The Coarse Baum-Connes Conjecuture for Relatively Hyperbolic Groups

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The category of Coarse Spaces

Category of Coarse spaces $\mathcal{C}$ consists by

- Objects: Coarse equivalence classes of proper metric spaces.
- Morphisms: $\mathrm{Hom}(X, Y) = \{f \colon X \to Y \text{coarse map}\}/\text{close}$.

There are two covariant functor from this category to the category of abelian groups $\mathbf{Ab}$.

- The coarse K-homology.
- The K-theory of the Roe-algebras.

What is Coarse K-homology?

- The coarse K-homology $KX_*(-)$ is a coarse version of the K-homology.
- $KX_* : \mathcal{C} \rightsquigarrow \mathbf{Ab}$ covariant functor.
- $KX_*(-)$ satisfies Mayer-Vietoris axiom.

Let X be a metric space.
$KX_*(X)$ represent a TOPOLOGICAL property of X.

Roe algebra and its K-theory

- For space X, we can associate X with a C^*-algebra $C^*(X)$, called the Roe algebra.
- $K_*(C^*(-))$: The K-theory of the Roe algebra
- $K_*(C^*(-)) : \mathcal{C} \rightsquigarrow \mathbf{Ab}$ covariant functor.
- $K_*(C^*(-))$ satisfies Mayer-Vietoris axiom.

Let X be a metric space.
$K_*(C^*(X))$ represent an Analytic property of X.

Coarse Baum-Connes conjecture

- There is a natural transformation μ from $KX_*(-)$ to $K_*(C^*(-))$.
- μ is called the coarse assembly map.

Conjecture

If X is a "good" metric space, then the coarse assembly map

$$\mu\colon KX_*(X) \to K_*(C^*(X))$$

is an isomorphism.

- Higson and Roe proved the conjecture for δ-hyperbolic spaces.

General Method for Coarse Baum-Connes conjecture

Let X be a metric space.

Guoliang Yu proved several sufficient condition of the coarse BC conjecture for X.

- The asymptotic dimension of X is finite.
    - Example: $\mathrm{asdim}(\mathbb{Z}^n) = \mathrm{asdim}(\mathbb{R}^n) = n$.
- X has the property A.
- X can be coarsely embedded into the Hilbert space.

“Definition” of Relatively Hyperbolic Groups

- Let G be a finitely generated group.
- Let $\mathbb{P} = \{P_1, \dots, P_k\}$ be a finite family of infinite subgroups.

$(G, \mathbb{P})$ is called a relatively hyperbolic group if
G is hyperbolic relative to $\mathbb{P}$,
or,
hyperbolic modulo $\mathbb{P}$,

- $P \in \mathbb{P}$ is called a parabolic subgroup.

Examples of relatively hyperbolic group

- Let A, B be finitely generated groups. Then $C = A * B$ is hyperbolic relative to $\{A, B\}$.
- Let M be a complete, finite volume Riemannian manifold with (pinched) negative sectional curvature

$$-b^2 < K(M) < -a^2 < 0.$$

Then $\pi_1(M)$ is hyperbolic relative to cusp subgroups.

  - A non-uniform lattice in $\mathbb{R}$-rank one simple Lie group.
  - Let K be a hyperbolic knot (i.e. $S^3 \setminus K$ admits hyperbolic metric). Then $\pi_1(S^3 \setminus K)$ is hyperbolic relative to $\mathbb{Z}^2$.

Known results

Theorem

Let $(G, \mathbb{P})$ be a relatively hyperbolic group.

- *If* $\operatorname{asdim} P < \infty$ *for all* $P \in \mathbb{P}$, *then* $\operatorname{asdim} G < \infty$ *(Osin).*
- *If* P *is exact for all* $P \in \mathbb{P}$, *then* G *is also exact (Ozawa).*
- *If* P *is coarsely embeddable in* l_2 *for all* $P \in \mathbb{P}$, *then* G *is also coarsely embeddable in* l_2 *(Dadarlat-Guentner).*

Due to Yu's work, those results imply the coarse Baum-Connes conjecture for such groups.

Main theorem

Theorem (Oguni-F)

Let $(G, \mathbb{P})$ be a relatively hyperbolic group. If all $P \in \mathbb{P}$ satisfies the following two conditions:

- *P admits a finite P-simplicial complex which is a universal space for proper actions.*
- *The coarse Baum-Connes conjecture for P holds.*

Then the coarse Baum-Connes conjecture for G also holds.

The combinatorial horoball

Definition

Let (P,d) be a metric space. The combinatorial horoball based on P, denoted by $\mathcal{H}(P)$, is the graph defined as follows:

- $\mathcal{H}(P)^{(0)} = P \times (\mathbb{N} \cup \{0\})$.
- $\mathcal{H}(P)^{(1)}$ contains the following two type of edges:
  1. For $l \in \mathbb{N} \cup \{0\}$ and $p, q \in P$, if $0 < d(p,q) \leq 2^l$ then there is a horizontal edge connecting (p,l) and (q,l).
  2. For $l \in \mathbb{N} \cup \{0\}$ and $p \in P$, there is a vertical edge connecting (p,l) and $(p,l+1)$.

Lemma

$\mathcal{H}(P)$ is δ-hyperbolic for some $\delta > 0$.

Notations

- Let G be a finitely generated group.
- Let $\mathbb{P} = \{P_1, \dots, P_k\}$ be a finite family of infinite subgroups.
- Choose a sequence $g_1, g_2, \dots$ in G such that for any $r = 1, \dots, k$, the map $\mathbb{N} \to G/P_r : a \mapsto g_{ak+r}P_r$ is bijective.
- For $i = ak + r \in \mathbb{N}$, let $P_{(i)}$ denote a subgroup P_r. Thus the set of all cosets $\bigsqcup_{r=1}^{k} G/P_r$ is indexed by the map $\mathbb{N} \ni i \mapsto g_i P_{(i)}$.

Notations

- Let $\mathcal{S}$ be a finite generating set of G.
- Let $d_{\mathcal{S}}$ be the word metric of G associated to $\mathcal{S}$.
- Each coset $g_i P_{(i)}$ has a proper metric d_i which is the restriction of $d_{\mathcal{S}}$.
- $\mathcal{H}(g_i P_{(i)})$ is the combinatorial horoball based on $(g_i P_{(i)}, d_i)$.
- The zero-th floor of $\mathcal{H}(g_i P_{(i)})$ can be embedded in $\Gamma = \mathrm{Cayley}(G, \mathcal{S})$.

The augmented space

Definition

The augmented space $X(G, \mathbb{P}, \mathcal{S})$ is obtained by pasting $\mathcal{H}(g_i P_{(i)})$ to Γ for all $i \in \mathbb{N}$.

$$X(G, \mathbb{P}, \mathcal{S}) = \Gamma \cup \bigcup_{i \in \mathbb{N}} \mathcal{H}(g_i P_{(i)}).$$

Definition (Groves-Manning)

G is hyperbolic relative to $\mathbb{P}$ if the augmented space $X(G, \mathbb{P}, \mathcal{S})$ is δ-hyperbolic for some $\delta \geq 0$.

Proof of the Main theorem

Theorem (Oguni-F)

Let $(G, \mathbb{P})$ be a relatively hyperbolic group. If all $P \in \mathbb{P}$ satisfies the following two conditions:

- *P admits a finite P-simplicial complex which is a universal space for proper actions.*
- *The coarse Baum-Connes conjecture for P holds.*

Then the coarse Baum-Connes conjecture for G also holds.

The keys to the proof is the following:

- Coarse Mayer-Vietoris exact sequences.
- Approximate discrete spaces by continuous spaces.

ω-excision

Definition

- Let M be a metric space.
- $M = A \cup B$.

$M = A \cup B$ is an ω-excisive decomposition, if for each $R > 0$ there exists some $S > 0$ such that

$$\mathrm{Pen}(A; R) \cap \mathrm{Pen}(B; R) \subset \mathrm{Pen}(A \cap B; S).$$

Here $\mathrm{Pen}(A; R) = \{p \in M : d(p, A) \leq R\}$.

Coarse Mayer-Vietoris sequences

Theorem (Higson-Roe-Yu)

Suppose that $M = A \cup B$ is an ω-excisive decomposition. Then the following diagram is commutative and horizontal sequences are exact:

$$\begin{array}{ccccc} \longrightarrow & KX_p(A \cap B) & \longrightarrow & KX_p(A) \oplus KX_p(B) & \longrightarrow \\ & \downarrow & & \downarrow & \\ \longrightarrow & K_p(C^*(A \cap B)) & \longrightarrow & K_p(C^*(A)) \oplus K_p(C^*(B)) & \longrightarrow \end{array}$$

$$\begin{array}{ccccc} KX_p(M) & \longrightarrow & KX_{p-1}(A \cap B) & \longrightarrow \\ \downarrow & & \downarrow & \\ K_p(C^*(M)) & \longrightarrow & K_{p-1}(C^*(A \cap B)) & \longrightarrow \end{array}$$

Here vertical arrows are coarse assembly maps.

Sketch of the Proof of the Main Theorem

- Let $(G, \mathbb{P})$ be a relatively hyperbolic group.
- $X_n := \Gamma \cup \bigcup_{i \geq n} \mathcal{H}(g_i P_{(i)})$.
- $X_\infty := \bigcap_{n \geq 1} X_n = \Gamma$.

- Since $X_1 = X(G, \mathbb{P}, \mathcal{S})$ is δ-hyperbolic, by the result of Higson-Roe ('93), the coarse assembly map

$$\mu_1 \colon KX_*(X_1) \to K_*(C^*(X_1))$$

is an isomorphism.

- Since $X_n = X_{n+1} \cup \mathcal{H}(g_n P_{(n)})$ by the induction and the coarse Mayer-Vietoris sequences,

$$\mu_n \colon KX_*(X_n) \to K_*(C^*(X_n))$$

is an isomorphism for all $n \geq 1$.

How to study $\mu_\infty : KX_*(X_\infty) \to K_*(C^*(X_\infty))$?

- Can we expect so-called Milnor sequence?

$$0 \to \varprojlim{}^1 KX_{p+1}(X_n) \to KX_p(X_\infty) \to \varprojlim KX_p(X_n) \to 0. \quad (1)$$

- No! In general, (1) is not exact!
- A counter example is $Y_n = \mathbb{R} \setminus [-n, n]$.
  - $Y_\infty := \cap_n Y_n = \emptyset$.
  - Y_n and $\mathbb{R}$ are coarsely equivalent for all $n \geq 0$.
  - $KX_p(Y_n) \cong KX_p(\mathbb{R}) = \mathbb{Z}$ if p is even.

Universal space for proper actions

- Let $\underline{E}G$ be a finite G-simplicial complex which is a universal space for proper actions.
- $\underline{E}G$ admits a proper metric such that Γ and $\underline{E}G$ are coarsely equivalent.
- Since $\underline{E}G$ is uniformly contractible, of bounded geometry, $KX_*(\underline{E}G) \cong K_*(\underline{E}G)$ (Higson-Roe).
- We have $KX_*(\Gamma) \cong K_*(\underline{E}G)$.

Contractible model

- $EX_n := \underline{E}G \cup \bigcup_{i \geq n} (g_i \underline{E}P_{(i)} \times [0, \infty))$
- $EX_\infty := \bigcap_{n \geq 1} EX_n = \underline{E}G$.

Proposition (Oguni-F)

For all $n \geq 0$,

$$KX_*(X_n) \cong K_*(EX_n).$$

Remark

EX_n admits a proper metric such that EX_n is coarsely equivalent to X_n. However, it is of unbounded geometry, so we cannot deduce the above proposition from a result of Higson-Roe.

Milnor exact sequence for K-homology

For the K-homology of a decreasing sequence of locally compact Hausdorff spaces EX_n, the following sequence is exact!

$$0 \to \varprojlim{}^1 K_{p+1}(EX_n) \to K_p(EX_\infty) \to \varprojlim K_p(EX_n) \to 0.$$

Milnor exact sequence for K-theory of C^*-algebra

Theorem (Phillips ('89))

- *Let $\{A_n\}$ be a projective system of C^*-algebras.*
- *We suppose that $A_\infty := \varprojlim A_n$ is a C^*-algebra.*

Then the following sequence is exact.

$$0 \to \varprojlim{}^1 K_{p+1}(A_n) \to K_p(A_\infty) \to \varprojlim K_p(A_n) \to 0.$$

We apply Phillips's theorem for $C^*(X_\infty) = \cap_{n\geq 1} C^*(X_n)$.

Final step

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim{}^1 K_{p+1}(EX_n) & \longrightarrow & K_p(EX_\infty) & \longrightarrow & \varprojlim K_p(EX_n) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varprojlim{}^1 K_{p+1}(C^*(X_n)) & \longrightarrow & K_p(C^*(X_\infty)) & \longrightarrow & \varprojlim K_p(C^*(X_n)) & \longrightarrow & 0.
\end{array}$$

By the five lemma, the vertical map of the center is an isomorphism. This implies that the coarse assembly map

$$\mu\colon KX_*(\Gamma) \to K_*(C^*(\Gamma))$$

is an isomorphism.