Application of Groebner basis to computing some homotopy invariants

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Outline

1. Brief Introduction to Gröbner basis
   - What can Gröbner basis do?
   - Monomial ordering and Division algorithm
   - Gröbner basis
2. Application to cup-length
   - Cup-length
   - Main Theorem
   - Sketch of Proof
   - LS-category
   - Immersion problem
3. Non commutative Gröbner basis
   - Steenrod algebra
   - Free resolution

What can Gröbner basis do?

Let k be a field, $R = k[x_1, \ldots, x_n]$ be a polynomial ring.
Let $f_1, \ldots, f_m \in R$ be polynomials and $I = \langle f_1, \ldots, f_m \rangle$ be an ideal of R.

Problem

- *Ideal Membership Problem: For given $f \in R$, determine if $f \in I$.*

Gröbner basis can solve these problems systematically.

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Monomial ordering
Definition

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We call an element $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ of R as a power product of R.

Definition (Monomial ordering)

Let $\geq$ be total ordering on the set of power products of R. $\geq$ is called Monomial ordering if and only if

- For any power products X and Y, if $X > Y$ then $XZ > YZ$ for any power product Z.
- If Y divides X, then $X \geq Y$.

If $\geq$ is a monomial ordering, then $\geq$ is well-ordering, that is, any subset of the set of power products has the least element.

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Monomial ordering

Example

Example (Lexicographic ordering)

Let $X = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $Y = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ be power products.

$$X > Y \overset{\text{def}}{\Leftrightarrow} \quad \text{the left-most non zero entry of } (a_1 - b_1, \dots, a_n - b_n) \text{ is positive.}$$

Then Lexicographic ordering is a monomial ordering.

Monomial ordering

Leading elements

Let $f = a_1X_1 + \cdots + a_nX_n$ for $a_1, \ldots, a_n \in k \setminus \{0\}$ and $X_1, \ldots, X_n$ are power products satisfying $X_1 > \cdots > X_n$

Definition

- $\mathrm{lp}(f) = X_1$ the leading power product of f.
- $\mathrm{lc}(f) = a_1$ the leading coefficient of f.
- $\mathrm{lt}(f) = a_1X_1$ the leading term of f.

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Division algorithm

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Theorem

INPUT

$f \in R$ and $F = (f_1, \ldots, f_s)$: ordered s-tuple of polynomials in R.

OUTPUT

$$f = a_1 f_1 + \cdots + a_s f_s + r.$$

where $a_1, \ldots, a_s \in k \setminus \{0\}$, and either $r = 0$ or $r = \sum c_i Y_i$, $c_i \in k \setminus \{0\}$, Y_i is a power product, satisfying none of the Y_i is divisible by any of $\mathrm{lp}(f_1), \ldots, \mathrm{lp}(f_s)$.

We denote above r as $r := \overline{f}^F$. Unfortunately, $\overline{f}^F$ is depend on the ordering on F and $f \in I = \langle F \rangle$ does not imply $\overline{f}^F = 0$.

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Gröbner basis

Definition

We can overcome this difficulty by choosing a Gröbner basis.

Definition

Let I be an ideal of R. A finite subset $G = \{g_1, \dots, g_s\} \subset I$ is a Gröbner basis of I if

$$\langle \{\mathrm{lt}(f) | f \in I\} \rangle = \langle \mathrm{lt}(g_1), \dots, \mathrm{lt}(g_s) \rangle.$$

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Gröbner basis

Basics properties

Proposition

Let I be an ideal of R and
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Then, for $f \in R$

- $\overline{f}^G$ is independent of an ordering on the G and $\overline{f}^G$ is unique.
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How to obtain Gröbner basis?

Let $f, g \in R$, $\mathrm{lp}(f) = x_1^{a_1} \cdots x_n^{a_n}$ and $\mathrm{lp}(g) = x_1^{b_1} \cdots x_n^{b_n}$.

Then the least common multiple of f and g is

$$\mathrm{lcm}(f, g) = x_1^{\max(a_1, b_1)} \cdots x_n^{\max(a_n, b_n)}.$$

The S-polynomial of f and g is

$$S(f, g) = \frac{\mathrm{lcm}(\mathrm{lp}(f), \mathrm{lp}(g))}{\mathrm{lt}(f)} f - \frac{\mathrm{lcm}(\mathrm{lp}(f), \mathrm{lp}(g))}{\mathrm{lt}(g)} g.$$

Theorem (Buchberger's test)

For $G = \{g_1, \dots, g_s\} \subset R$, $I = \langle G \rangle$, the followings are equivalent:

- *G is a Gröbner basis of I.*
- *$\overline{S(g_i, g_j)}^G = 0$ for each pair i, j s.t. $i \neq j$.*

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Gröbner basis

Buchberger's algorithm

Theorem

Let $I = \langle f_1, \ldots, f_s \rangle$ be a ideal. Then a Gröbner basis for I can be constructed in a finite number of steps by the following algorithm:

INPUT $F = (f_1, \ldots, f_s)$

OUTPUT a Gröbner basis $G = \{g_1, \ldots, g_t\}$ *for* I

$G := F$

REPEAT

  $G' := G$

  FOR each pair $\{p, q\}, p \neq q$ *in* G' *DO*

  $S := \overline{S(p, q)}^{G'}$

  IF $S \neq 0$ *THEN* $G := G \cup \{S\}$

UNTIL $G = G'$

Cup-length

cup-length and LS-category

Let R be a commutative ring. We define the cup-length of R as

$$\mathrm{cup}(R) := \max\left\{ n \middle|^{\exists} x_1, \dots, x_n \in R \setminus R^\times \text{ s.t. } x_1 \cdots x_n \neq 0 \right\}.$$

This invariant is useful in algebraic topology.
Let X be a space and A be a commutative ring.

$$\mathrm{cup}_A(X) := \mathrm{cup}(\tilde{H}^*(X; A)).$$

Theorem

$\mathrm{cup}_A(X) \leq \mathrm{cat}(X)$.

Where $\mathrm{cat}(X)$ is LS-category of X normalized as $\mathrm{cat}(*) = 0$.

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Cup-length
Oriented Grassmann manifolds

We will compute a cup-length of oriented Grassmann manifolds.

$$\widetilde{G}_{n,k} := SO(n+k)/SO(n) \times SO(k)$$

consists of oriented k-dimensional vector subspace in $\mathbb{R}^{n+k}$.

- When $k = 2$, the cohomology of $\widetilde{G}_{n,2}$ is well-known.
- When $k \geq 3$, that of $\widetilde{G}_{n,k}$ is in vague.

In this talk, we will compute the $\mathbb{Z}/2$ cup-length of oriented Grassmann manifold for

$$k = 3;\ n = 2^{m+1} - 4\ (m \geq 2).$$

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Main Theorem

Notion of Theorem

Theorem

$\mathrm{cup}_{\mathbb{Z}/2}(\widetilde{G}_{n,3}) = n + 1$ *for* $n = 2^{m+1} - 4\,(m \geq 2)$.

Sketch of Proof

Cohomology of oriented Grassmann manifolds

- There are the double covering map $p_n \colon \widetilde{G}_{n,k} \to G_{n,k}$.
- $\mathrm{Im} p_n^* \cong H^*(\widetilde{G}_{n,3}; \mathbb{Z}/2)\ (* < n)$.
  $\because\ \widetilde{G}_{n,3} \simeq_n BSO(3);\quad p_\infty^*, i^*$:epi.
- $\mathrm{Im} p_n^* \cong \mathbb{Z}/2[w_2, w_3]/J_n$, where $J_n = p_\infty^*(I_n)$.

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Sketch of Proof

Cohomology of oriented Grassmann manifolds

- There are the double covering map $p_n \colon \widetilde{G}_{n,k} \to G_{n,k}$.
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- $\mathrm{Im}\, p_n^* \cong \mathbb{Z}/2[w_2, w_3]/J_n$.
- By degree reason, $\mathrm{cup}(\mathrm{Im}\, p_n^*)$ determines $\mathrm{cup}_{\mathbb{Z}/2}(\widetilde{G}_{n,3})$.
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- By Borel, the generator of J_n is given.
- Computing Gröbner basis of J_n, we can solve the Ideal Membership Problem of J_n.
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LS-category

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For topological space X,

$$cat(X) = \min \left\{ n \middle| {}^{\exists} U_0, \dots, U_n \subset X \text{ open, contractible;} \right. \\ \left. \text{s.t. } X = \bigcup_{i=0}^{n} U_i \right\}.$$

- Any smooth fanction on a manifold X has at least $\mathrm{cat}(X) + 1$ critical points.
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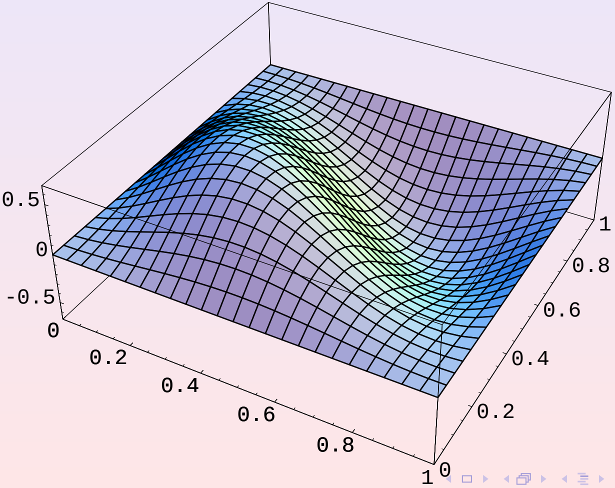
A smooth function on T^2 which has three critical points. $(\mathrm{cat}(T^2) = 2)$



LS-category

Lower and upper bounds

- By Main Theorem, immediately we have a lower bound of $\mathrm{cat}(\widetilde{G}_{n,3})$.
- Using some obstruction theory and Main Theorem, we also have a upper bound of it.

Theorem

$n+1 \leq \mathrm{cat}(\widetilde{G}_{n,3}) < \frac{3}{2}n$ for $n = 2^{m+1} - 4 (m \geq 2)$.
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We give an another application of our Main Theorem to Immersion of $\widetilde{G}_{n,3}$ into a Euclidean space.

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- *$\widetilde{G}_{n,3}$ immerses into $\mathbb{R}^{6n-3}$ but not into $\mathbb{R}^{3n+8}$ when $n = 2^{m+1} - 4\,(m \geq 3)$.*
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Proof: Lower bounds

- λ: canonical bundle over $\widetilde{G}_{n,3}$.
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- $T\widetilde{G}_{n,3} \cong \mathrm{Hom}(\lambda, \lambda^{\perp}) \cong \lambda \otimes \lambda^{\perp}$.
- $T\widetilde{G}_{n,3} \oplus \lambda \otimes \lambda \cong \lambda \otimes (\lambda \oplus \lambda^{\perp}) \cong (n+3)\lambda$.
- $(1 + w_2 + w_3)^{n+4} = 1$. $\quad\because$ Main Theorem.

$$\begin{aligned} w(\nu) &= \frac{1}{w(T\widetilde{G}_{n,3})} = \frac{w(\lambda \otimes \lambda)}{w((n+3)\lambda)} \\ &= 1 + w_2 + w_3 + w_2^2 + w_2^3 + w_3^2 + w_2^2 w_3 + w_2 w_3^2 + w_3^3. \end{aligned}$$

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Proposition (Hirsch)

Let M^m be m-dim manifold. The followings are equivalent.

- M^m can immerse into $\mathbb{R}^{m+p}$.
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- The classifying map $\nu\colon M^m \to BO$ lifts $\nu_p\colon M^m \to BO(p)$.

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Steenrod algebra

Motivation

- I would like to do calculations on the Steenrod algebra and modules over it, with computer.
- We consider the Steenrod algebra $\mathcal{A}_2$ as follow.

Let

$$A = \mathbb{Z}/2\langle \mathrm{Sq}^1, \ldots, \mathrm{Sq}^i, \ldots \rangle$$

be a free associative non commutative algebra. Let

$$I_{\mathrm{Adem}} = \left\langle \mathrm{Sq}^a \mathrm{Sq}^b - \sum_{i=0}^{[a/2]} \binom{b-1-i}{a-2i} \mathrm{Sq}^{a+b-i} \mathrm{Sq}^i \,\middle|\, a < 2b \right\rangle$$

be a two-side ideal of A generated by the Adem relations. Then

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Steenrod algebra

Non commutative Gröbner basis

$$\begin{aligned} A &= \mathbb{Z}/2\langle \mathrm{Sq}^1, \ldots, \mathrm{Sq}^i, \ldots \rangle \\ I_{\mathrm{Adem}} &= \left\langle \mathrm{Sq}^a \mathrm{Sq}^b - \sum_{i=0}^{[a/2]} \binom{b-1-i}{a-2i} \mathrm{Sq}^{a+b-i} \mathrm{Sq}^i \,\middle|\, a < 2b \right\rangle \\ \mathcal{A}_2 &= A / I_{\mathrm{Adem}}. \end{aligned}$$

- There exists the theory of a non commutative Gröbner basis and we can define the Gröbner basis of I_{Adem}.
- It is well-known that admissible products Sq^I forms a basis of the $\mathbb{Z}/2$-vector space $\mathcal{A}_2$, it follows that the Adem relations are Gröbner basis of I_{Adem}.

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Free resolution

Commutative and non Commutative case

- Let $R = k[x_1, \ldots, x_n]$ be free commutative ring.
- It is well-known that there is an algorithm using Gröbner basis for calculating a free resolution of R-module M.

$$\cdots \longrightarrow R^{a_2} \longrightarrow R^{a_1} \longrightarrow M.$$

- I generalized the above algorithm for a module over non commutative ring $A = k\langle x_1, \ldots, x_n\rangle$ using non commutative Gröbner basis.
- I would like to apply it to free resolution of $\mathcal{A}_2$-module and computing E_2 terms of the Adams spectral sequence.

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Free resolution

Calculation example

- I wrote a computer program which compute the Free resolution of $\mathbb{F}_2$.
- The following is the free resolution of $\mathbb{F}_2$ in degree less than 8

$$\cdots \xrightarrow{\varphi_9} \mathcal{A}_2^6 \xrightarrow{\varphi_8} \mathcal{A}_2^6 \xrightarrow{\varphi_7} \mathcal{A}_2^7 \xrightarrow{\varphi_6} \mathcal{A}_2^{14} \xrightarrow{\varphi_5} \mathcal{A}_2^9 \xrightarrow{\varphi_4} \mathcal{A}_2^8 \xrightarrow{\varphi_3} \mathcal{A}_2^4 \xrightarrow{\varphi_2} \mathcal{A}_2 \xrightarrow{\varphi_1} \mathbb{F}_2$$

$\varphi_i = \varphi_8$ for $i \geq 8$.

Now I am trying to compute $E_2^{*,*} = \mathrm{Ext}_{\mathcal{A}_2}^{*,*}(H^*(X), H^*(Y))$ which converges to $\{Y, X\}_*$, for $X, Y = S^n, \mathbb{R}P^n, \mathbb{C}P^n, O(n), U(n)$ etc...

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For Further Reading I

William W. Adams and Philippe Loustaunau.
An introduction to Gröbner bases, volume 3 of *Graduate Studies in Mathematics*.
American Mathematical Society, Providence, RI, 1994.

Huishi Li.
Noncommutative Gröbner bases and filtered-graded transfer, volume 1795 of *Lecture Notes in Mathematics*.
Springer-Verlag, Berlin, 2002.