Introduction	Non-commutative Gröbner basis	Free resolution	Application
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00	000000	00000	
0000	000	00000	

Application of non-commutative Gröbner basis to calculations of E_2 terms of the Adams spectral sequence

Tomohiro Fukaya

Department of Mathematics Kyoto University

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Free resolution of the Steenrod Algebra

Introduction	Non-commutative Gröbner basis	Free resolution	Application
00	00	000	
00	000000	00000	
0000	000	00000	

Outline

Introduction

Steenrod Algebra The Problem We Study and The Basic Tool Adams spectral sequence

Non-commutative Gröbner basis

Non-commutative ring Review of Non-commutative Gröbner basis Example: Steenrod Algebra

Free resolution

Syzygy module Syzgy for Gröbner basis Syzygy for not Gröbner basis

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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00	000000	00000	
0000	000	00000	

- I am interested in the application of Gröbner basis to the study of algebraic topology.
- The theory of Gröbner bases for polynomial rings was developed by Bruno Buchberger in 1965.
- One can view it as a multivariate, non-linear generalization of:
 - the Euclidean algorithm for computation of greatest common divisors,
 - Gaussian elimination for linear systems, and
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Introduction	Non-commutative Gröbner basis oo ooocooo ooo	Free resolution ooo ooooo ooooo	Application

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Introduction	Non-commutative Gröbner basis oo ooooooo ooo	Free resolution 000 00000 00000	Application

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Introduction	Non-commutative Gröbner basis oo ooooooo ooo	Free resolution 000 00000 00000	Application

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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00	000000	00000	
0000	000	00000	

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- For example, I studied the ring structrure of the cohomology of the Oriented Grassmann manifolds using Gröbner basis. I obtaind the exact values of the cup-length for an infinite family of the Oriented Grassmann manifolds.
- Today, I will talk on the algorithm of the calculation of the E₂ terms of the Adams spectral sequence using non-commutative Gröbner bases.

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
• 0	00	000	
00	000000	00000	
0000	000	00000	

Steenrod Algebra

Outline

Introduction

Steenrod Algebra

The Problem We Study and The Basic Tool Adams spectral sequence

Non-commutative Gröbner basis

Non-commutative ring

Review of Non-commutative Gröbner basis

Example: Steenrod Algebra

Free resolution

Syzygy module Syzgy for Gröbner basis Syzygy for not Gröbner basis

Introduction ○● ○○○○	Non-commutative Gröbner basis oo oooooo ooo	Free resolution 000 00000 00000	Application
Steenrod Algebra			

Steenrod algebra, A_2 , which is generated by operations ($i \ge 0$)

$$\operatorname{Sq}^{i} \colon H^{*}(\ ; \mathbb{F}_{2}) \to H^{*+i}(\ ; \mathbb{F}_{2}),$$

• $Sq^0 =$ the identity homomorphism.

• If
$$x \in H^n(X; \mathbb{F}_2)$$
, then $\operatorname{Sq}^n x = x^2$.

- ► $x, y \in H^*(X; \mathbb{F}_2),$ Sq^k $(x \cup y) = \sum_{i=0}^k Sq^i x \cup Sq^{k-i} y,$ the Cartan formula.
- ▶ The following relations hold among the generators: if 0 < a < 2b

$$\mathrm{Sq}^{a}\mathrm{Sq}^{b} = \sum_{i=0}^{[a/2]} \binom{b-1-i}{a-2i} \mathrm{Sq}^{a+b-i}\mathrm{Sq}^{i}$$

These relations are called the Adem relations.

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
00	00	000	
•0	000000	00000	
0000	000	00000	

The Problem We Study and The Basic Tool

Outline

Introduction

Steenrod Algebra

The Problem We Study and The Basic Tool

Adams spectral sequence

Non-commutative Gröbner basis

Non-commutative ring

Review of Non-commutative Gröbner basis

Example: Steenrod Algebra

Free resolution

Syzygy module Syzgy for Gröbner basis Syzygy for not Gröbner basis

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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00	000000	00000	
0000	000	00000	

The Problem We Study and The Basic Tool

Problem

For topological space *X* and *Y*, compute the groups of morphisms

$$\{Y, X\}_k := \lim_{n \to \infty} [\Sigma^{n+k} Y, \Sigma^n X].$$

If $Y = S^0$, then $\{Y, X\}_k = \pi_k^S(X)$; the stable k-th homotopy group of X.

Introduction	Non-commutative Gröbner basis	Free resolution	Application
00 00 •000	00 000000 000	000 00000 00000	
0000	000	00000	

Adams spectral sequence

Outline

Introduction

Steenrod Algebra The Problem We Study and The Basic Tool

Adams spectral sequence

Non-commutative Gröbner basis

Non-commutative ring Review of Non-commutative Gröbner basis Example: Steenrod Algebra

Free resolution

Syzygy module Syzgy for Gröbner basis Syzygy for not Gröbner basis

Introduction	Non-commutative Gröbner basis	Free resolution	Application
00	00	000	
00	000000	00000	
0●00	000	00000	

Adams spectral sequence

Theorem (Adams spectarl sequence)

There is a spectral sequence, converging to $_{(2)}\{Y,X\}_*,$ with E_2 -terms given by

$$E_2^{s,t} \cong \operatorname{Ext}_{\mathcal{A}_2}^{s,t}(H^*(X; \mathbb{F}_2), H^*(Y; \mathbb{F}_2))$$
(1)

and differentials d_r of bi-degree (r, r - 1).

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Free resolution of the Steenrod Algebra

Kyoto university

Introduction ○○ ○○●○	Non-commutative Gröbner basis oo ooocooo ooo	Free resolution 000 00000 00000	Application
Adams spectral seque	ence		

*E*₂-terms of Adams spectral sequence are given by A_2 -free resolution of A_2 -module $M = H^*(X; \mathbb{F}_2)$

$$\cdots \xrightarrow{d_{i+1}} R_i \xrightarrow{d_i} R_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} R_0 \xrightarrow{d_0} M \to 0$$

where R_i is free A_2 -module ($\oplus A_2$) and A_2 is the Steenrod algebra. For another A_2 -module $N = H^*(Y; \mathbb{F}_2)$, we obtain a cochain complex

$$\operatorname{Hom}(M,N) \xrightarrow{d^0} \operatorname{Hom}(R_0,N) \xrightarrow{d^1} \operatorname{Hom}(R_1,N) \xrightarrow{d^2} \cdots$$

The cohomology of this complex defines $Ext_{A_2}^{*,*}(M, N)$.

Kyoto university

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Introduction ○○ ○○●○	Non-commutative Gröbner basis oo ooocooo ooo	Free resolution 000 00000 00000	Application
Adams spectral seque	ence		

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Introduction ○○ ○○●○	Non-commutative Gröbner basis oo ooocooo ooo	Free resolution 000 00000 00000	Application
Adams spectral seque	ence		

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Introduction ○○ ○○○●	Non-commutative Gröbner basis oo oooooo ooo	Free resolution ooo ooooo ooooo	Application
Adams spectral seque	ence		

The key to calculate such a free resolution is

Non-commutative Gröbner basis and Syzygy module.

Remark

Mr. Euiyong Park (Seoul National University) ,and his co-workers, are using the non-commutative Gröbner basis to study the representation of Lie algebras and their universal enveloping algebras.

Introduction ○○ ○○●	Non-commutative Gröbner basis oo oooooo ooo	Free resolution 000 00000 00000	Application
Adams spectral sequ	ence		

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Introduction ○○ ○○○	Non-commutative Gröbner basis oo oooooo ooo	Free resolution 000 00000 00000	Application
Adams spectral sequer	ICE		

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
00	•0	000	
00	000000	00000	
0000	000	00000	

Non-commutative ring

Outline

Introduction

Steenrod Algebra The Problem We Study and The Basic Tool Adams spectral sequence

Non-commutative Gröbner basis

Non-commutative ring

Review of Non-commutative Gröbner basis Example: Steenrod Algebra

Free resolution

Syzygy module Syzgy for Gröbner basis Syzygy for not Gröbner basis

Introduction 00 0000	Non-commutative Gröbner basis ○ ○○○○○ ○○○	Free resolution 000 00000 00000	Application
Non-commutative ring			

▶ k: field.

► R = k < x₁, x₂,..., x_i, ··· > non-commutative free associative ring

the set of polynomials.

►
$$\mathcal{B} = \{x_{i_1}x_{i_2}\cdots x_{i_k}\} \cup \{1\}$$

► the set of monomials

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Free resolution of the Steenrod Algebra

Introduction	Non-commutative Gröbner basis	Free resolution	Application
oo	○●	000	
oo	○○○○○○	00000	
oooo	○○○	00000	
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
oo	○●	000	
oo	○○○○○○	00000	
oooo	○○○	00000	
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
oo	○●	000	
oo	○○○○○○	00000	
oooo	○○○	00000	
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
oo	○●	000	
oo	○○○○○○	00000	
oooo	○○○	00000	
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
00	○○	000	
00	●○○○○○	00000	
0000	○○○	00000	

Outline

Introduction

Steenrod Algebra The Problem We Study and The Basic Tool Adams spectral sequence

Non-commutative Gröbner basis

Non-commutative ring

Review of Non-commutative Gröbner basis

Example: Steenrod Algebra

ree resolution Syzygy module Syzgy for Gröbner basis Syzygy for not Gröbner basi

Introduction	Non-commutative Gröbner basis	Free resolution	Application
00 00 0000	00 00000 000	000 00000 00000	

Definition (Monomial order)

Let \leq be a well-ordering on \mathcal{B} . \leq is said to be a monomial ordering on R if the following two conditions are satisfied.

- ▶ if $u, v, w, s \in B$ with $w \le u$, then $vws \le vus$.
- For $u, w \in \mathcal{B}$, if u = vws for some $v, s \in \mathcal{B}$ with $v \neq 1$ or $s \neq 1$, then $w \leq u$.

Hence $1 \leq u$ for all $u \in \mathcal{B}$.

Introduction	Non-commutative Gröbner basis	Free resolution	Application
00 00 0000	00 00000 000	000 00000 00000	

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
00 00 0000	00 00000 000	000 00000 00000	

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
00	○○	000	
000	○○●○○○	00000	
0000	○○○	00000	

From here, We fix a monomial ordering.

For
$$f \in R$$
, $f \neq 0$, we may write

 $f = c_1 u_1 + \cdots + c_m u_m$

where $c_i \in k \setminus \{0\}$, $u_i \in \mathcal{B}$, $u_1 > u_2 > \cdots > u_m$. $\blacktriangleright \operatorname{Im}(f) = u_1$, the leading monomial of *f*.

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Free resolution of the Steenrod Algebra

Kyoto university

Introduction	Non-commutative Gröbner basis	Free resolution	Application
00 00 0000	00 00●000 000	000 00000 00000	

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
00	00	000	
00	000000	00000	
0000	000	00000	

Theorem (Division on f by \mathcal{H})

INPUT
$$f, \mathcal{H} = \{h_1, \dots, h_m \dots\} \subset R$$

OUTPUT $f = \sum_{\alpha=0}^{d} \mu_{\alpha} \mathbf{v}_{\alpha} h_{i_{\alpha}} \mathbf{s}_{\alpha} + r$

 $\blacktriangleright \ \mu_{\alpha} \in k \setminus \{0\}, \ v_{\alpha}, s_{\alpha} \in \mathcal{B}. \ \mathsf{Im}(\mu_{\alpha} v_{\alpha} h_{i_{\alpha}} s_{\alpha}) \leq \mathsf{Im}(f).$

▶ r = 0 or $r = \sum_{i} c_{i}t_{j}$ where $c_{i} \in k \setminus \{0\}$, $t_{i} \in B$,

• $\operatorname{Im}(r) \leq \operatorname{Im}(f)$, none of the t_i is divisible by any $\operatorname{Im}(h_i)$.

 \triangleright r is depend on the ordering on \mathcal{H} .

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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00	000000	00000	
0000	000	00000	

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Theorem (Division on f by \mathcal{H})

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Introduction oo oo ooooo	Non-commutative Gröbner basis ○○ ○○○○	Free resolution ooo ooooo ooooo	Application

- ► The *r* appeared in above OUTPUT is called a remainder of *f* on the division by \mathcal{G} , dented $\overline{f}^{\mathcal{H}}$.
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Definition

Let *I* be two-side ideal of *R*. $\mathcal{G} = \{g_1, g_2, ...\} \subset I$ is Gröbner basis, if and only if

 $(\operatorname{\mathsf{Im}}(g)|g\in\mathcal{G})=(\operatorname{\mathsf{Im}}(g)|g\in I)$

▶ If $\mathcal{G} = \{g_1, g_2, ...\} \subset R$ is Gröbner basis, then $f^{\mathcal{G}}$ is independent of the order on \mathcal{G} and $f^{\mathcal{G}}$ is unique $f \in I = \{g_1, g_2, ...\} \oplus f^{\mathcal{G}} = 0.$

Introduction oo ooo oooo	Non-commutative Gröbner basis ○○ ○○○○	Free resolution ooo ooooo ooooo	Application

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• $f \in I = (g_1, g_2, \ldots, g_i, \ldots) \Leftrightarrow \overline{f}^{\mathcal{G}} = 0.$

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Introduction oo ooo oooo	Non-commutative Gröbner basis ○○ ○○○○	Free resolution ooo ooooo ooooo	Application

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Introduction oo ooo oooo	Non-commutative Gröbner basis ○○ ○○○○	Free resolution ooo ooooo ooooo	Application

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Introduction oo ooo oooo	Non-commutative Gröbner basis ○○ ○○○○	Free resolution ooo ooooo ooooo	Application

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Outline

Introduction

Steenrod Algebra The Problem We Study and The Basic Tool Adams spectral sequence

Non-commutative Gröbner basis

Non-commutative ring Review of Non-commutative Gröbner basis Example: Steenrod Algebra

Free resolution Syzygy module Syzgy for Gröbner basis Syzygy for not Gröbner basis

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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We can consider the Steenrod algebra as a quotient ring.

- $\blacktriangleright A = \mathbb{Z}_2 \langle Sq^1, Sq^2, \dots \rangle.$
- ► $\mathcal{G} = \{ Sq^aSq^b \sum_{i=0}^{[a/2]} {b-1-i \choose a-2i} Sq^{a+b-i}Sq^i | 0 < a < 2b \}.$ Set of Adem relations.
- ▶ I = (G): A two-side Ideal generated by G.
- ► $\mathcal{A}_2 = \mathcal{A}/I.$

Introduction oo oo oooo	Non-commutative Gröbner basis ○○○○○○○ ○●○	Free resolution ooo ooooo ooooo	Application

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Introduction oo ooo oooo	Non-commutative Gröbner basis ○○ ○○○○○○ ○○●	Free resolution ooo ooooo ooooo	Application
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Definition (A monomial Ordering of Steenrod Algebra) Let $u = Sq^{a_1} \cdots Sq^{a_k}$, $v = Sq^{b_1} \cdots Sq^{b_l} \in \mathcal{B}$. Then, $u \ge v$ if and only if

▶ k > l or,

k = l and the right-most nonzero entry of (a₁ − b₁,..., a_k − b_k) is positive.

Im(Sq^aSq^b - $\sum_{i=0}^{[a/2]} {b-1-i \choose a-2i}$ Sq^{a+b-i}Sqⁱ) = Sq^aSq^b. Proposition *G* is a Gröbner basis of *L*.



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Definition (A monomial Ordering of Steenrod Algebra) Let $u = \operatorname{Sg}^{a_1} \cdots \operatorname{Sg}^{a_k}$, $v = \operatorname{Sg}^{b_1} \cdots \operatorname{Sg}^{b_l} \in \mathcal{B}$. Then, u > v if and only if k > l or.k = l and the right-most nonzero entry of $(a_1 - b_1, \ldots, a_k - b_k)$ is positive. $\operatorname{Im}(\operatorname{Sq}^{a}\operatorname{Sq}^{b} - \sum_{i=0}^{[a/2]} {\binom{b-1-i}{2}} \operatorname{Sq}^{a+b-i}\operatorname{Sq}^{i} = \operatorname{Sq}^{a}\operatorname{Sq}^{b}.$ Proposition

 \mathcal{G} is a Gröbner basis of I.

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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0000	000	00000	

Syzygy module

Outline

Introduction Steenrod Algebra The Problem We Study and The Basic Tool Adams spectral sequence Non-commutative Gröbner basis Non-commutative ring Review of Non-commutative Gröbner basis Example: Steenrod Algebra

Free resolution

Syzygy module

Syzgy for Gröbner basis Syzygy for not Gröbner basis

Introduction 00 00 0000	Non-commutative Gröbner basis oo oooooo ooo	Free resolution ○●○ ○○○○○	Application

Syzygy module

Again, let $R = k < x_1, x_2, ..., x_i, ... >$.

We consider the algorithm of computing a free-resolution of left-*R*-module M,

$$\cdots \xrightarrow{d_{i+1}} R_i \xrightarrow{d_i} R_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} R_0 \xrightarrow{d_0} M \to 0$$

where R_i 's are free R-modules.

The key to compute the free resolutions is the

syzygy modules.

Free resolution of the Steenrod Algebra

Kyoto university

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Syzygy module			

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Free resolution of the Steenrod Algebra

Kyoto university

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
oo	oo	○●○	
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Introduction 00 0000	Non-commutative Gröbner basis oo ooooooo ooo	Free resolution oo● ooooo	Application

Definition

Syzygy module

The kernel of the map $\phi: \mathbb{R}^s \to M$ given by

$$(h_1,\ldots,h_s)\mapsto \sum_{i=1}^s h_i\mathbf{f}_i, \quad \mathbf{f}_i\in M$$

is called the syzygy module of the matrix ${}^{t}[\mathbf{f}_{1}\cdots\mathbf{f}_{s}]$. It is denoted Syz $(\mathbf{f}_{1},\ldots,\mathbf{f}_{s})$.

Free resolution of the Steenrod Algebra

Kyoto university

Introduction	Non-commutative Gröbner basis	Free resolution	Application
00	00	000	
00	000000	00000	
0000	000	00000	

Outline

Introduction Steenrod Algebra The Problem We Study and The Basic Tool Adams spectral sequence Non-commutative Gröbner basis Non-commutative ring Review of Non-commutative Gröbner basis Example: Steenrod Algebra

Free resolution

Syzygy module Syzgy for Gröbner basis Syzygy for not Gröbner basis

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Introduction 00 0000	Non-commutative Gröbner basis oo ooooooo ooo	Free resolution ○○ ○●○○○ ○○○○○	Application
Syzgy for Gröbner bas	is		

We consider only the case M = R.

Step 1: we consider

$$\operatorname{Syz}(g_1,\ldots,g_t) = \operatorname{Ker}[R^t \xrightarrow{\times^t[g_1\cdots g_t]} R]$$

where $\{g_1, \ldots, g_t\}$ is Gröbner basis.

Free resolution of the Steenrod Algebra

Introduction 00 0000	Non-commutative Gröbner basis oo ooooooo ooo	Free resolution ○○ ○●○○○ ○○○○○	Application
Syzgy for Gröbner bas	is		

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Free resolution of the Steenrod Algebra

Introduction	Non-commutative Gröbner basis	Free resolution	Application
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We let $Im(g_i) = X_i$ and $X_{ij} = Icm(X_i, X_j)$. Then the S-polynomial of g_i and g_j is, given by

$$egin{aligned} \mathcal{S}(g_i,g_j) &= rac{X_{ij}}{X_i}g_i - rac{X_{ij}}{X_j}g_j, \quad \mathsf{lp}(\mathcal{S}(g_i,g_j)) < \mathsf{lp}(rac{X_{ij}}{X_j}g_i), \; \mathsf{lp}(rac{X_{ij}}{X_j}g_j), \end{aligned}$$

$$S(g_i,g_j)=\sum_{
u=t}h_{ij
u}g_
u,\quad {\sf lp}(S(g_i,g_j))\geq ({\sf lp}(h_{ij
u}g_
u)),$$

Let

$$\mathbf{s}_{ij} := rac{X_{ij}}{X_i} \mathbf{e}_i - rac{X_{ij}}{X_j} \mathbf{e}_j - (h_{ij1}, \ldots, h_{ijt}) \in \mathbf{R}^t.$$

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Free resolution of the Steenrod Algebra

Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Free resolution of the Steenrod Algebra

Introduction	Non-commutative Gröbner basis	Free resolution	Application
00	oo	○○○	
000	oooooo	○○●○○	
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Free resolution of the Steenrod Algebra

Introduction	Non-commutative Gröbner basis	Free resolution	Application
00	oo	○○○	
000	oooooo	○○●○○	
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Free resolution of the Steenrod Algebra

Introduction	Non-commutative Gröbner basis	Free resolution	Application
00	oo	○○○	
000	oooooo	○○●○○	
0000	ooo	○○○○○	

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Free resolution of the Steenrod Algebra

Introduction	Non-commutative Gröbner basis	Free resolution	Application
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We note that $\mathbf{s}_{ij} \in \operatorname{Syz}(g_1, \ldots, g_t)$, since

$$\mathbf{s}_{ij} \begin{bmatrix} g_1 \\ \vdots \\ g_t \end{bmatrix} = \left(\frac{X_{ij}}{X_i} \mathbf{e}_i - \frac{X_{ij}}{X_j} \mathbf{e}_j - (h_{ij1}, \dots, h_{ijt}) \right) \begin{bmatrix} g_1 \\ \vdots \\ g_s \end{bmatrix}$$
$$= S(g_i, g_j) - \sum_{\nu=t}^t h_{ij\nu} g_{\nu} = 0.$$

Theorem

With notation above, the collection $\{\mathbf{s}_{ij}|1 \le i \le j \le t\}$ is generating set for $Syz(g_1, \ldots, g_t)$.

Free resolution of the Steenrod Algebra

Kyoto university

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Theorem

There exists an algorithm to compute a generating set of

 $Syz(g_1, \ldots, g_t)$

using S-polynomials and the division algorithm for $\{g_1, \ldots, g_t\}$.

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
00	00	000	
00	000000	00000	
0000	000	00000	

Outline

Introduction Steenrod Algebra The Problem We Study and The Basic Tool Adams spectral sequence Non-commutative Gröbner basis Non-commutative ring Review of Non-commutative Gröbner basis Example: Steenrod Algebra

Free resolution

Syzygy module Syzgy for Gröbner basis Syzygy for not Gröbner basis

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
00	00	000	
00	000000	00000	
0000	000	0000	

Step 2: we consider

$$\operatorname{Syz}(f_1,\ldots,f_s) = \operatorname{Ker}[R^s \xrightarrow{\times^t[f_1\cdots f_s]} R]$$

where $\{f_1, \ldots, f_s\}$ is not Gröbner basis.

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Free resolution of the Steenrod Algebra

Introduction	Non-commutative Gröbner basis	Free resolution	Application
00	00	000	
00	000000	00000	
0000	000	00000	

• $\{f_1, \ldots, f_s\} \subset R$: not Gröbner basis.

- $\{g_1, \ldots, g_t\}$: Gröbner basis of $\{f_1, \ldots, f_s\}$.
- $\triangleright F = {}^t[f_1 \ldots f_s], \ G = {}^t[g_1 \ldots g_t].$

• $\exists S: s \times t$ matrix, $\exists T: t \times s$ matrix s.t. F = SG and G = TF.

- compute a generating set $\{\mathbf{s}_1, \ldots, \mathbf{s}_r\}$ for $Syz(g_1, \ldots, g_t)$
- $\bullet \ 0 = \mathbf{s}_i G = \mathbf{s}_i (TF) = (\mathbf{s}_i T)F,$

• $\langle \mathbf{s}_i T | i = 1, \dots, r \rangle \subset \operatorname{Syz}(f_1, \dots, f_s).$

► I_s : $s \times s$ identity matrix, $(I_s - ST)F = F - STF = F - SG = F - F = 0$

• the rows $\mathbf{r}_1, \ldots, \mathbf{r}_s$ of $I_s - TS$ are also in Syz (f_1, \ldots, f_s) .

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Introduction 00 00 0000	Non-commutative Gröbner basis oo oooooo ooo	Free resolution	Application

- $\{f_1, \ldots, f_s\} \subset R$: not Gröbner basis.
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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oooo	ooo	○○●○○	

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
oo	oo	○○○	
oo	oooooo	○○○○○	
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- \triangleright 0 = s_iG = s_i(TF) = (s_iT)F.
- \triangleright I_{a} : $s \times s$ identity matrix.

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Introduction 00 0000	Non-commutative Gröbner basis oo oooooo ooo	Free resolution	Application

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Introduction 00 0000	Non-commutative Gröbner basis oo oooooo ooo	Free resolution	Application

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Introduction oo oo oooo	Non-commutative Gröbner basis oo oooooo ooo	Free resolution	Application
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 - ► $\exists S: s \times t \text{ matrix}, \exists T: t \times s \text{ matrix}$ s.t. F = SG and G = TF.
- compute a generating set $\{\mathbf{s}_1, \ldots, \mathbf{s}_r\}$ for $Syz(g_1, \ldots, g_t)$
- $\bullet \ 0 = \mathbf{s}_i G = \mathbf{s}_i (TF) = (\mathbf{s}_i T)F,$
 - $\langle \mathbf{s}_i T | i = 1, \dots, r \rangle \subset \operatorname{Syz}(f_1, \dots, f_s).$

► I_s : $s \times s$ identity matrix, $(I_s - ST)F = F - STF = F - SG = F - F = 0$

the rows $\mathbf{r}_1, \ldots, \mathbf{r}_s$ of $I_s - TS$ are also in $Syz(f_1, \ldots, f_s)$.

Introduction oo oo oooo	Non-commutative Gröbner basis oo ooooooo ooo	Free resolution	Application

- $\{f_1, \ldots, f_s\} \subset R$: not Gröbner basis.
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Using the Gröbner basis for module, we can extend above theorem for $Syz(f_1, ..., f_s) = Ker[R^s \xrightarrow{\times^t [f_1 \cdots f_s]} R^d].$

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Free resolution of the Steenrod Algebra

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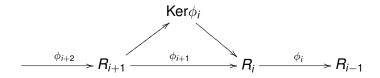
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Kyoto university

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Introduction 00 0000	Non-commutative Gröbner basis oo oooooo ooo	Free resolution ○○○ ○○○○○ ○○○○●	Application

Using above theorem, we can compute the free resolution.



Free resolution of the Steenrod Algebra

Introduction	Non-commutative Gröbner basis	Free resolution	Application
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- ► By above method, we can compute the free resolution of A₂-module H*(X; Z/2).
- Moreover, using the division algorithm, we have the minimal resolution.
- I am writing a computer program to compute the E₂-terms of Adams spectral sequence. The program will be available from the web page (http://www.math.kyoto-u.ac.jp/~tomo_xi).

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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Summary

Our method works for any space X, Y

$$E_2^{s,t} \cong \mathsf{Ext}_{\mathcal{A}_2}^{s,t}(H^*(X;\mathbb{F}_2),H^*(Y;\mathbb{F}_2)) \Rightarrow_{(2)} \{Y,X\}_*$$

On the other hand, There are many calculations for the special case $X, Y = S^0$

$$\mathsf{E}^{s,t}_2\cong\mathsf{Ext}^{s,t}_{\mathcal{A}_2}(\mathbb{F}_2,\mathbb{F}_2)\Rightarrow\pi^S_*$$

the stable homotopy groups of the sphere

Introduction	Non-commutative Gröbner basis	Free resolution	Application
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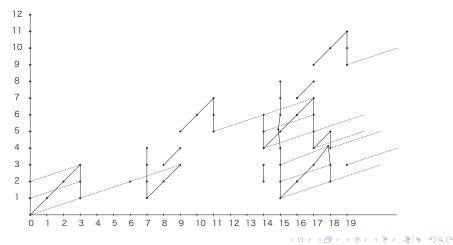
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Introduction	Non-commutative Gröbner basis	Free resolution	Application
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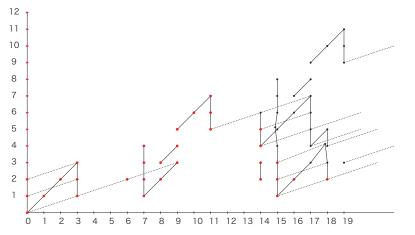
(a part of) Known results



Free resolution of the Steenrod Algebra

Introduction	Non-commutative Gröbner basis	Free resolution	Application
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Our results



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For Further Reading

For Further Reading I



Nilliam W. Adams and Philippe Loustaunau. An introduction to Gröbner bases, volume 3 of Graduate Studies in Mathematics.

American Mathematical Society, Providence, RI, 1994.



🕨 Huishi Li

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