2$ is the Steenrod algebra.

For another $\mathcal{A}_2$-module $N = H^*(Y; \mathbb{F}_2)$, we obtain a cochain complex

$$\mathrm{Hom}(M, N) \xrightarrow{d^0} \mathrm{Hom}(R_0, N) \xrightarrow{d^1} \mathrm{Hom}(R_1, N) \xrightarrow{d^2} \cdots .$$

The cohomology of this complex defines $\mathrm{Ext}^{*,*}_{\mathcal{A}_2}(M, N)$.

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Non-commutative Gröbner basis and Syzygy module.

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We consider the non-commutative polynomial ring.

- k: field.
- $R = k < x_1, x_2, \ldots, x_i, \cdots >$ non-commutative free associative ring
  - the set of polynomials.
- $\mathcal{B} = \{x_{i_1} x_{i_2} \cdots x_{i_k}\} \cup \{1\}$
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Definition (Monomial order)

Let $\leq$ be a well-ordering on $\mathcal{B}$. $\leq$ is said to be a monomial ordering on R if the following two conditions are satisfied.

- if $u, v, w, s \in \mathcal{B}$ with $w \leq u$, then $vws \leq vus$.
- For $u, w \in \mathcal{B}$, if $u = vws$ for some $v, s \in \mathcal{B}$ with $v \neq 1$ or $s \neq 1$, then $w \leq u$.

Hence $1 \leq u$ for all $u \in \mathcal{B}$.

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From here, We fix a monomial ordering.

- For $f \in R$, $f \neq 0$, we may write

$$f = c_1 u_1 + \cdots + c_m u_m$$

where $c_i \in k \setminus \{0\}$, $u_i \in \mathcal{B}$, $u_1 > u_2 > \cdots > u_m$.

- $\mathrm{lm}(f) = u_1$, the leading monomial of f.

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Theorem (Division on f by $\mathcal{H}$)

$$\text{INPUT} \qquad f,\ \mathcal{H} = \{h_1, \ldots, h_m \ldots\} \subset R$$

$$\text{OUTPUT} \qquad f = \sum_{\alpha=0}^{d} \mu_\alpha v_\alpha h_{i_\alpha} s_\alpha + r$$

- $\mu_\alpha \in k \setminus \{0\}$, $v_\alpha, s_\alpha \in \mathcal{B}$. $\mathrm{Im}(\mu_\alpha v_\alpha h_{i_\alpha} s_\alpha) \leq \mathrm{Im}(f)$.
- $r = 0$ or $r = \sum_j c_j t_j$ where $c_j \in k \setminus \{0\}$, $t_j \in \mathcal{B}$,
  - $\mathrm{Im}(r) \leq \mathrm{Im}(f)$, none of the t_j is divisible by any $\mathrm{Im}(h_i)$.
- r is depend on the ordering on $\mathcal{H}$.

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- The r appeared in above OUTPUT is called a remainder of f on the division by $\mathcal{G}$, dented $\overline{f}^{\mathcal{H}}$.
- Then $\overline{f}^{\mathcal{H}}$ is depend on the order on $\mathcal{H}$.
- To overcome this difficulty, we need Gröbner basis.

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Definition

Let I be two-side ideal of R. $\mathcal{G} = \{g_1, g_2, \dots\} \subset I$ is Gröbner basis, if and only if

$$(\mathrm{lm}(g) | g \in \mathcal{G}) = (\mathrm{lm}(g) | g \in I)$$

- If $\mathcal{G} = \{g_1, g_2, \dots\} \subset R$ is Gröbner basis, then
  - $f^{\mathcal{G}}$ is independent of the order on $\mathcal{G}$ and $f^{\mathcal{G}}$ is unique.
  - $f \in I = (g_1, g_2, \dots, g_i, \dots) \Leftrightarrow f^{\mathcal{G}} = 0$.
- There exists an algorithm to compute a Gröbner basis of $\mathcal{H} = \{h_1, \dots, h_s\}$.

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Outline

We can consider the Steenrod algebra as a quotient ring.

- $A = \mathbb{Z}_2\langle \mathrm{Sq}^1, \mathrm{Sq}^2, \dots \rangle$.
- $\mathcal{G} = \{\mathrm{Sq}^a\mathrm{Sq}^b - \sum_{i=0}^{[a/2]} \binom{b-1-i}{a-2i} \mathrm{Sq}^{a+b-i}\mathrm{Sq}^i | 0 < a < 2b\}$. Set of Adem relations.
- $I = (\mathcal{G})$: A two-side Ideal generated by $\mathcal{G}$.
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Definition (A monomial Ordering of Steenrod Algebra)

Let $u = Sq^{a_1} \cdots Sq^{a_k}$, $v = Sq^{b_1} \cdots Sq^{b_l} \in \mathcal{B}$.
Then, $u \geq v$ if and only if

- $k > l$ or,
- $k = l$ and the right-most nonzero entry of $(a_1 - b_1, \ldots, a_k - b_k)$ is positive.

$\mathrm{Im}(Sq^a Sq^b - \sum_{i=0}^{[a/2]} \binom{b-1-i}{a-2i} Sq^{a+b-i} Sq^i) = Sq^a Sq^b$.

Proposition

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Outline

Again, let $R = k < x_1, x_2, \dots, x_i, \cdots >$.
We consider the algorithm of computing a free-resolution of left-R-module M,

$$\cdots \xrightarrow{d_{i+1}} R_i \xrightarrow{d_i} R_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} R_0 \xrightarrow{d_0} M \to 0$$

where R_i's are free R-modules.

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where R_i's are free R-modules.

The key to compute the free resolutions is the

syzygy modules.

Again, let $R = k < x_1, x_2, \ldots, x_i, \cdots >$.
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Definition

The kernel of the map $\phi\colon R^s \to M$ given by

$$(h_1, \ldots, h_s) \mapsto \sum_{i=1}^{s} h_i \mathbf{f}_i, \quad \mathbf{f}_i \in M$$

is called the syzygy module of the matrix ${}^t[\mathbf{f}_1 \cdots \mathbf{f}_s]$. It is denoted $\mathrm{Syz}(\mathbf{f}_1, \ldots, \mathbf{f}_s)$.

Outline

- Introduction
  - Steenrod Algebra
  - The Problem We Study and The Basic Tool
  - Adams spectral sequence
- Non-commutative Gröbner basis
  - Non-commutative ring
  - Review of Non-commutative Gröbner basis
  - Example: Steenrod Algebra
- Free resolution
  - Syzygy module
  - Syzygy for Gröbner basis
  - Syzygy for not Gröbner basis

We consider only the case $M = R$.

Step 1: we consider

$$\mathrm{Syz}(g_1, \ldots, g_t) = \mathrm{Ker}[R^t \xrightarrow{\times^t[g_1 \cdots g_t]} R]$$

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We let $\mathrm{lm}(g_i) = X_i$ and $X_{ij} = \mathrm{lcm}(X_i, X_j)$. Then the S-polynomial of g_i and g_j is, given by

$$S(g_i, g_j) = \frac{X_{ij}}{X_i} g_i - \frac{X_{ij}}{X_j} g_j, \quad \mathrm{lp}(S(g_i, g_j)) < \mathrm{lp}(\frac{X_{ij}}{X_i} g_i), \ \mathrm{lp}(\frac{X ij}{X_j} g_j),$$

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$$\mathbf{s}_{ij} := \frac{X_{ij}}{X_i} \mathbf{e}_i - \frac{X_{ij}}{X_j} \mathbf{e}_j - (h_{ij1}, \ldots, h_{ijt}) \in R^t.$$

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We note that $\mathbf{s}_{ij} \in \mathrm{Syz}(g_1, \ldots, g_t)$, since

$$\begin{aligned} \mathbf{s}_{ij} \begin{bmatrix} g_1 \\ \vdots \\ g_t \end{bmatrix} &= \left(\frac{X_{ij}}{X_i} \mathbf{e}_i - \frac{X_{ij}}{X_j} \mathbf{e}_j - (h_{ij1}, \ldots, h_{ijt}) \right) \begin{bmatrix} g_1 \\ \vdots \\ g_s \end{bmatrix} \\ &= S(g_i, g_j) - \sum_{\nu=t}^{t} h_{ij\nu} g_\nu = 0. \end{aligned}$$

Theorem

With notation above, the collection $\{\mathbf{s}_{ij} | 1 \leq i \leq j \leq t\}$ *is generating set for* $\mathrm{Syz}(g_1, \ldots, g_t)$.

Theorem

There exists an algorithm to compute a generating set of

$$\mathrm{Syz}(g_1, \dots, g_t)$$

using S-polynomials and the division algorithm for $\{g_1, \dots, g_t\}$.

Outline

Step 2: we consider

$$\mathrm{Syz}(f_1, \ldots, f_s) = \mathrm{Ker}[R^s \xrightarrow{\times {}^t[f_1 \cdots f_s]} R]$$

where $\{f_1, \ldots, f_s\}$ is not Gröbner basis.

Syzygy for not Gröbner basis

- $\{f_1, \ldots, f_s\} \subset R$: not Gröbner basis.
- $\{g_1, \ldots, g_t\}$: Gröbner basis of $\{f_1, \ldots, f_s\}$.
- $F = {}^t[f_1 \ldots f_s]$, $G = {}^t[g_1 \ldots g_t]$.
  - $^\exists S$: $s \times t$ matrix, $^\exists T$: $t \times s$ matrix s.t. $F = SG$ and $G = TF$.
- compute a generating set $\{\mathbf{s}_1, \ldots, \mathbf{s}_r\}$ for $\mathrm{Syz}(g_1, \ldots, g_t)$
- $0 = \mathbf{s}_i G = \mathbf{s}_i(TF) = (\mathbf{s}_i T)F$,
  - $\langle \mathbf{s}_i T | i = 1, \ldots, r \rangle \subset \mathrm{Syz}(f_1, \ldots, f_s)$.
- I_s: $s \times s$ identity matrix,
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Theorem

With the notation above we have
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Using the Gröbner basis for module, we can extend above theorem for $\mathrm{Syz}(\mathbf{f}_1, \ldots, \mathbf{f}_s) = \mathrm{Ker}[R^s \xrightarrow{\times {}^t[\mathbf{f}_1 \cdots \mathbf{f}_s]} R^d]$.

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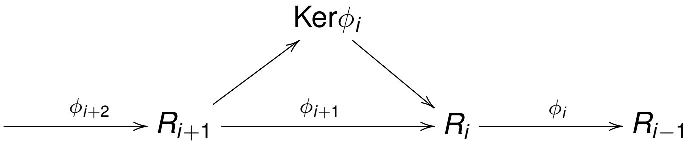
Using above theorem, we can compute the free resolution.



- By above method, we can compute the free resolution of $\mathcal{A}_2$-module $H^*(X; \mathbb{Z}/2)$.
- Moreover, using the division algorithm, we have the minimal resolution.
- I am writing a computer program to compute the E_2-terms of Adams spectral sequence. The program will be available from the web page (http://www.math.kyoto-u.ac.jp/˜tomo_xi).

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Summary

Our method works for any space X, Y

$$E_2^{s,t} \cong \mathrm{Ext}^{s,t}_{\mathcal{A}_2}(H^*(X;\mathbb{F}_2), H^*(Y;\mathbb{F}_2)) \Rightarrow {}_{(2)}\{Y, X\}_*$$

On the other hand,
There are many calculations for the special case $X, Y = S^0$

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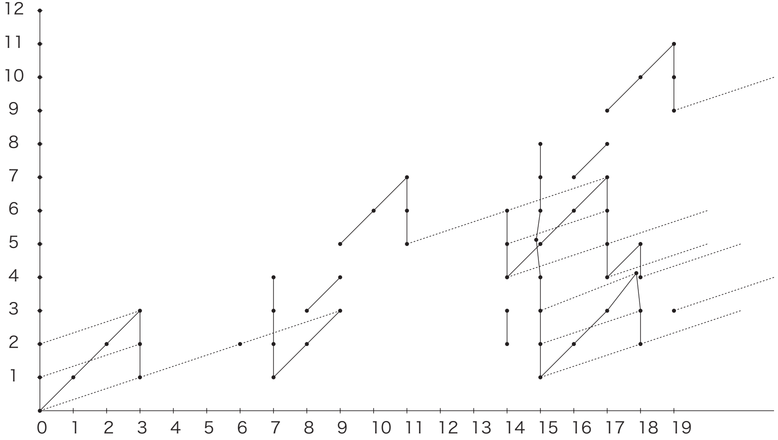
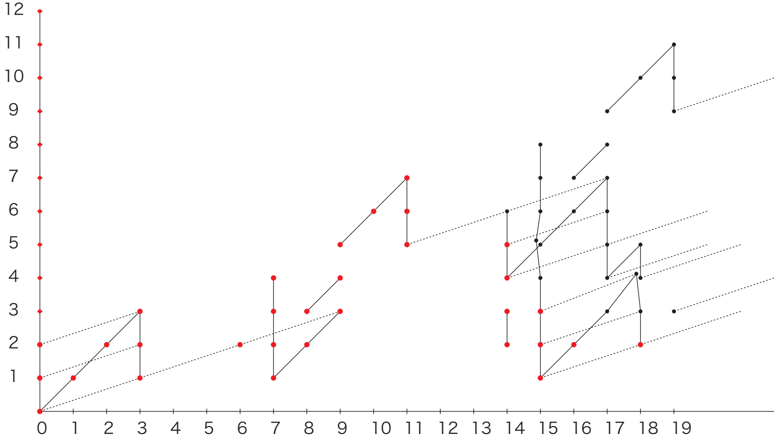
(a part of) Known results



Our results



For Further Reading I

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