Computer Algebra and Algebraic Topology

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Department of Mathematics Kyoto university
Tomohiro Fukaya

Today's Talk.

Experimental Mathematics

http://www.expmath.org/

Statement of Philosophy & Publishing Criteria

Experiment has always been, and increasingly is, an important method of mathematical discovery. (Gauss declared that his way of arriving at mathematical truths was "through systematic experimentation.") Yet this tends to be concealed by the tradition of presenting only elegant, fully developed, and rigorous results.

Topology

Very soft Geometry

- Object (Space) : topological space
- Morphism : continuous map

We consider that two spaces are "same" when we can deform one to the other in continuous way.

Example

- A mug cup and a solid torus are “same”.

Wikipedia

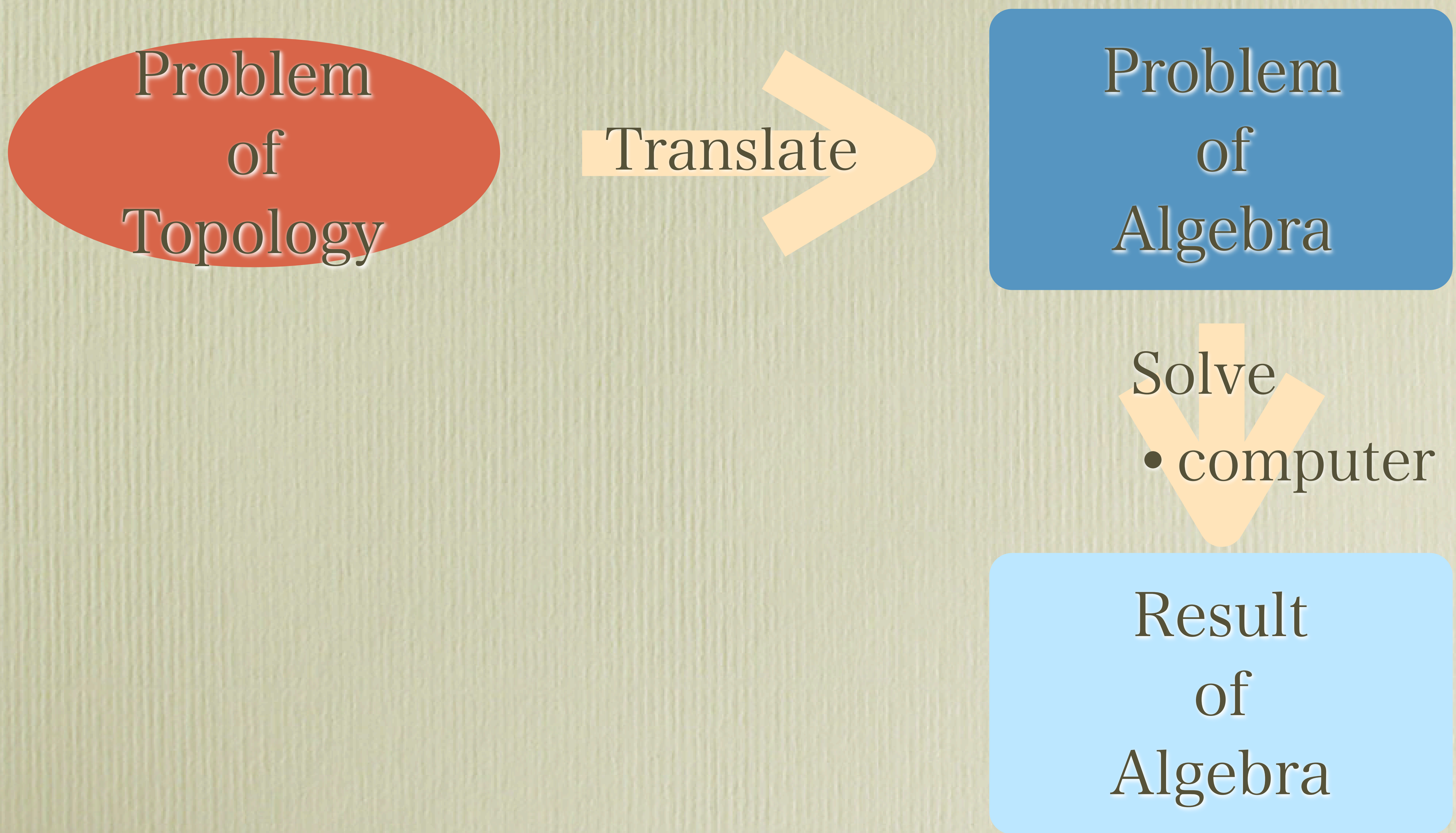
It is difficult (for me) to imagine such a continuous deformation in my head.

Translation

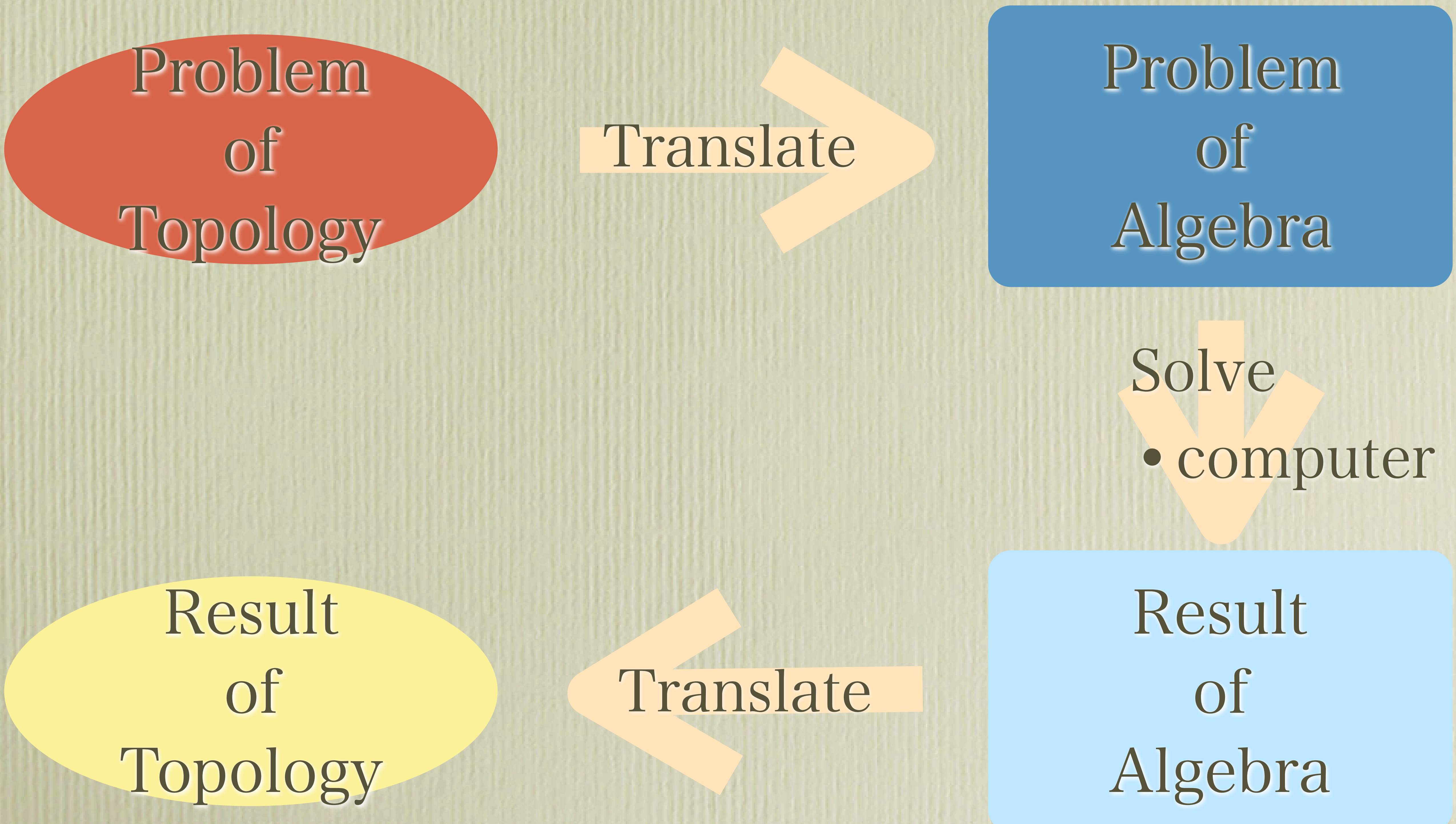
Problem of Topology

Translation

Translation



Translation



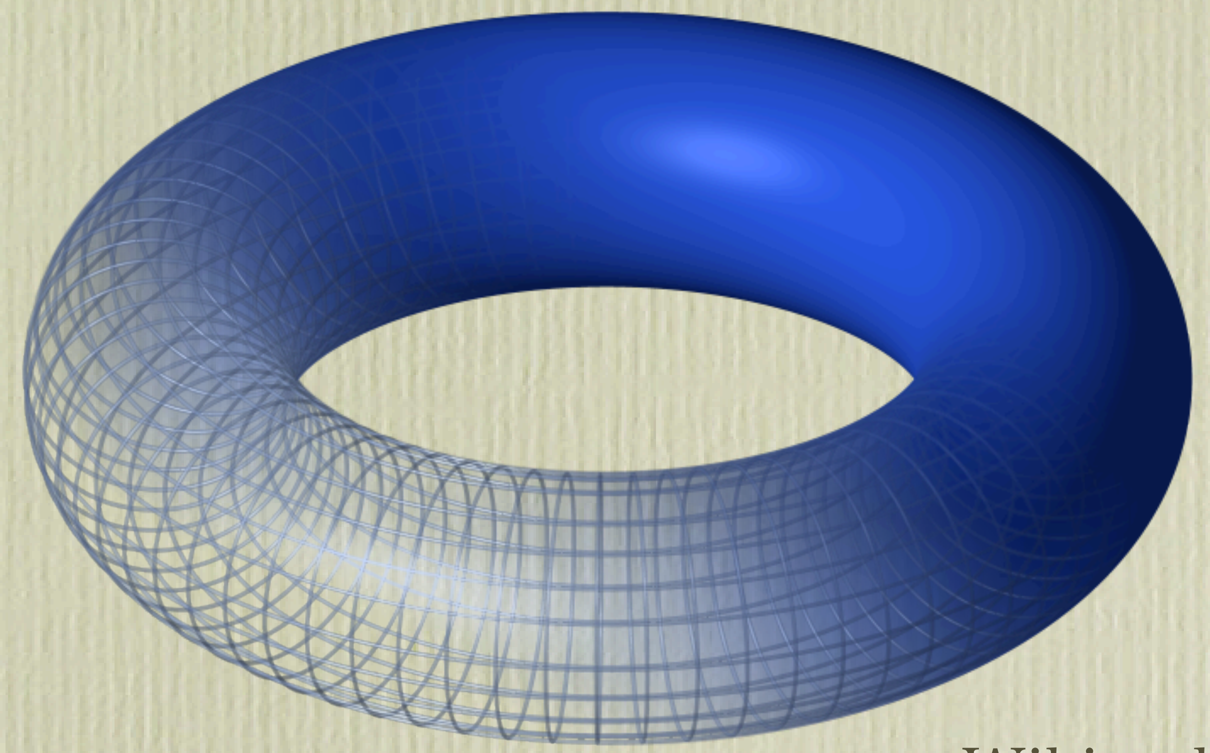
Dictionary

Translate the problem of TOPOLOGY into the problem of ALGEBRA.

Space		Algebra
X		$H^*(X)$: cohomology
X		$\pi_*(X)$: homotopy

Example

T : torus



Wikipedia

$$H^*(T;\mathbf{Z}) = \mathbf{Z}[t_1, t_2]/\langle t_1^2, t_2^2 \rangle$$

LS-category

Lusternik-Schnirelmann category

Simple but difficult problem of Topology

Definition X : topological space

$A \subset X$: contractible in X

$\Leftrightarrow$ We can deform A to a single point in X

$\Leftrightarrow {}^{\exists} H : A \times [0, 1] \to X$ continuous s.t.

$H(x, 0) = x,$

$H(x, 1) = p_0$ $\quad p_0 \in X$ base point

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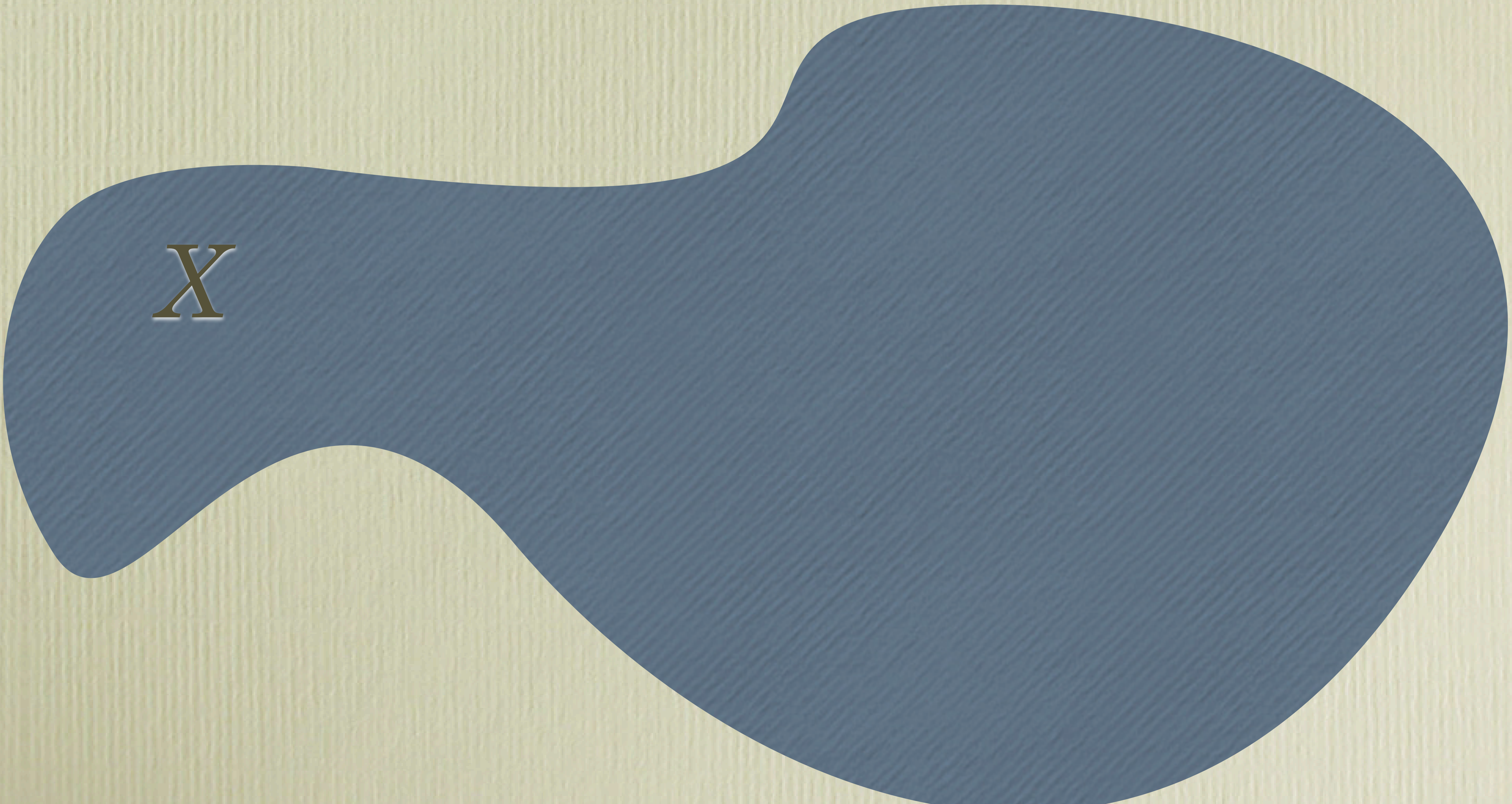

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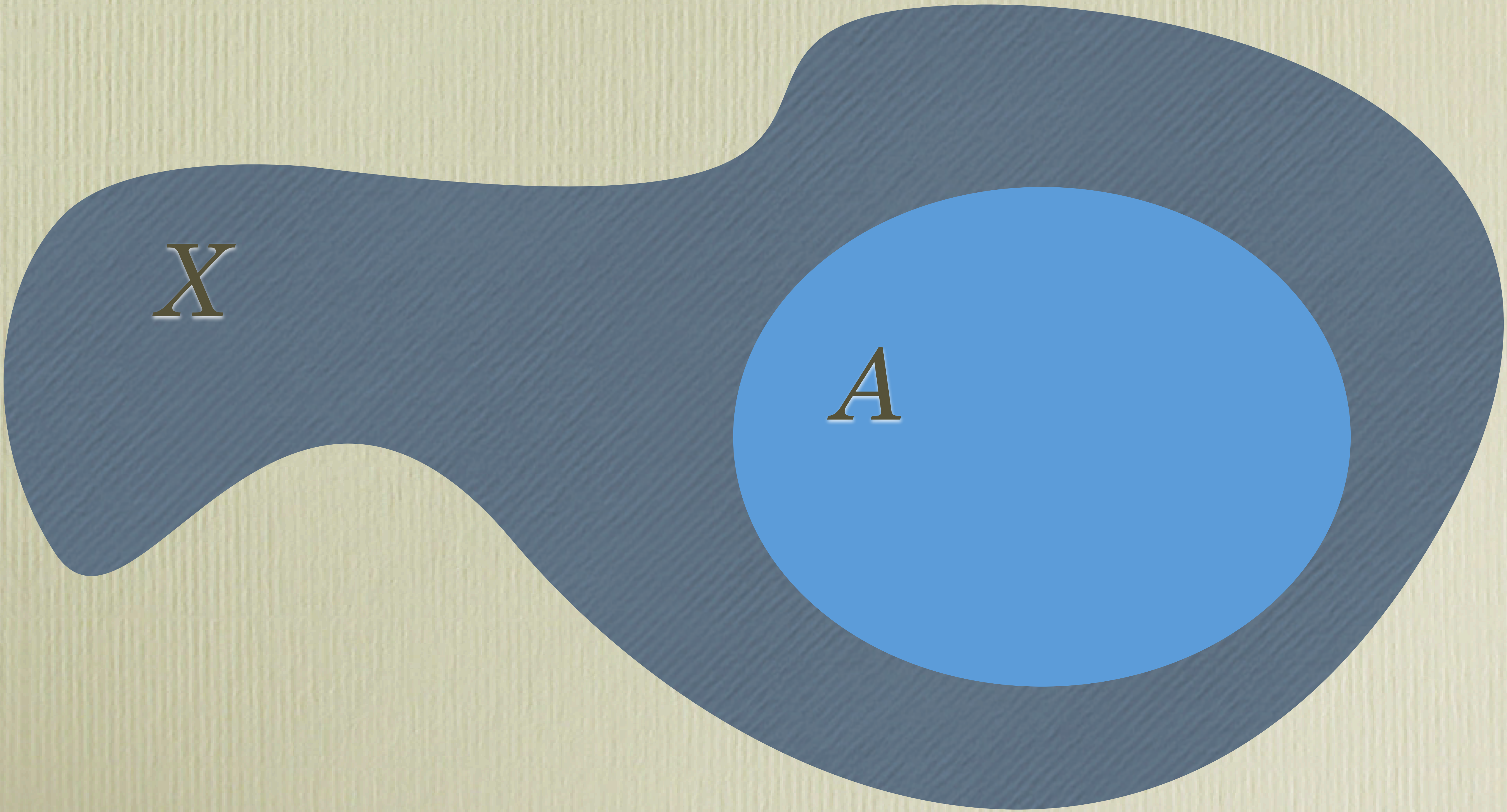

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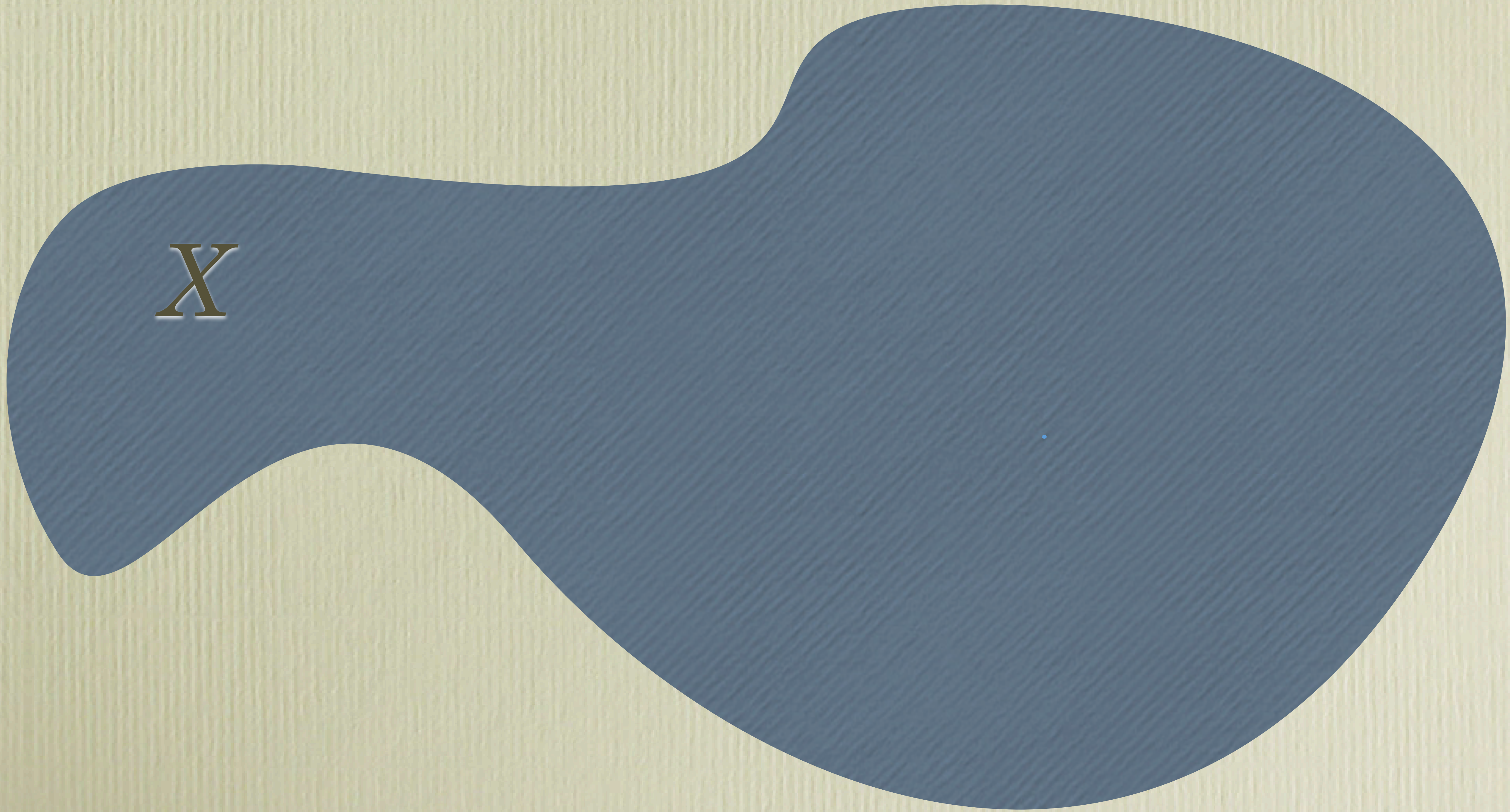

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Definition $X:$ topological space

$$\mathrm{cat}(X) := \min \left\{ n \;\middle|\; \begin{array}{c} {}^{\exists}U_1, \ldots, U_{n+1} \subset X \\ \text{open, contractible in } X \text{ s.t.} \\ X = \bigcup_{i=1}^{n+1} U_i \end{array} \right\}$$

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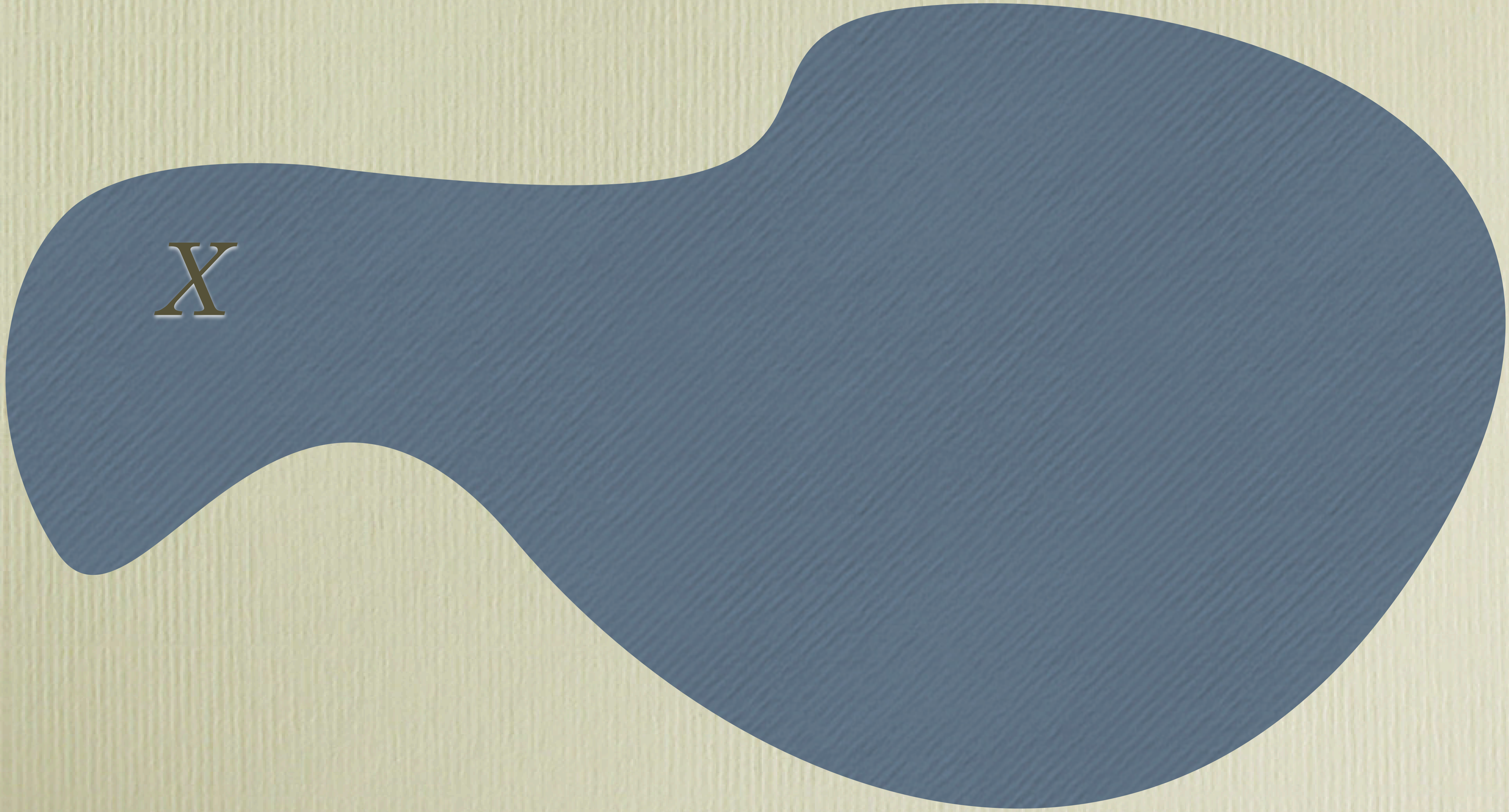

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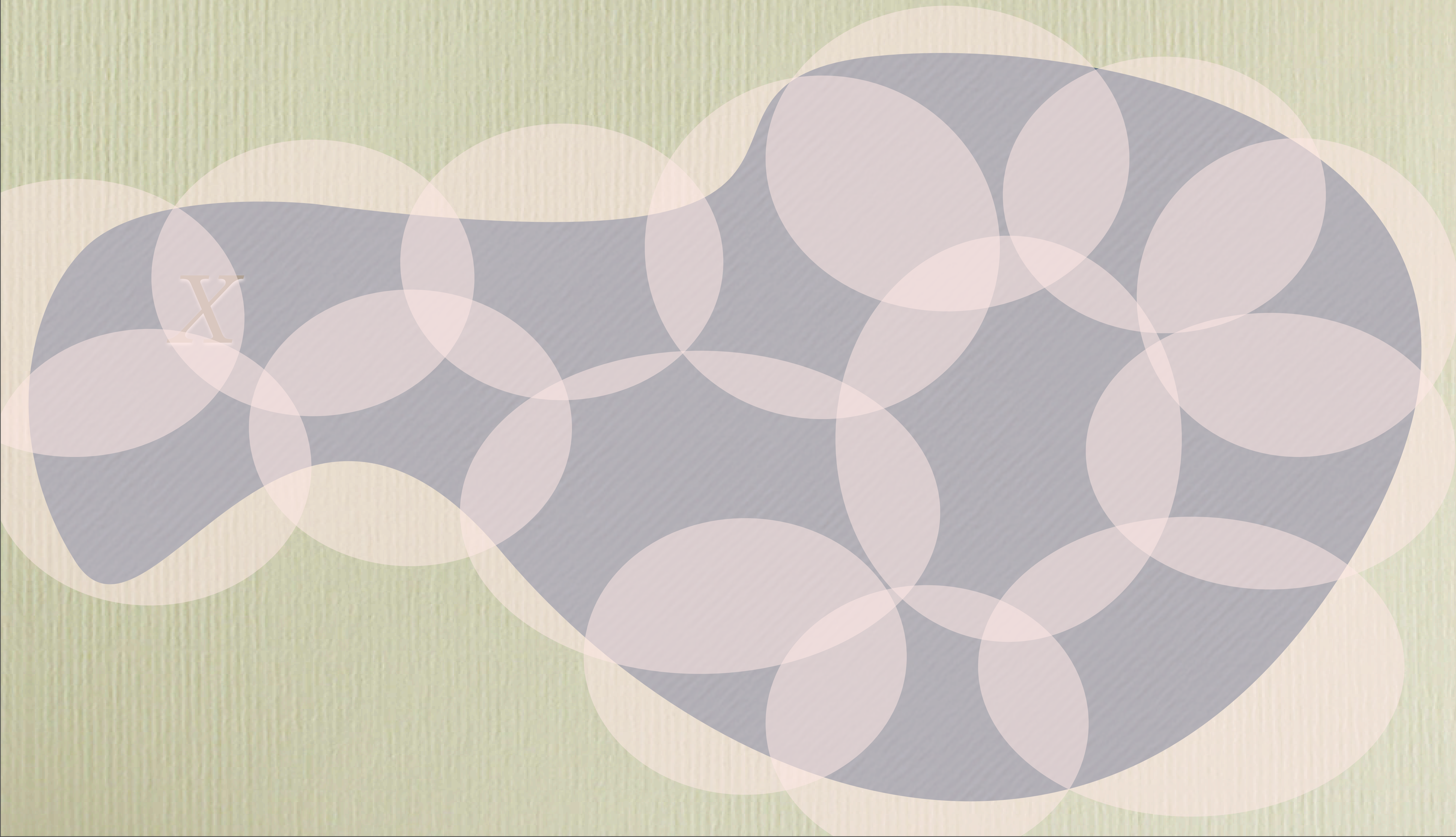

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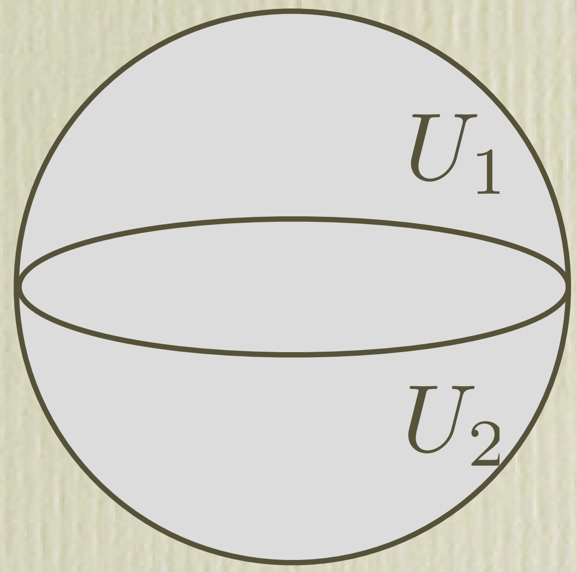
Example

$$\mathrm{cat}(S^2) = 1$$



LS-category

Lusternik-Schnirelmann category

- It looks quite difficult to compute cat(X) for arbitrary space from the definition.
- We translate this problem into the problem of an algebra.

Cup-length

Definition

$$\mathrm{cup}(X) := \max\left\{ n \;\middle|\; \begin{array}{c} {}^{\exists}x_1, \ldots, x_n \in H^*(X) \\ \text{s.t. } x_1 \cup x_2 \cup \cdots \cup x_n \neq 0 \end{array} \right\}$$

Cup-length

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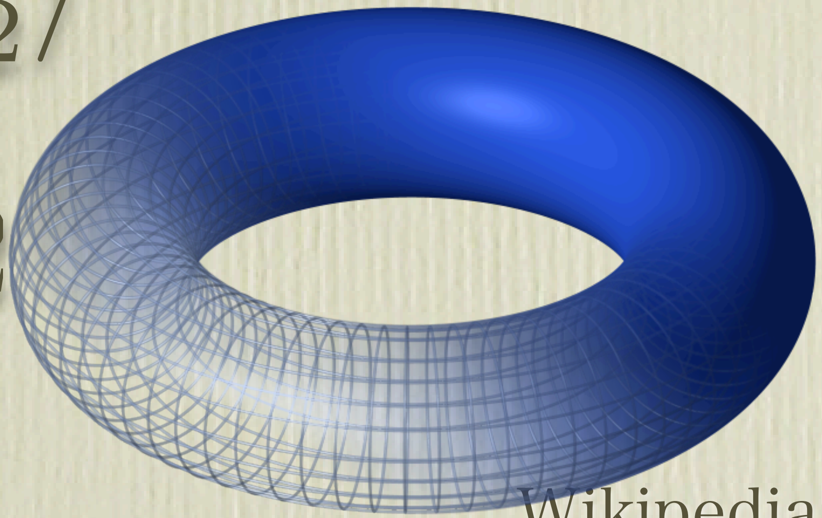
T : torus

$$H^*(T; \mathbf{Z}) = \mathbf{Z}[t_1, t_2]/\langle t_1^2, t_2^2 \rangle$$

$$t_1 \cup t_2 \neq 0 \Rightarrow \mathrm{cup}(T) = 2$$



Wikipedia

Cup-length

Definition

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Theorem

$$\mathrm{cup}(X) \leq \mathrm{cat}(X)$$

This theorem translate the problem of TOPOLOGY into the problem of ALGEBRA.

What I did

- I studied the LS-category of oriented Grassmann manifold $\tilde{\mathrm{Gr}}_{n,k}$ consists of oriented k-dimensional sub spaces in Euclidean $(n+k)$-space $\mathbf{R}^{n+k}$.

- Especially, for the case $k = 3$, I studied cup$(\tilde{\mathrm{Gr}}_{n,3})$ using computer

Cohomology of $\tilde{\mathrm{G}}\mathrm{r}_{n,3}$

There exists subring which isomorphic to

a polynomial ring.

$$H^*(\tilde{\mathrm{G}}\mathrm{r}; \mathbf{Z}/2) \supset \mathbf{Z}/2[w_2, w_3]/J_n$$

Cohomology of $\tilde{\mathrm{Gr}}_{n,3}$

There exists subring which isomorphic to

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$$H^*(\tilde{\mathrm{Gr}}; \mathbf{Z}/2) \supset \mathbf{Z}/2[w_2, w_3]/J_n$$

J_n is generated by g_n, g_{n+1}, g_{n+2}

$$g_r = \sum_{\frac{r}{3} \leq s \leq \frac{r}{2}} \binom{s}{3s-r} w_2^{3s-r} w_3^{r-2s}$$

Cohomology of $\tilde{\mathrm{G}}\mathrm{r}_{n,3}$

$$\mathrm{cup}(\tilde{\mathrm{G}}\mathrm{r}_{n,3}) = \max\{a+b | w_2^a w_3^b \notin J_n\} + 1$$

- To compute the cup-length of $\tilde{\mathrm{G}}\mathrm{r}_{n,k}$ we have to know the ring structure of

$$\mathbf{Z}/2[w_2, w_3]/J_n$$

- To study such multi-variable polynomial ring and it's quotient ring, Gröbner basis is a powerful method.

Gröbner basis

- There are many software which calculate the Gröbner basis (Macaulay, SINGULAR, CoCoA, etc...)
- I used "Asir" http://www.math.kobe-u.ac.jp/Asir/asir-ja.html
- First, I calculated $\mathrm{cup}(\tilde{\mathrm{G}}\mathrm{r}_{n,3})$ for

$$0 \leq n \leq 200$$

Table of cup-length

Table of $\mathrm{cup}(\tilde{\mathrm{G}}\mathrm{r}_{n,3})$ for $4 \leq n \leq 66$

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
cup	5	5	5	6	7	8	10	11	13	13	13	13	13	13	13	13	14	15	20	21	22

n	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
cup	23	26	27	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	29	30	31

n	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66
cup	40	41	42	43	44	45	46	47	52	53	54	55	58	59	61	61	61	61	61	61	61

Observation

- Each "period" begins when $n = 2^{m+1} - 4$.
- I investigated the generator of J_n for

$$n = 2^{m+1} - 4$$

$$H^*(\tilde{\mathrm{Gr}}_{n,3}; \mathbf{Z}/2) \supset \mathbf{Z}/2[w_2, w_3]/J_n$$

$$J_n = \langle g_n, g_{n+1}, g_{n+2} \rangle$$

On g_n

- By computer calculation, I saw that

$$g_n = 0 \quad \text{when } n = 2^{m+1} - 4.$$

- it can be proved easily by induction.

g_{n+1}, g_{n+2}

Binary expansion of coefficients of

$$g_{n+1} = \sum_{\frac{n+1}{3} \leq s \leq \frac{n+1}{2}} \binom{s}{3s-(n+1)} w_2^{3s-(n+1)} w_3^{n+1-2s}$$

$n = 2^6 - 4$

List of integers s with $\frac{n+2}{3} \leq s \leq \frac{n+2}{2}$ and $\binom{s}{3s-(n+1)}$ is odd.

S	binary expansion of s	S	binary expansion of s
42	101010	55	110111
43	101011	58	111010
45	101101	59	111011
46	101110	61	111101
47	101111	62	111110
53	110101	63	111111
54	110110		

Observation

Let $s = \sum 2^{s_i}$ (binary expansion of s).

$\binom{s}{3s-(n+1)}$ is odd $\Rightarrow$ if $s_j = 0$, then $s_{j+1} = 1$.

This property characterize the coefficients of

$$g_{n+1} = \sum_{\frac{n+1}{3} \leq s \leq \frac{n+1}{2}} \binom{s}{3s-(n+1)} w_2^{3s-(n+1)} w_3^{n+1-2s}$$

Similar things holds for g_{n+2} .

Conclusion

- This enable us to calculate the Gröbner basis of $J_n = \langle g_n, g_{n+1}, g_{n+2} \rangle$.
- Then we determin $\mathrm{cup}(\tilde{\mathrm{Gr}}_{n,3})$.

Theorem

$$\mathrm{cup}(\tilde{\mathrm{Gr}}_{n,3}) = n + 1 \text{ for } n = 2^{m+1} - 4$$

Application

- Upper bound of LS-category.

$$n + 1 \leq \mathrm{cat}(\tilde{\mathrm{G}}\mathrm{r}_{n,3}) < \frac{3}{2}n.$$

- Immersion into Euclidean space.

$\tilde{\mathrm{G}}\mathrm{r}_{n,3}$ immerses into $\mathbf{R}^{6n-3}$ but not into $\mathbf{R}^{3n+8}$
when $n = 2^{m+1} - 4\,(m \geq 3)$
and $\tilde{\mathrm{G}}\mathrm{r}_{4,3}$ immerses into $\mathbf{R}^{21}$ but not into $\mathbf{R}^{17}$.

- To prove both corollary, we need to show vanishing of some “obstructions”.
- Such “obstructions” are represented by cohomology class $w \in H^*(\tilde{\mathrm{Gr}}_{n,3})$.
- As above, we can investigate the vanishing of those “obstructions” using computer.

Thank you!